ELASTIC-PLASTIC TORSION OF A SQUARE BAR(1)

BY
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1. **Introduction.** Consider a cylindrical bar twisted by terminal couples. According to the theory for elastic-plastic solids [2], [3], the twisted bar will behave elastically, if the angle of twist per unit length is sufficiently small. Under such circumstances, one calls it purely elastic torsion. However, as the applied torque increases to a certain critical value, some portion of the bar will become plastic, i.e., the maximum shearing stress there exceeds a definite value. Furthermore, it is assumed that as the applied torque increases the plastic portion of the bar will continue to grow. It is in this situation that one calls it elastic-plastic torsion.

Apparently, von Mises was the first to give a complete description of the elastic-plastic torsion problem. He described it as follows: "Find a continuous function ψ with given constant $\Delta\psi$ in some interior region, given $|\operatorname{grad}\psi|$ in the outer region and given a constant value on the closed contour." In [2] the same problem is described in a more detailed way as follows: "Find a function $\psi(x, y)$ which vanishes on C and, together with its first derivatives, is continuous in the domain bounded by C; nowhere on C or its interior must the gradient of ψ have an absolute value larger than a given constant k; wherever the absolute value of $\nabla\psi$ is smaller than k, the function ψ must satisfy the differential equation $\Delta\psi = -2\mu\theta$."

From what has been described in [1] and [2] together with the results for completely plastic torsion [4], it is not difficult to see that elastic-plastic torsion is a free-boundary-value problem for the Poisson equation, $\Delta\psi = -2\mu\theta$. In what follows, attention will be restricted only to the simple case of a square bar. Also, as a method for overcoming certain difficulties, we shall formulate it as a variational minimum problem. The object of this paper is to exhibit a smooth solution for the minimum problem and to establish the uniqueness of the smooth solution and its continuous monotone dependence upon parameters.

2. **Statement of the problem.** Let Q be a square plane domain. Let $\Psi(x,y)$ be the function defined by the formula,

$$\Psi(q) = k\rho(q, \partial Q), \qquad q \in \bar{Q},$$

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where $\rho(q, \partial Q)$ stands for the distance from the point q to the boundary of Q, ∂Q, and where k is a positive constant. The problem to be considered is to find the function $\psi(x, y)$ such as to minimize the integral,

$$I[u] = \iint_Q [(\nabla u)^2 - 4\mu\theta u]\, dxdy \tag{2.1}$$

with μ, θ being positive constants. The admissible class consists of those functions which are continuous and are less than or equal to Ψ in $\bar{Q}$ and which vanish on ∂Q and possess finite Dirichlet integrals over Q.

REMARK 1. Clearly, the values of the expressions in (2.1) are bounded from below. For if u^* is the solution of the Dirichlet problem,

$$\Delta u^* = -2\mu\theta \text{ in } Q, \quad u^* = 0 \text{ on } \partial Q, \tag{2.2}$$

then for every admissible function, $I[u] \geqq I[u^*]$. Furthermore, from the integral representation for the solution u^* of (2.2),

$$u^*(x, y) = 2\mu\theta \iint_Q G(x, y; \xi, \eta)\, d\xi d\eta$$

with G being Green's function of (2.2) in Q, we see that $u^* > \Psi$ somewhere in Q provided that $\mu\theta$ is sufficiently large. Therefore, the admissibility condition, $u \leqq \Psi$ in $\bar{Q}$, is essential.

REMARK 2. If $\mu\theta$ is so small that the solution u^* of the Dirichlet problem (2.2) is less than or equal to Ψ in $\bar{Q}$, then it is a solution of the minimum problem. Thus the present formulation of the elastic-plastic torsion problems does include purely elastic torsion problems as a special class.

REMARK 3. Consider the integral,

$$\frac{I[u]}{4\mu\theta} = \iint_Q \left[\frac{(\nabla u)^2}{4\mu\theta} - u\right] dxdy.$$

If we let $\theta \to \infty$, i.e., the angle of twist becomes infinite, then the minimum problem is that of maximizing the integral,

$$\iint_Q u(x, y)\, dxdy \tag{2.3}$$

subject to the condition that $u \leqq \Psi$ in $\bar{Q}$. Accordingly, the present formulation also includes completely plastic torsion as a limiting case [4].

REMARK 4. The minimum problem (2.1) may be regarded as a principle of minimizing the energy,

$$\iint_Q (\nabla u)^2 \, dx dy, \tag{2.4}$$

under the additional isoperimetric condition that the integral (2.3) is equal to a given constant. That is, the applied torque is given. For these two minimum problems are equivalent if we identify the constant, $-4\mu\theta$, as a Lagrange multiplier. In both formulations, the yield condition, i.e., that the maximum shearing stress be less than or equal to the constant k, has been replaced by the somewhat stronger inequality, $u \leqq \psi$ in $\bar{Q}$. However, such a replacement will not only simplify the analysis, it also insures that the extremal will actually solve the elastic-plastic problem.

The main object is now to show that the minimum problem has a unique smooth solution which fulfills all the requirements as listed in [1], [2]

3. **Minimizing sequence and symmetrizations.** Let

$$d = \inf_{\{u\}} I[u]. \tag{3.1}$$

Suppose that no extremal can be found, then there is a minimizing sequence $\{\psi_n\}$ such that

$$\iint_Q [(\nabla \psi_n)^2 - 4\mu\theta\psi_n] \, dx dy = d_n \to d. \tag{3.2}$$

To select a pointwise convergent subsequence from the minimizing sequence, we apply the following symmetrization processes:

a. *Steiner symmetrization.* In what follows, we choose the center of Q as the origin of a rectangular coordinate system, (x, y), with the coordinate axes coinciding with the diagonals of Q. We apply Steiner symmetrization [5] with respect to the plane, $x = 0$, to the body B_n bounded by the plane $z = 0$ and the surface $z = \psi_n(x, y), (x, y) \in \bar{Q}$. This symmetrization process leaves the volume of B_n and its base Q unchanged and it diminishes (does not increase) the surface area of B_n. Accordingly, the function $\psi_n^*(x, y)$ obtained from $\psi_n(x, y)$ by such a symmetrization process yields a smaller (not larger) Dirichlet integral over Q [5]. Also, it is clear that ψ_n^* is continuous and is less than or equal to Ψ in $\bar{Q}$. Hence, it is an admissible function.

Since Steiner symmetrization about the plane $y = 0$ transforms a body which has already been symmetrized about the plane $x = 0$ into one retaining that property, we conclude that no generality is lost if we assume that each ψ_n in the minimizing sequence has been symmetrized with respect to both of the planes, $x = 0$ and $y = 0$. We note that such a function $\psi_n(x, y)$ is monotone nonincreasing in $|x|$ and $|y|$ in its domain of definition.

b. *Partial Steiner symmetrization.* Denote by 1, 2, 3, 4 the vertices of Q

and by 0 the center of Q. Let Q_{12} be the subdomain of Q bounded by the line segments 01, 02 and 12. Consider the body b_n bounded by the three planes, $x=0$, $y=0$ and

$$z = \Psi(x,y), \qquad (x,y) \in \bar{Q}_{12},$$

and that portion of the surface, $z=\psi_n(x,y)$, where $(x,y)\in\bar{Q}_{12}$ and $\psi_n(x,y) < \Psi(x,y)$. The body b_n may consist of many pieces. We shall apply "partial Steiner symmetrization" to this body b_n so as to define a function $\psi_n^*(x,y),(x,y)\in\bar{Q}_{12}$, such that ψ_n^* will yield smaller value of the integral (2.1) and that both the functions ψ_n^* and $\Psi-\psi_n^*$ are monotone in $|x|$ and $|y|$.

To this end, we choose the point with the coordinates,

$$x = y = 0, \qquad z = \Psi(0,0),$$

as the origin of a coordinate system (ξ,η,ζ) such that the ξ-axis is perpendicular to the plane $y=0$ and that the positive ζ-axis contains the vertex 1 of Q (see Figure 1). Let A_n be the orthogonal projection of the body b_n upon the plane

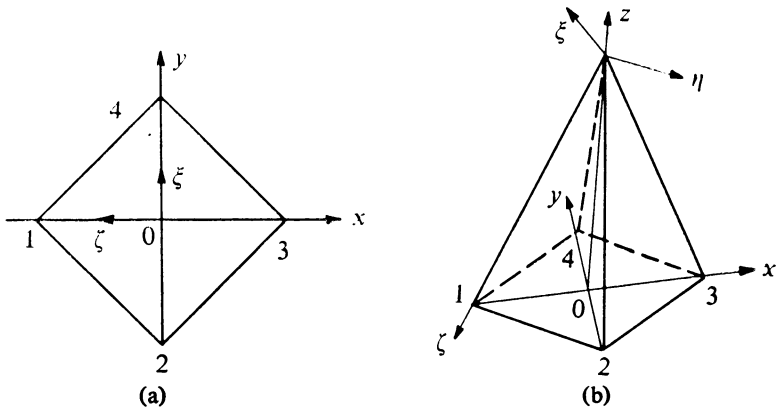

FIGURE 1

$\zeta=0$. We construct the body b_n^* from the body b_n by the following geometrical relations: A straight line through a point (ξ,η) in A_n and parallel to the ζ-axis intersects ∂b_n at $2m$ points,

$$(\xi,\eta,\zeta_1),\ (\xi,\eta,\zeta_2),\cdots,(\xi,\eta,\zeta_{2m}) \tag{3.3}$$

with

$$\zeta_1(\xi,\eta) > \zeta_2(\xi,\eta) > \cdots > \zeta_{2m}(\xi,\eta); \tag{3.4}$$

the same straight line intersects ∂b_n^* at two and only two points,

$$(\xi,\eta,\zeta_0), \qquad (\xi,\eta,\zeta),$$

where the point (ξ,η,ζ_0) lies on the plane $x=0$ and

$$\zeta(\xi,\eta) = \zeta_0(\eta) + \sum_{j=1}^{2m} (-1)^{j-1}\zeta_j(\xi,\eta). \tag{3.5}$$

We emphasize that $\zeta_{2m} \geqq \zeta_0$ and that ζ_0 is a function of η alone. The body b_n^* so obtained is bounded by the three planes, $x = 0$, $y = 0$ and $z = \Psi(x,y)$, (x,y) in $\bar{Q}_{12}$ and the surface defined in (3.5), i.e., the surface

$$\zeta = \zeta(\xi,\eta), \qquad (\xi,\eta) \in A_n. \tag{3.6}$$

The procedure we have just outlined is meaningful if the surface $\psi_n(x,y)$ is sufficiently smooth, but the construction remains valid for all admissible functions also because they can be obtained as the limiting cases of sufficiently smooth functions. Denote by $v(b_n)$, $v(b_n^*)$ respectively the volume of b_n and b_n^*. Then $v(b_n) = v(b_n^*)$. Indeed,

$$\begin{aligned} v(b_n^*) &= \iint_{A_n} [\zeta(\xi,\eta) - \zeta_0(\xi,\eta)]\, d\xi d\eta \\ &= \iint_{A_n} \left[\sum_{j=1}^{2m} (-1)^{j-1}\zeta_j(\xi,\eta)\right] d\xi d\eta = v(b_n), \end{aligned}$$

where the relations in (3.3)–(3.5) have been used.

Let S_n be the surface area of the surface

$$z = \psi_n(x,y), \qquad (x,y) \in \bar{Q}_{12} \text{ and } \psi_n(x,y) < \Psi(x,y). \tag{3.7}$$

Let S_n^* be the surface area of the surface defined by (3.5) and (3.6). We assert that

$$S_n^* \leqq S_n. \tag{3.8}$$

To prove this inequality, we first add the following remark:

REMARK 5. Steiner symmetrization leaves unchanged the area of the intersection of the body with any cylinder whose generator is perpendicular to the plane of symmetrization [5].

The same proof as shown in [5] also shows that the above remark is equally applicable to the partial Steiner symmetrization. Let σ_n be the area of those cylindrical parts of the surface in (3.7), where its tangential planes are all perpendicular to the plane $\zeta = 0$. According to Remark 5, this portion of the surface (3.7) is only translated without changing its area by the partial Steiner symmetrization and it becomes a part of the surface (3.6). Let σ_n^* be the area of those cylindrical parts of the surface (3.6) where its tangential planes are normal to the plane $\zeta = 0$. It is easy to see that σ_n^* is caused by jump discontinuities of the functions $\zeta_j(\xi,\eta)$, $j = 1,2,\cdots,2m$. Thus, we conclude that

$$\sigma_n^* = \sigma_n. \tag{3.9}$$

We proceed now to calculate the value of $S_n - \sigma_n$ by using the coordinate system (ξ,η,ζ) and the functions $\zeta_j(\xi,\eta)$. To do this we note that the points of intersections, (ξ,η,ζ_j), $j = 1,2,\cdots,2m-1$, all belong to the surface (3.7). However, whether the point, (ξ,η,ζ_{2m}), lies on the plane $x = 0$ depends on whether the equality $\zeta_{2m} = \zeta_0$ holds. If $\zeta_{2m} > \zeta_0$, the point (ξ,η,ζ_{2m}) also belongs to the surface (3.7). Thus.

$$S_n - \sigma_n = \iint_{A_n} \left\{ \sum_{j=1}^{N} \left[1 + \left(\frac{\partial \zeta_j}{\partial \xi}\right)^2 + \left(\frac{\partial \zeta_j}{\partial \eta}\right)^2 \right]^{1/2} \right\} d\xi d\eta, \tag{3.10}$$

where to exclude the area contributed by the plane $x = 0$ we set

$$N(\xi,\eta) = 2m(\xi,\eta) - 1 \quad \text{if } \zeta_{2m} - \zeta_0 = \frac{\partial \zeta_{2m}}{\partial \xi} = \frac{\partial}{\partial \eta}(\zeta_{2m} - \zeta_0) = 0, \tag{3.10a}$$

$$N(\xi,\eta) = 2m(\xi,\eta) \quad \text{otherwise.} \tag{3.10b}$$

Correspondingly, we have

$$\begin{aligned} S_n^* - \sigma_n^* = \iint_{A_n} \Bigg\{ 1 &+ \left[\frac{\partial}{\partial \xi}\left(\zeta_0 + \sum_{j=1}^{2m} (-1)^{j-1} \zeta_j \right) \right]^2 \\ &+ \left[\frac{\partial}{\partial \eta}\left(\zeta_0 + \sum_{j=1}^{2m} (-1)^{j-1} \zeta_j \right) \right]^2 \Bigg\}^{1/2} d\xi d\eta. \end{aligned} \tag{3.11}$$

We proceed to show that the value of the integrand in (3.10) is not less than that in (3.11) for almost all points in A_n where these values are defined. Now, if (3.10a) holds, then the integrand in (3.10) becomes

$$\sum_{j=1}^{2m-1} \left[1 + \left(\frac{\partial \zeta_j}{\partial \xi}\right)^2 + \left(\frac{\partial \zeta_j}{\partial \eta}\right)^2 \right]^{1/2}; \tag{3.12}$$

while the integrand in (3.11) can be written as

$$\left\{ 1 + \left[\frac{\partial}{\partial \xi} \sum_{j=1}^{2m-1} (-1)^{j-1} \zeta_j \right]^2 + \left[\frac{\partial}{\partial \eta} \sum_{j=1}^{2m-1} (-1)^{j-1} \zeta_j \right]^2 \right\}^{1/2}. \tag{3.13}$$

On the other hand, if (3.10b) holds, then the integrand in (3.10) becomes

$$\sum_{j=1}^{2m} \left[1 + \left(\frac{\partial \zeta_j}{\partial \xi}\right)^2 \quad \left(\frac{\partial \zeta_j}{\partial \eta}\right)^2 \right]^{1/2}; \tag{3.14}$$

while the integrand in (3.11) can be written as

$$\left\{ 1 + \left[\sum_{j=1}^{2m-1} (-1)^{j-1} \frac{\partial \zeta_j}{\partial \xi} \right]^2 + \left[\left(\frac{\partial \zeta_0}{\partial \eta} - \frac{\partial \zeta_{2m}}{\partial \eta} \right) + \sum_{j=1}^{2m-1} (-1)^{j-1} \frac{\partial \zeta_j}{\partial \eta} \right]^2 \right\}^{1/2}, \tag{3.15}$$

because $\partial\zeta_0/\partial\xi = 0$. Moreover, $\psi_n(x,y)$ is a single-valued function of x and y, geometrical consideration shows that

$$\left| \frac{\partial\zeta_0}{\partial\eta} - \frac{\partial\zeta_{2m}}{\partial\eta} \right| \leqq \left| \frac{\partial\zeta_{2m}}{\partial\eta} \right| .$$

Thus, by the same proof as for Steiner's theorem about surface area [5], our assertion (3.8) follows from (3.9), (3.12) and (3.13) or from (3.9), (3.14) and (3.15).

Since $\psi_n(x,y)$ is monotone nonincreasing in $|x|$ and $|y|$, it is easy to show that $\zeta(\xi,y)$ defined in (3.5) is monotone nonincreasing in $|\eta|$. Consequently, the surface (3.6) has the nonparametric representation,

$$z = \psi_n^*(x,y), \qquad (x,y) \in q_{12},$$

with respect to the coordinate system (x,y,z), where q_{12} is a subset of $\bar{Q}_{12}$. However, we can extend $\psi_n^*(x,y)$ to the closed domain $\bar{Q}_{12}$ by putting

$$\psi_n^*(x,y) = \Psi(x,y)$$

wherever ψ_n^* is not defined by the relation in (3.5). It is easy to see that ψ_n^* so extended to $\bar{Q}_{12}$ is continuous and it is less than or equal to Ψ in $\bar{Q}_{12}$ and it vanishes along the segment 12. Moreover, S_n^* is precisely the area of that portion of the surface,

$$z = \psi_n^*(x,y), \qquad (x,y) \in \bar{Q}_{12} \text{ and } \psi_n^* < \Psi .$$

Let $\hat{S}_n$ be the surface area of that portion of the surface $z = \psi_n(x,y)$ where $(x,y) \in \bar{Q}_{12}$ and $\psi_n(x,y) = \Psi(x,y)$. Similarly, let $\hat{S}_n^*$ be the surface area of that portion of the surface $z = \psi_n^*(x,y)$ where $(x,y) \in \bar{Q}_{12}$ and $\psi_n^* = \Psi$. Let S be the surface area of the surface $z = \Psi(x,y)$, (x,y) in $\bar{Q}_{12}$ and V the volume under this surface. Let $\hat{\sigma}_n, \hat{\sigma}_n^*$ be respectively the area of the intersection of b_n and b_n^* with the plane $z = \Psi(x,y)$, (x,y) in $\bar{Q}_{12}$. In view of Remark 5, we have $\hat{\sigma}_n = \hat{\sigma}_n^*$. Hence,

$$\hat{S}_n = S - \hat{\sigma}_n = S - \hat{\sigma}_n^* = \hat{S}_n^* .$$

This equality together with the inequality (3.8) implies that

$$S_n + \hat{S}_n \geqq S_n^* + \hat{S}_n^* .$$

That is, the surface area of the surface, $z = \psi_n(x,y), (x,y)$ in $\bar{Q}_{12}$, is greater than (not less than) that of the surface, $z = \psi_n^*(x,y)$, (x,y) in $\bar{Q}_{12}$. From the fact that $v(b_n) = v(b_n^*)$, we also conclude that the volumes under these two surfaces are equal. For they are equal to $V - v(b_n) = V - v(b_n^*)$.

Finally, we extend the function $\psi_n^*(x,y)$ to the whole closed square $\bar{Q}$ by two successive reflections about the lines $x = 0$ and $y = 0$ respectively. Then $\psi_n^*(x,y)$ so extended is *continuous* in $\bar{Q}$ and vanishes on ∂Q. Because of the symmetry of

$\Psi(x,y), \psi_n^*$ is less than or equal to Ψ in $\bar{Q}$ and hence it is an admissible function. Also, from the fact that $\psi_n(x,y)$ has been symmetrized about both of the planes $x=0$ and $y=0$, we conclude that the surface area of the surface,

$$z = \psi_n^*(x,y), \qquad (x,y) \in \bar{Q},$$

is less than or equal to that of the surface,

$$z = \psi_n(x,y), \qquad (x,y) \in \bar{Q},$$

and that the volumes under these two surfaces are equal. Since both ψ_n, ψ_n^* vanish along ∂Q, it follows [5] that

$$I[\psi_n^*] \leqq I[\psi_n].$$

In passing, we note that (i) $\psi_n^*(x,y)$ *is also monotone nonincreasing in* $|x|$. *and* $|y|$, (ii) $\Psi(x,y) - \psi_n^*(x,y)$ *is monotone nonincreasing in* $|x|$.

Now, we proceed to define another admissible function ψ_n^{**} from the function ψ_n^*. To this end, we choose the point with coordinates,

$$x = y = 0, \qquad z = \Psi(0,0),$$

as the origin of a coordinate system (ξ', η', ζ') such that the ξ'-axis is perpendicular to the plane $x=0$ and the positive ζ'-axis contains the vertex 2 of Q. Let A_n^* be the orthogonal projection of b_n^* upon the plane $\zeta'=0$. We transform the body b_n^* into a body b_n^{**} by the following relations: A straight line through a point (ξ', η') in A_n^* and parallel to the ζ'-axis intersects ∂b_n^* at $2m$ points,

$$(\xi', \eta', \zeta_1'), (\xi', \eta', \zeta_2'), \cdots, (\xi', \eta', \zeta_{2m}')$$

with

$$\zeta_1'(\xi', \eta') > \zeta_2'(\xi', \eta') > \cdots > \zeta_{2m}'(\xi', \eta');$$

it intersects ∂b_n^{**} at two and only two points,

$$(\xi', \eta', \zeta_0'), \quad (\xi', \eta', \zeta'),$$

where the point (ξ', η', ζ_0') lies on the plane $y=0$ and

$$\zeta'(\xi', \eta') = \sum_{j=1}^{2m} (-1)^{j-1} \zeta_j'(\xi', \eta').$$

By construction completely analogous to what has just been done, we obtain a function $\psi_n^{**}(x,y), (x,y)$ in $\bar{Q}$, from the function $\psi_n^*(x,y), (x,y)$ in $\bar{Q}$, such that ψ_n^{**} is an admissible function with

$$I[\psi_n^{**}] \leqq I[\psi_n^*].$$

Furthermore, (i) $\psi_n^{**}(x,y)$ *is also monotone nonincreasing in* $|x|$ *and* $|y|$ and (ii) $\Psi(x,y) - \psi_n^{**}(x,y)$ *is monotone nonincreasing in* $|x|$ *and* $|y|$.

REMARK 6. The property (ii) of ψ_n^{**} insures that ψ_n^{**} will be invariant under further partial Steiner symmetrization with respect to the plane $\zeta = 0$ or the plane $\zeta' = 0$. Furthermore, since ψ_n^{**} is, by its construction, symmetric with respect to the planes $x = 0$ and the plane $y = 0$, the property (i) of ψ_n^{**} insures that it is invariant under Steiner symmetrization about the plane $x = 0$ or the plane $y = 0$.

REMARK 7. We may also expect that ψ_n^{**} will also be invariant under Steiner symmetrization about the plane $x + y = 0$ or the plane $x - y = 0$. Indeed, if this were not the case, we can easily construct a function ψ_n^{***} with this additional property. To do this, let m be the middle point of the segment 12. Let Δ_{m1}, Δ_{m2} denote respectively the triangles $0m1$ and $0m2$. Suppose that

$$\iint_{\Delta_{m1}} [(\nabla\psi_n^{**})^2 - 4\mu\theta\psi_n^{**}]\,dxdy \leqq \iint_{\Delta_{m2}} [(\nabla\psi_n^{**})^2 - 4\mu\theta\psi_n^{**}]\,dxdy.$$

Then we define

$$\psi_n^{***}(x,y) = \psi_n^{**}(x,y), \qquad (x,y) \in \Delta_{m1},$$

and extend ψ_n^{***} to the whole closed square $\bar{Q}$ by repeated reflections about the lines of symmetry of Q. The function ψ_n^{***} so defined will have the following properties: (i) it is monotone nonincreasing in $|x|$ and $|y|$, (ii) it is symmetric with respect to each of the planes, $x = 0$, $y = 0$, $x + y = 0$ and $x - y = 0$ and (iii) $\Psi(x,y) - \psi_n^{***}(x,y)$ is monotone nonincreasing in $|x|$ and $|y|$. Without loss of generality, we shall assume that each function ψ_n in the original minimizing sequence $\{\psi_n\}$ has these three properties.

REMARK 8. The above conclusions (i)–(iii) obtained by partial Steiner symmetrizations can also be reached by the following method. Let $\{\psi_n\}$ be the minimizing sequence obtained from a minimizing sequence by Steiner symmetrizations with respect to both of the planes, $x = 0$ and $y = 0$. Consider the functions,

$$\chi_n(x,y) = \Psi(x,y) - \psi_n(x,y), \qquad n = 1,2,3,\cdots,$$

for (x,y) in $\bar{Q}$. We may apply Steiner symmetrizations with respect to the planes, $x = 0$ and $y = 0$, to the body, b_n, bounded by the surface, $z = \chi_n(x,y)$, (x,y) in $\bar{Q}$ and the plane $z = 0$. In this way, we can construct a body b_n^* which is bounded by the surface $z = \chi_n^*(x,y)$ and the plane $z = 0$. The interesting point is that for all integers n the function,

$$\psi_n^*(x,y) = \Psi(x,y) - \chi_n^*(x,y), \qquad (x,y) \in \bar{Q},$$

satisfies the inequality,

$$I[\psi_n^*] \leqq I[\psi_n].$$

Furthermore, geometrical consideration also shows that this geometrical operation is actually a partial Steiner symmetrization.

4. **Existence of extremal.** Based on what has been proved, we shall assume with no loss of generality that for each n the function $\psi_n(x,y)$ in the minimizing sequence, $\{\psi_n\}$, has the properties (i)–(iii) in Remark 7. Hence, it is possible to deduce from Helly's selection principle that the sequence $\{\psi_n\}$ includes a convergent subsequence. However, the monotonicity of the functions, ψ_n and $\Psi - \psi_n$, yields much stronger results. Indeed, for $0 \leqq x_1 \leqq x_2$ or $x_2 \leqq x_1 \leqq 0$, we have

$$\begin{aligned} 0 &\leqq \psi_n(x_1,y) - \psi_n(x_2,y) \\ &\leqq [\Psi(x_1,y) - \Psi(x_2,y)] - \{[\Psi(x_1,y) - \psi_n(x_1,y)] - [\Psi(x_2,y) - \psi_n(x_2,y)]\} \\ &\leqq \Psi(x_1,y) - \Psi(x_2,y) \leqq k|x_1 - x_2|. \end{aligned}$$

Thus, for all integers n, ψ_n satisfies the single Lipschitz condition

$$|\psi_n(x_1,y_1) - \psi_n(x_2,y_2)| \leqq k[|x_1 - x_2| + |y_1 - y_2|].$$

Hence, the minimizing sequence, $\{\psi_n\}$, is equicontinuous and by Arzela's theorem it must, therefore, contain a subsequence that converges uniformly to the Lipschitz continuous limit, $\psi(x,y)$, which is less than or equal to Ψ in $\bar{Q}$. As is well known, the absolute continuity of ψ implies that its partial derivatives exist almost everywhere and they are locally summable and hence square summable over $\bar{Q}$. Therefore, ψ is an admissible function of our minimum problem. Furthermore, the monotonicity of the functions, ψ_n and $\Psi - \psi_n$, is still preserved in the limit. Consequently, both ψ and $\Psi - \psi$ are nonnegative and monotone nonincreasing in $|x|$ and $|y|$ for (x,y) in $\bar{Q}$.

We proceed to show that the limit function ψ actually minimizes the integral $I[u]$. To do this, we introduce the functions,

$$\begin{aligned} \phi(x,y) &= \psi(x,y) + \frac{1}{2}\mu\theta r^2, \quad (x,y) \text{ in } \bar{Q}, \\ \Phi(x,y) &= \Psi(x,y) + \frac{1}{2}\mu\theta r^2, \quad (x,y) \text{ in } \bar{Q}, \end{aligned} \tag{4.1}$$

where $r^2 = x^2 + y^2$. These functions are clearly absolutely continuous in $\bar{Q}$. Hence, we may apply a divergence theorem to derive that

$$I[\psi] = \iint_Q (\nabla\phi)^2\, dxdy + \text{constant}. \tag{4.2}$$

Thus, to show that ψ is an extremal it suffices to show that ϕ minimizes the integral

$$J[v] = \iint_Q (\nabla v)^2\, dxdy, \tag{4.3}$$

among the class of functions v which are continuous and are less than or equal

to Φ in $\bar{Q}$, which have finite Dirichlet integrals over Q and which are equal to Φ on ∂Q.

In what follows, we shall write $\|f\|^2$ for the Dirichlet integral of f over Q whenever it exists. Clearly, the sequence,

$$\{\phi_n\} = \left\{\psi_n + \frac{1}{2}\mu\theta r^2\right\},$$

is a minimizing sequence for the integral $J[v]$ and that the parallelogram law,

$$\left\| \frac{\phi_m + \phi_n}{2} \right\|^2 + \left\| \frac{\phi_m - \phi_n}{2} \right\|^2 = \frac{\|\phi_m\|^2 + \|\phi_n\|^2}{2}, \tag{4.4}$$

holds for all integers m, n. For any $\varepsilon > 0$, we can choose m, n so large that

$$\|\phi_m\|^2 \leqq \hat{d}^2 + \varepsilon, \ \|\phi_n\|^2 \leqq \hat{d}^2 + \varepsilon, \ \hat{d} \equiv \inf_{\{v\}} J[v]. \tag{4.5}$$

On the other hand, for the admissible function, $(\phi_m + \phi_n)/2$, for $J[v]$, we must have

$$\left\| \frac{\phi_m + \phi_n}{2} \right\|^2 \geqq \hat{d}^2. \tag{4.6}$$

It follows from (4.4)–(4.6) that

$$\hat{d}^2 + \left\| \frac{\phi_m - \phi_n}{2} \right\|^2 \leqq \frac{\hat{d}^2 + \varepsilon + \hat{d}^2 + \varepsilon}{2} = \hat{d}^2 + \varepsilon.$$

After subtraction of $\hat{d}^2$ from both sides, this is enough to prove the desired convergence of the sequence, $\{\phi_n\}$, with respect to the Dirichlet norm.

According to the Riesz-Fischer theorem, there exist two limit functions, ϕ_x, ϕ_y, which are uniquely defined almost everywhere in Q which have summable squares over Q and which satisfy the relations

$$\lim_{n\to\infty} \iint_Q \left(\frac{\partial\phi_n}{\partial x} - \phi_x\right)^2 dxdy = \lim_{n\to\infty} \iint_Q \left(\frac{\partial\phi_n}{\partial y} - \phi_y\right)^2 dxdy = 0. \tag{4.7}$$

In view of the Schwarz inequality,

$$\left[\iint_Q \left(\frac{\partial x_n}{\partial x} - \phi_x\right) u\, dxdy\right]^2 \leqq \iint_Q \left(\frac{\partial\phi_n}{\partial x} - \phi_x\right)^2 dxdy \iint_Q u^2\, dxdy,$$

we see that

$$\begin{aligned} \lim_{n\to\infty} \iint_Q \left(\frac{\partial\phi_n}{\partial x} u\right) dxdy &= \iint_Q \phi_x u\, dxdy, \\ \lim_{n\to\infty} \iint_Q \left(\frac{\partial\phi_n}{\partial y} u\right) dxdy &= \iint_Q \phi_y u\, dxdy, \end{aligned} \tag{4.8}$$

for any function u with a summable square over Q. In particular, for u being

continuously differentiable and with compact support in Q, we have, by integration by parts of the left-hand sides in (4.8),

$$\lim_{n\to\infty} \iint_Q \phi_n \frac{\partial u}{\partial x} dxdy = \iint_Q \phi \frac{\partial u}{\partial x} dxdy = \iint_Q \phi_x u \, dxdy,$$

(4.9)

$$\lim_{n\to\infty} \iint_Q \phi_n \frac{\partial u}{\partial y} dxdy = \iint_Q \phi \frac{\partial u}{\partial y} dxdy = \iint_Q \phi_y u \, dxdy.$$

This proves that the functions ϕ_x, ϕ_y are the weak partial derivatives of ϕ. But $\phi(x,y)$ is absolutely continuous in Q. The last two equalities in (4.9) also show that ϕ_x, ϕ_y are actually the partial derivatives of ϕ.

Since for all integers n,

$$\| \phi - \phi_n \|^2 \geqq | \, \| \phi \|^2 - \| \phi_n \|^2 \, |,$$

we conclude from the limit relations in (4.7) that

$$J[\phi] = \| \phi \|^2 = \iint_Q (\phi_x^2 + \phi_y^2) \, dxdy = \inf_{\{v\}} J[v].$$

That is, ϕ does minimize the integral $J[v]$. Hence ψ minimizes the integral $I[u]$ and it is an extremal to be sought.

5. **Elastic-plastic partition of Q.** It has been proved that the partial derivatives of the extremal ψ exist almost everywhere. However, we wish to show that ψ is actually smooth in Q. To achieve this purpose, we need to consider the following sets associated with the extremal ψ:

$$E = \{(x,y) \,|\, (x,y) \in \bar{Q}, \ \psi(x,y) < \Psi(x,y)\},$$

(5.1)

$$P = \{(x,y) \,|\, (x,y) \in \bar{Q}, \ \psi(x,y) = \Psi(x,y)\}.$$

From these definitions, it follows that $E + P = \bar{Q}$, E is open and P is closed. We shall call E the elastic region and P the plastic portion of $\bar{Q}$. Although these definitions seemingly differ from the usual ones, it will be shown that they are actually equivalent to those as used in the theory of plasticity, [2], [3].

Since the diagonals of Q are precisely the sets where $\operatorname{grad} \Psi$ is discontinuous, we intend to show that the diagonals of Q are contained in E. To prove this we shall assume the contrary and then derive a contradiction. Let p_1 be a point on the segment 01 (see Figure 2(a)) such that

$$\rho(0,1) \geqq \rho(1,p_1) = y_0' > 0.$$

Suppose that $p_1 \in P$. Then the monotone nonincreasing property of the function $\Psi - \psi$ implies that the triangle $1p_1'p_1''$ is contained in P, where the segment $p_1'p_1''$

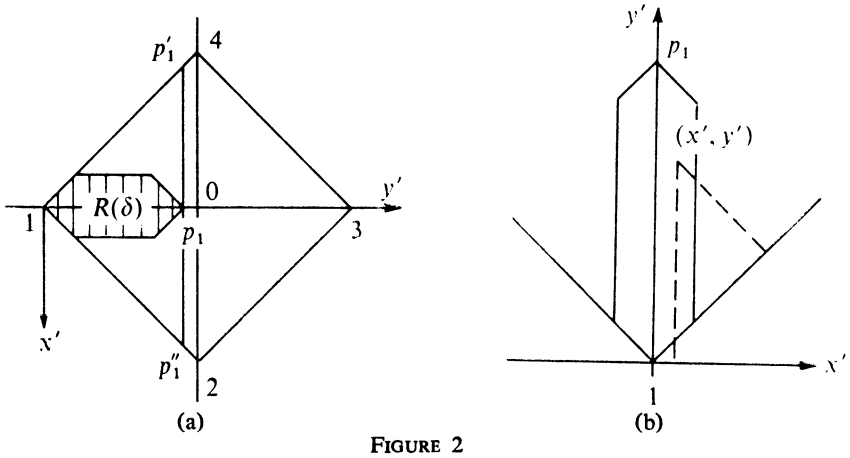

FIGURE 2

passes through p_1 normal to segment $1p_1$ with its end points p_1', p_1'' on ∂Q. Consider a diamond-shaped region $R(\delta)$ which stays in the triangle $1'_{p1}p_1''$ and contains the segment $1p_1$. This region $R(\delta)$ is bounded by two straight lines parallel to the segment $1p_1$ with distance δ from it and by another four straight lines through the points 1 and p_1 respectively and parallel to the sides of Q.

Now we define a function ψ_δ in $\overline{R(\delta)}$ to be such that it is equal to Ψ on $\partial R(\delta)$ and it remains to be constant along any segment perpendicular to the segment $1p_1$. By choosing the vertex 1 of Q as the origin of a coordinate system (x', y'), with the positive y'-axis containing the segment $1p_1$, then we have for (x', y') in $\overline{R(\delta)}$,

$$\Psi(x', y') = 2^{-1/2}k(y' - |x'|), \qquad \psi_\delta(x', y') = 2^{-1/2}k(y' - \delta).$$

The area of $R(\delta)$ is given by

$$A = \iint_{R(\delta)} dx'dy' = 2\int_0^\delta \left[\int_{x'}^{y'_0 - x'} dy' \right] dx' = 2\delta(y_0' - \delta).$$

Direct computation gives

$$\begin{aligned} I_{R(\delta)}[\psi_\delta] &\equiv \iint_{R(\delta)} [(\nabla\psi_\delta)^2 - 4\mu\theta\psi_\delta]\, dx'dy' \\ &= \frac{1}{2}k^2A - 4(2)^{1/2}k\mu\theta\left[\frac{1}{2}y_0'^2\delta - \frac{3}{2}y_0'\delta^2 + \delta^3\right], \\ I_{R(\delta)}[\Psi] &\equiv \iint_{R(\delta)} [(\nabla\Psi)^2 - 4\mu\theta\Psi]\, dx'dy' \\ &= k^2A - 4(2)^{1/2}k\mu\theta\left[\frac{1}{2}y_0'x' - y_0'\delta^2 + \frac{2}{3}\delta^3\right]. \end{aligned}$$

Hence

$$I_{R(\delta)}[\Psi] - I_{R(\delta)}[\psi_\delta] = k\delta \left[\frac{4}{3}(2)^{1/2}\mu\theta\delta^2 - (k + 2^{3/2}\mu\theta y_0')\delta + ky_0'\right].$$

Consequently if we choose δ to be sufficiently small, say

$$0 < \delta < \frac{ky_0'}{k + 2^{3/2}\mu\theta y_0'},$$

then $I_{R(\delta)}[\Psi] > I_{R(\delta)}[\psi_\delta]$. Accordingly, if we define

$$\psi^* = \begin{cases} \psi & \text{in } \bar{Q} - R(\delta), \\ \psi_\delta & \text{in } R(\delta), \end{cases}$$

then ψ^* is also an admissible function and $I[\psi^*] < I[\psi]$. But this contradicts that ψ is an extremal and hence the point p_1 must be contained in E. Since the point p_1 was arbitrarily chosen, we reach the conclusion which was set to prove.

Consider the frontier,

$$l = \bar{E} \cap P \cap Q = Q \cap \partial E = Q \cap \partial P,$$

which we shall call the elastic-plastic boundary. Let l_{12} be that portion of l contained in the subdomain Q_{12} of Q. Because of the monotonicity of the function $\Psi - \psi$, any line parallel to the x- or y-axis intersects l_{12} not more than one point. Thus l_{12} has a nonparametric representation, $y = y_{12}(x)$, with y_{12} being *monotone*. As is well known, the monotone property of l_{12} implies that its tangent is defined almost everywhere and that it is a *rectifiable* curve. It may also be noted that $l - l_{12}$ consists of the mirror images of l_{12} with respect to the diagonals of Q. Accordingly, l consists of exactly four monotone arcs.

From what it has been shown it is now clear that E is a simple connected region containing the diagonals of Q and that the interior of P, if it is nonempty, consists of exactly four simply connected components each of which can be covered by the inward normals of ∂Q.

6. **Smoothness of the extremal in** E. Since $\psi = \Psi$ in the plastic portion P and P does not contain the diagonals of Q, if the interior of P is nonempty, ψ is evidently smooth there. We intend to show that ψ is twice continuously differentiable in the elastic region E and it satisfies the Poisson equation (2.2) there. The proof will be essentially the same as that for the Dirichlet problem. The only additional care which has to be taken is that all admissible functions must be less than or equal to Ψ. To avoid this difficulty as well as for simplicity, we follow the technique introduced in [6] to make proper choice of the variations so as to transform the variational condition into an integral representation for ψ, which will exhibit more clearly the nature of the extremal.

To construct the desired variation, we take the fundamental solution of the Laplace equation,

$$S(x,y;\xi,\eta) = \frac{1}{2\pi}\log\frac{1}{r}, \quad r = [(x-\xi)^2+(y-\eta)^2]^{1/2}. \tag{6.1}$$

For any point (ξ,η) in E, let $\lambda > 0$ be small enough so that the disk E_λ of radius λ around (ξ,η) is contained in E. Consider the function

$$S_\lambda(x,y;\xi,\eta) = \iint_{E_\lambda} S(x,y;\xi',\eta')\,d\xi'd\eta'. \tag{6.2}$$

Direct evaluation [7] gives that

$$S_\lambda(x,y;\xi,\eta) = \begin{cases} -\frac{1}{2}\lambda^2\log r + \frac{1}{4}(\lambda^2 - r^2), & r \leqq \lambda, \\ -\frac{1}{2}\lambda^2\log r, & r \geqq \lambda. \end{cases} \tag{6.3}$$

Thus for any point (x,y) in the exterior of E_λ,

$$\Delta S_\lambda(x,y;\xi,\eta) = 0; \tag{6.4}$$

while for (x,y) in E_λ,

$$\Delta S_\lambda(x,y;\xi,\eta) = -1, \tag{6.5}$$

where the Laplacian operator applies at the point (x,y).

We shall multiply S_λ by an auxiliary factor so as to transform it into a localized variation of ψ. To do this, let E_1, E_2 be two concentric disks such that $E \supset E_2 \supset E_1 \supset E_\lambda$. Let $\omega(x,y)$ be a sufficiently smooth function which vanishes for (x,y) in the exterior of E_2, reduces to one for (x,y) in E_1 and takes on the values between 0 and 1 otherwise. For example,

$$\omega = \begin{cases} 0 \quad \text{for } r \geqq r_2, \\ \exp\{(r-r_1)^2/(r^2-r_2^2)\} \quad \text{for } r_1 < r < r_2, \\ 1 \quad \text{for } r \leqq r_1, \end{cases} \tag{6.6}$$

where r_1 and r_2 stand for the radii of E_1 and E_2 respectively. The localized variations which we set out to construct are essentially

$$v(x,y;\xi,\eta) = \omega(x,y)S_\lambda(x,y;\xi,\eta). \tag{6.7}$$

Indeed, $\Psi - \psi$ is positive and continuous in E; if we choose the radii of E_2 and E_λ to be *sufficiently small*, then it follows from (6.3), (6.6) and (6.7) that for some $\varepsilon_0 > 0$,

$$\Psi(x,y) - [\psi(x,y) + \varepsilon v(x,y;\xi,\eta)] \geqq 0 \text{ in } E \tag{6.8}$$

for all values of ε, $-\varepsilon_0 \leqq \varepsilon \leqq \varepsilon_0$. Thus, for such choice of r_2 and λ, the functions,

$$\psi(x,y) + \varepsilon v(x,y;\xi,\eta), \qquad |\varepsilon| < \varepsilon_0,$$

are all admissible for our variational problem. Consequently, we have the variational condition,

$$\iint_Q [\nabla\psi(x,y)\cdot\nabla v(x,y;\xi,\eta) - 2\mu\theta v(x,y;\xi,\eta)]\,dxdy = 0, \tag{6.9}$$

where the gradient operator, ∇, applies at the point (x,y).

From (6.3)–(6.7) we see that v is continuously differentiable and piecewise twice continuously differentiable in Q and it vanishes in the exterior of E_2. Under these circumstances, we may apply a divergence theorem to transform (6.9) into the form,

$$\iint_Q [\psi(x,y)\Delta v(x,y;\xi,\eta) + 2\mu\theta v(x,y;\xi,\eta)]\,dxdy = 0. \tag{6.10}$$

Since the auxiliary factor ω which appears in (6.6) is constant outside the shell, $E_2 - E_1$, by taking (6.4) and (6.5) into account in E_λ and its complement respectively, we transform (6.10) into the equivalent result,

$$\iint_{E_\lambda} \psi(x,y)\,dxdy = \iint_{E_2-E_1} [\psi\Delta(\omega S_\lambda) + 2\mu\theta\omega S_\lambda]\,dxdy + 2\partial\theta \iint_{E_1} S_\lambda dxdy, \tag{6.11}$$

where the operator, Δ, applies at the point (x,y).

From (6.3) and (6.6) we see that if the inequality (6.8) holds for some $\lambda > 0$, then it holds for all smaller values of λ. Thus dividing the equation (6.11) by $\pi\lambda^2$ and letting $\lambda \to 0$, we find

$$\begin{aligned} \psi_0(\xi,\eta) &= \lim_{\lambda\to 0} \frac{1}{\pi\lambda^2} \iint_{E_\lambda} \psi(x,y)\,dxdy \\ &= \iint_{E_2-E_1} \{\psi(x,y)\Delta[\omega(x,y)S(x,y;\xi,\eta) + 2\mu\theta\omega S]\}\,dxdy \\ &\quad + 2\mu\theta \iint_{E_1} S(x,y;\xi,\eta)\,dxdy. \end{aligned} \tag{6.12}$$

For the definition (6.2) shows that as $\lambda \to 0$ the quotient, $S_\lambda/\pi\lambda^2$, converges to S uniformly in E together with all its first and second derivatives. Furthermore, the integral of ψ_0 over E_λ has the value given by the expression on the right of (6.11). Thus, we have

$$\iint_{E_\lambda} [\psi(x,y) - \psi_0(x,y)]\,dxdy = 0$$

for any disk E_λ lying inside E_1. This implies that ψ and ψ_0 are identical in E_1

because both of them are continuous there. This proves that $\psi(\xi,\eta)$ has throughout E_1 the integral representation (6.12) and hence it possesses continuous first and second derivatives in E_1 [8]. Furthermore, the conclusion is valid throughout the entire elastic region E because the location there of the disk E_1 is quite arbitrary.

Now we are able to apply a divergence theorem to the variational condition like the one in (6.9) to obtain

$$\iint_E [\Delta\psi + 2\mu\theta]v^* \, dxdy = 0 \tag{6.13}$$

for every continuously differentiable variation v^* which vanishes in some neighborhood of ∂E. We emphasize that here v^* is not necessarily of the form (6.7). Since $\Psi - \psi$ is strictly positive in E, the fundamental lemma of the calculus of variation insures that the equation, $\Delta\psi = -2\mu\theta$, must hold throughout the entire elastic region E. This conclusion can also be proved by directly differentiating the integral representation of ψ given in (6.12).

7. **Smoothness of the extremal in** Q. It has been shown that the extremal ψ satisfies the Poisson equation in the elastic region E. By definition it is equal to Ψ in the plastic portion P of $\bar{Q}$. Since the region E contains the diagonals of Q where $\operatorname{grad}\psi$ is discontinuous, to show that ψ is smooth in Q it suffices to establish that it is smooth along the entire elastic-plastic boundary l. For an interior point q of l. we can choose the coordinate axes coinciding with the diagonals of Q in such a way that q falls in the first quadrant with an abscissa $x_0 > 0$. Since q is an interior point of Q and it is off the diagonals of Q, there are two points q_1, q_2 with abscissas x_1 and x_2 respectively such that $0 < x_1 < x_0 < x_2$ and that the arc q_1q_2 of l stays in the first quadrant. Let

$$y = a - x, \qquad a > x > 0,$$

be the equation of that portion of ∂Q staying in the first quadrant and let

$$y = f(x), \qquad x_1 \leqq x \leqq x_2, \tag{7.1}$$

be the nonparametric representation of the rectifiable arc q_1q_2. Then

$$0 < \varepsilon_1 = f(x_2), \quad 0 < \varepsilon_2 = \min_{x_1 \leqq x \leqq x_2} (a - x - f(x)).$$

Let $\eta(x)$ be a smooth function defined in the closed interval, $[x_1, x_2]$, such that $\eta(x) > 0$ for $x_1 < x < x_2$ and $\eta(x_1) = \eta(x_2) = 0$. Let C_1, C_2, C_3 be three rectifiable curves defined respectively by the equations,

$$\begin{aligned} &C_1\colon\ y = f(x) - \varepsilon\eta(x), \\ &C_2\colon\ y = f(x) + \varepsilon\eta(x), \qquad x_1 \leqq x \leqq x_2, \quad \varepsilon > 0. \\ &C_3\colon\ y = f(x) + 2\varepsilon\eta(x), \end{aligned} \tag{7.2}$$

If ε_0 is the greatest number such that

$$\max_{x_1 \leqq x \leqq x_2} (2\varepsilon_0 \eta(x)) \leqq \min(\varepsilon_1, \varepsilon_2),$$

then for all values of $\varepsilon, 0 < \varepsilon \leqq \varepsilon_0$, the curve C_1 lies in the elastic region E while the curves C_2 and C_3 stay in the plastic portion P. However, as $\varepsilon \to 0$, all these three curves approach the arc $q_1 q_2$ as limit. Let C_0 be a fixed rectifiable curve defined by

$$C_0 : y = f(x) - 2\varepsilon_0 \eta(x), \quad x_1 \leqq x \leqq x_2. \tag{7.3}$$

For all positive values of $\varepsilon \leqq \varepsilon_0$, the region R_1 enclosed by C_0 and C_1 is contained in E while the region R_2 enclosed by C_2 and C_3 is contained in P and they are separated by the region R enclosed by C_1 and C_2 (see Figure 3).

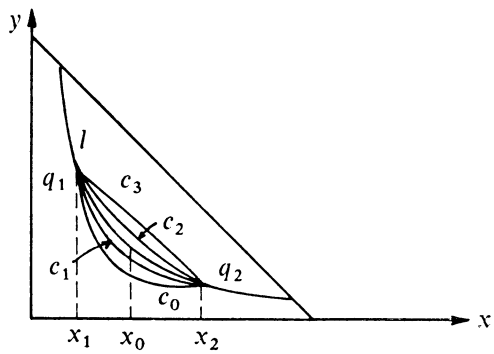

FIGURE 3

Let $\omega_0(x)$ be a non-negative smooth function defined on the interval $[x_1, x_2]$ such that $\omega_0(x)/\eta(x)$ vanishes at the end points x_1 and x_2. Now, define the continuous function $\omega(x, y)$ in $\bar{R}_1 + \bar{R} + \bar{R}_2$ by the following rules:

(i) for (x, y) in R_1, ω is the solution to the Dirichlet problem,

$$\Delta\omega + 0 \text{ in } R_1, \quad \omega = 0 \text{ on } C_0 \text{ and } \omega = \omega_0(x) \text{ on } C_1; \tag{7.4}$$

(ii) for (x, y) in R,

$$\omega(x, y) = \omega_0(x); \tag{7.5}$$

(iii) for (x, y) in R_2, it is given by the formula,

$$\omega(x, y) = \omega_0(x)[f(x) + 2\varepsilon\eta(x) - y]/\varepsilon\eta(x). \tag{7.6}$$

Clearly, the function ω is continuous and piece-smooth in $\bar{R}_1 + \bar{R} + \bar{R}_2$ and it vanishes on $C_0 + C_3 = \partial(\bar{R}_1 + \bar{R} + \bar{R}_2)$. From (7.5) and (7.6) we also see that $\omega(x, y)$ is non-negative in $\bar{R}_2 + \bar{R}$. Further, as a consequence of the maximum principle for the solution of (7.4), it follows that ω is non-negative in $\bar{R}_1$. Accordingly, for all values of ε, $0 < \varepsilon \leqq \varepsilon_0$, the function ψ_ε defined by the equations

$$\psi_\varepsilon(x,y) = \begin{cases} \psi(x,y) \text{ for } (x,y) \text{ in } Q-(R_1+R+R_2), \\ \psi(x,y)-\varepsilon\omega(x,y) \text{ for } (x,y) \text{ in } R_1+R+R_2, \end{cases} \tag{7.7}$$

is an *admissible* function of our minimum problem; and it converges to ψ as $\varepsilon \to 0$. However, we emphasize that $\omega_y(x,y)$ is of the order $O(1/\varepsilon)$ for (x,y) in R_2 as $\varepsilon \to 0$.

Since ψ minimizes the integral $I[u]$ among all admissible functions u, we have

$$I[\psi_\varepsilon] - I[\psi] \geqq 0 \text{ for all values of } \varepsilon,\ 0 < \varepsilon \leqq \varepsilon_0. \tag{7.8}$$

Now for ε being sufficiently small we have

$$\begin{aligned} I[\psi_\varepsilon] - I[\psi] &= \iint_{R_1+R+R_2} [(\nabla\psi_\varepsilon)^2 - (\nabla\psi)^2 - 4\mu\theta(\psi_\varepsilon - \psi)]\,dxdy \\ &= -2\varepsilon \iint_{R_1} [\nabla\psi\cdot\nabla\omega - 2\mu\theta\omega]\,dxdy \\ &\quad -2\varepsilon \iint_{R_2} [\nabla\Psi\cdot\nabla\omega - 2\mu\theta\omega]\,dxdy + O(\varepsilon^2). \end{aligned} \tag{7.9}$$

For the area of R is of the order of $O(\varepsilon)$; $\omega = \omega_0(x)$ in R; and for almost all values of x and y with $x > 0$ and $y > 0$,

$$0 \geqq \psi_x \geqq \Psi_x = -k/(2)^{1/2}, \qquad 0 \geqq \psi_y \geqq \Psi_y = -k/(2)^{1/2}, \tag{7.10}$$

which follows from the fact that $\Psi - \psi$ is monotone nonincreasing in x and y there. Since the function $f(x)$ in equation (7.1) is monotone nonincreasing, it is easy to see from the equations in (7.2) that R_1, R and R_2 are Jordan domains. Hence, the divergence theorem [9] can be applied to the two integrals in equation (7.9). After taking into account the facts that

$$\Delta\psi = -2\mu\theta \text{ in } R_1, \quad \Delta\Psi = 0 \text{ in } R_2,$$

we find that

$$I[\psi_\varepsilon] - I[\psi] = -2\varepsilon \int_{C_1} \omega_0 \frac{\partial\psi}{\partial n}\,ds - 2\varepsilon \int_{C_2} \omega_0 \frac{\partial\Psi}{\partial n}\,ds + O(\varepsilon^2), \tag{7.11}$$

where $\partial\psi/\partial n$, $\partial\Psi/\partial n$ denote the *outward* normal derivatives and the increasing direction of the arc length S will be so chosen that if we travel along C_1 or C_2 in the increasing direction of S then the region R_1 or R_2 stays on our left. Furthermore, using the defining equation (7.2) for C_1, we find

$$\begin{aligned} \int_{C_1} \omega_0 \frac{\partial\psi}{\partial n}\,dS &= \int_{x_2}^{x_1} \omega_0(x)\left[-\psi_y(x,f(x)-\varepsilon\eta(x)) + \psi_x(x,f(x)-\varepsilon\eta(x))\frac{df(x)}{dx}\right]\,dx \\ &\quad -\varepsilon\int_{x_2}^{x_1} \omega_0(x)\psi_x(x,f(x)-\varepsilon\eta(x))\frac{d\eta(x)}{dx}\,dx. \end{aligned}$$

But the value of the last integral on the right is bounded uniformly in ε in view of the restrictions on $\omega_0(x)$ and $\eta(x)$ and the inequalities in (7.10). It follows that

$$\int_{C_1} \omega_0 \frac{\partial \psi}{\partial n} dS = -\int_{x_1}^{x_2} \omega_0(x) \left[-\psi_y(x, f(x) - \varepsilon\eta(x)) + \psi_x(x, f(x) - \varepsilon\eta(x)) \frac{df}{dx} \right] dx + O(\varepsilon).$$

Similarly, from the second equation in (7.2) and the inequalities in (7.10), there follows

$$\begin{aligned} \int_{C_2} \omega_0 \frac{\partial \Psi}{\partial n} dx &= \int_{x_1}^{x_2} \omega_0(x) \left[-\Psi_y(x, f(x) + \varepsilon\eta(x)) + \Psi_x(x, f(x) + \varepsilon\eta(x)) \frac{df}{dx} \right] dx + O(\varepsilon) \\ &= \int_{x_1}^{x_2} \omega_0(x) \left[-\Psi_y(x, f(x) - \varepsilon\eta(x)) + \Psi_x(x, f(x) - \varepsilon\eta(x)) \frac{df}{dx} \right] dx + O(\varepsilon). \end{aligned}$$

By substituting the last two expressions into the the equation (7.11), it gives

$$\begin{aligned} I[\psi_e] - I[\psi] = 2\varepsilon \int_{x_1}^{x_2} \Bigg\{ &\left[\Psi_y\Big(x, f(x) - \varepsilon\eta(x)\Big) - \psi_y\Big(x, f(x) - \varepsilon\eta(x)\Big) \right] \\ &- \left[\Psi_x\Big(x, f(x) - \varepsilon\eta(x)\Big) - \psi_x\Big(x, f(x) - \varepsilon\eta(x)\Big) \right] \frac{df}{dx} \Bigg\} \omega_0(x)\, dx + O(\varepsilon^2). \end{aligned} \tag{7.12}$$

Since for almost all values of $x, x_1 \leqq x \leqq x_2$,

$$df(x)/dx \leqq 0,$$

this inequality together with that in (7.10) insures that for almost all values of $x, x_1 \leqq x \leqq x_2$, and for all values of ε, $0 < \varepsilon \leqq \varepsilon_0$, the integrand in (7.12) is, almost everywhere, nonpositive. Hence, for the inequality in (7.8) to hold for all positive values of ε, it is necessary that

$$\lim_{\varepsilon \to 0} \left[\Psi_y(x, f(x) - \varepsilon\eta(x)) - \psi_y(x, f(x) - \varepsilon\eta(x)) \right] = 0,$$

$$\lim_{\varepsilon \to 0} \left[\Psi_x(x, f(x) - \varepsilon\eta(x)) - \psi_x(x, f(x) - \varepsilon\eta(x)) \right] = 0,$$

for almost all values of x, $x_1 \leqq x \leqq x_2$. In particular, this implies that the normal derivatives

$$\frac{\partial \psi}{\partial n} \to \frac{\partial \Psi}{\partial n} \tag{7.13}$$

almost everywhere along any curve which approaches l in the way as described in (7.2) by letting $\varepsilon \to 0$.

Our object is to establish that $\partial\psi/\partial n = \partial\Psi/\partial n$ everywhere on l so as to insure the smoothness of ψ in the entire domain Q. However, to do this we shall first establish the analyticity of l in the next section by applying the relation (7.13).

8. **Analyticity of elastic-plastic boundary.** In this section we shall use the splendid idea developed in [10]–[12] to establish the analyticity of the elastic-plastic boundary and then prove that ψ is continuously differentiable across l. Since analytic continuation will be an essential tool, we shall adopt the usual notations for complex numbers. Throughout this section we shall choose the center of Q as the origin of a coordinate system with the coordinate axes coinciding with the diagonals of Q. Because of symmetry, we need only to establish the analyticity of that portion of the elastic-plastic boundary l which stays in the first quadrant.

Instead of the extremal ψ we shall be concerned with the function $u(x,y)$ given by the formula,

$$u(x,y) = \Psi(x,y) - \psi(x,y) - \frac{1}{2}\mu\theta(x^2+y^2). \tag{8.1}$$

Clearly u is harmonic in E except on the diagonals of Q and it takes on the continuous boundary values on l,

$$-\frac{1}{2}\mu\theta(x^2+y^2), \tag{8.2}$$

and almost everywhere on l,

$$\operatorname{grad} u = -\operatorname{grad}\left[\frac{1}{2}\mu\theta(x^2+y^2)\right]. \tag{8.3}$$

Consider the fundamental singularity,

$$S(z,\bar{z};\zeta,\bar{\zeta}) = \log(z-\zeta)(\bar{z}-\bar{\zeta}), \tag{8.4}$$

of the Laplace equation where

$$z = x+iy,\ \bar{z} = x-iy;\ \zeta = \xi+i\eta,\ \bar{\zeta} = \xi-i\eta. \tag{8.5}$$

Let E_1 be that portion of the elastic region E lying in the first quadrant. If Γ is any closed rectifiable Jordan curve in the region E_1, we have from Green's formula

$$\int_\Gamma \left(u\frac{\partial S}{\partial n} - S\frac{\partial u}{\partial n}\right) ds = 0 \tag{8.6}$$

for points ζ outside the closed curve Γ. Let t_0 be an interior point of the elastic-plastic boundary l which stays in the first quadrant. That is, t_0 is not on ∂Q. We denote by E_0 and P_0 respectively the intersections of E_1 and P with the disk, $|z-t_0|<\rho$. If we let one arc of Γ approach l while restricting the remainder of Γ to lie in E_1 but outside the disk, $|z-t_0|<\rho$, we obtain

$$(8.7)\qquad \int_l \left\{U(z,\bar z)\,\frac{\partial S(z,\bar z;\zeta,\bar\zeta)}{\partial n} - S(z,\bar z;\zeta,\bar\zeta)\frac{\partial U(z,\bar z)}{\partial n}\right\} ds \;=\; P(\zeta,\bar\zeta)$$

for points $\zeta \in P_0$, where

$$(8.8)\quad U(z,\bar z) = u\left(\frac{z+\bar z}{2},\ \frac{z-\bar z}{2i}\right), \qquad P(\zeta,\bar\zeta) = \int_{\Gamma-l}\left(S\,\frac{\partial U}{\partial n} - U\,\frac{\partial S}{\partial n}\right) ds.$$

Since S is analytic in the real variables ξ and η, we see that $P(\zeta,\bar\zeta)$ is a regular analytic function of ξ and η in P_0. We can extend S analytically by letting ξ and η take independent complex values; and due to the principle of permanence of functional relations, the identity (8.7) also holds for the independent complex variables, $\zeta = \xi + i\eta$, $\zeta^* \equiv \bar\zeta = \xi - i\eta$, where ξ and η are now independent complex variables in P_0. After this analytic continuation, we let $\bar\zeta^* \to t_0$ on l for ζ in P_0 and obtain the relation,

$$(8.9)\qquad \int_l \left\{U(z,\bar z)\frac{\partial S(z,\bar z;\zeta,\bar t_0)}{\partial n} - S(z,\bar z;\zeta,\bar t_0)\,\frac{\partial U(z,\bar z)}{\partial n}\right\} ds \;=\; P(\zeta,\bar t_0).$$

Since $S = \log(z-\zeta)(\bar z - \bar t_0)$, we have to state the branch of the logarithm used in (8.9). For this purpose, we draw an arbitrary curve from ζ to t_0, which meets l only at t_0 and which lies in the disk, $|z - t_0| < \rho$. Outside this branch line the above logarithm is a single-valued function in the whole z-plane. We may determine it in a unique way by requiring, for example, that its imaginary part at a given point $t \in l$ be greater than or equal to zero but less than 2π.

For points $\zeta \in E_0$, we now define an analytic function $\hat U(\zeta,\bar t_0)$ by the formula,

$$(8.10)\qquad \hat U(\zeta,\bar t_0) = \int_l S(z,\bar z;\zeta,\bar t_0)\frac{\partial}{\partial n}U(z,\bar z)\,ds - \int_l U(z,\bar z)\frac{\partial}{\partial n}S(z,\bar z;\zeta,\bar t_0)\,ds - P(\zeta,\bar t_0).$$

We use again the above convention in the determination of S. Here as well as in what follows, we use n to stand for the inward normals and the direction of increasing s will be so chosen that the arc length s increases if we proceed along Γ with the region enclosed by Γ on our left. We proceed to determine the limiting values of $\hat U(\zeta,\bar t_0)$ as ζ approaches a point t on l from E_0.

Clearly the function $P(\zeta,\bar t_0)$ is continuous in ζ across l. However, the jumps of the integrals on the right of (8.10) have to be carefully evaluated so as to determine the boundary values of $\hat U(\zeta,\bar t_0)$. The expression,

$$(8.11)\qquad \int_l \frac{\partial U}{\partial n} S\,ds = i\int_l [\log(z-\zeta)(\bar z - \bar t_0)](U_z dz - U_{\bar z} d\bar z),$$

jumps, as ζ across l into E_0 at the point t, by the integral along the elastic-plastic arc l,

$$(8.12)\qquad -2\pi i\int_{t_0}^{t}\frac{\partial U(z,\bar z)}{\partial n}ds = 2\pi\int_{t_0}^{t}[U_z(z,\bar z)dz - U_{\bar z}(z,\bar z)d\bar z],$$

where we have adopted the notations,

$$2\frac{\partial}{\partial z} = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y},\quad 2\frac{\partial}{\partial \bar z} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}.$$

For $\partial U/\partial n$ is independent of ζ and the value of S jumps by $2\pi i$ on that portion of l between t_0 and t. To evaluate the jumps of the second integral on the right of (8.10), we first calculate that

$$(8.13)\qquad \int_l U\frac{\partial S}{\partial n}\,ds = i\int_l U(z,\bar z)\left(\frac{dz}{z-\zeta} - \frac{d\bar z}{\bar z - \bar t_0}\right).$$

Since U assumes the continuous boundary values as given in (8.1), it follows from Plemelj formula [13] that the jump of the expression in (8.13), as ζ across l into E_0 at t, is equal to

$$(8.14)\qquad -2\pi[U(t,\bar t) + U(t_0,\bar t_0)].$$

From (8.10), (8.12) and (8.14) we see that $\hat U(\zeta,\bar t_0)$ assumes the continuous boundary values along l,

$$(8.15)\qquad \hat U(t,\bar t_0) = 2\pi[U(t,\bar t) + U(t_0,\bar t_0)] + 2\pi\int_{t_0}^{t}\{U_z\,dz - U_{\bar z}d\bar z\},$$

where the integral is to be evaluated along l.

Since $\hat U(z,z^*)/4\pi$ is, for z, z^* in E_0, an analytic continuation of the function $U(x,y)$ through the formula,

$$\frac{1}{4\pi}\hat U(z,z^*) = u\left(\frac{z+z^*}{2},\ \frac{z-z^*}{2i}\right),$$

by letting the variables x and y take independent complex values in E_0. Formula (8.15) is itself also interesting. Moreover, upon an integration by parts, it goes over to

$$(8.16)\qquad \hat U(t,\bar t_0) = 4\pi U(t_0,\bar t_0) + 4\pi\int_{t_0}^{t}U_z(z,\bar z)\,dz,$$

where the integral is still to be evaluated along l. This shows that for fixed t_0 on l, $\hat U(z,\bar t_0)$ and the analytic function, defined by the line integral

$$4\pi\int_{t_0}^{z}U_\zeta(\zeta,\bar\zeta)\,d\zeta$$

along any rectifiable path in E_0, have the same boundary values up to the additive negative constant, $4\pi\cup(t_0\bar t_0)$. Furthermore, we may apply Cauchy's integral theorem

to replace the path of integration in (8.16) by any rectifiable simple curve lying in E_0. For our purpose, formula (8.15) remains more useful and the Helmholtz representation (8.10) will play the essential role.

The elastic-plastic boundary l has the parametric representation, $t = t(s)$. We shall write

$$t' = dt/ds, \qquad \bar{t}' = d\bar{t}/ds.$$

Our intention is to derive an integral equation satisfied by the tangent vector t' to the elastic-plastic boundary l. To this end we first differentiate the equation (8.15) purely formally with respect to t to obtain there the relation

$$\hat{U}_t(t,\bar{t}_0) = 4\pi U_t(t,\bar{t}) - 2\pi U_{\bar{t}}(t,\bar{t})\bar{t}'^2. \tag{8.17}$$

We proceed to show that the analytic function $\hat{U}(\zeta,\bar{t}_0)$ has, indeed, derivatives almost everywhere on l as given by the expression on the right of (8.17).

From (8.10), (8.11) and (8.13) we see that for ζ in E_0

$$\hat{U}(\zeta,\bar{t}_0) = i\int_l [\log(z-\zeta)(\bar{z}-\bar{t}_0)](U_z(z,\bar{z})dz - U_{\bar{z}}(z,\bar{z})d\bar{z})$$
$$- i\int_l U(z,\bar{z})\left(\frac{dz}{z-\zeta} - \frac{d\bar{z}}{\bar{z}-\bar{t}_0}\right) - P(\zeta,\bar{t}_0).$$

Hence, for ζ in E_0, we have

$$\hat{U}_k(\zeta,\bar{t}_0) = -i\int_l \frac{U_z(z,\bar{z})dz - U_{\bar{z}}(z,\bar{z})d\bar{z}}{z-\zeta} - i\int_l \frac{U(z,\bar{z})dz}{(z-\zeta)^2} - P_\zeta(\zeta,\bar{t}_0). \tag{8.18}$$

Thus, our next step is to evaluate the expressions on the right as ζ approaches almost all t on l from E_0. Since $P_\zeta(\zeta,\bar{t}_0)$ is continuous in ζ across l, it suffices to evaluate the jumps of the two integrals as ζ crosses l into E_0 at any point t where the function, $U_t\bar{t}'$, is uniquely defined.

From now on we restrict l to be that portion of l in the disk, $|z - t_0| < \rho$. On l we define a function

$$A(t) = \int_{t_0}^{t} U_{\bar{z}}(z,\bar{z})\,d\bar{z}, \tag{8.19}$$

where the integration is to be carried out along l. We know that almost everywhere on l the derivatives

$$t' = dt/ds,\ U_t(t,\bar{t}),\ U_{\bar{t}}(t,\bar{t}), \tag{8.20}$$

exist. Hence

$$dA(t)/dt = U_{\bar{t}}(t,\bar{t})\bar{t}'^2 \tag{8.21}$$

wherever the right-hand side is uniquely defined. Since l is a monotone curve,

let t_1 be a point on l where the relations (8.20) and (8.21) hold and let t_1' denote the tangent vector to l at that point. As has been shown, l descends monotonically in the first quadrant. Thus points $z = x + iy$, on a line making an angle 45° with the horizontal and intercepting l at $t_1 = x_1 + iy_1$, lie in E_0 for $y < y_1$ and in P_0 for $y > y_1$. We now denote by z a point in E_0 which lies on such a line through t_1 and denote by

$$w = 2t_1 - z \tag{8.22}$$

the point in P_0 on the same line and at equal distance

$$\varepsilon = |z - t_1| \tag{8.23}$$

from t_1. We wish to prove the jump condition,

$$\lim_{z \to t_1} \left\{ \int_l \frac{U_{\bar{t}}(t,\bar{t})\, d\bar{t}}{t - z} - \int_l \frac{U_{\bar{t}}(t,\bar{t})\, d\bar{t}}{t - w} \right\} = 2\pi i U_{\bar{t}_1}(t_1, \bar{t}_1)\bar{t}_1'^2, \tag{8.24}$$

where $t_1 = x_1 + iy_1$. By Cauchy's integral formula, we have immediately

$$\lim_{z \to t_1} \left\{ \int_l \frac{U_{\bar{t}_1}\bar{t}_1'^2 dt}{t - z} - \int_l \frac{U_{\bar{t}_1}\bar{t}_1'^2 dt}{t - w} \right\} = 2\pi i U_{\bar{t}_1}(t_1, \bar{t}_1)\bar{t}_1'^2.$$

Therefore, an integration by parts shows that equation (8.24) is equivalent to

$$\lim_{z \to t_1} \left\{ \int_l \left[\frac{A(t) - A(t_1)}{t - t_1} - U_{\bar{t}_1}(t_1, \bar{t}_1)\bar{t}_1'^2 \right] \frac{(z - w)^2 (t - t_1)^2}{(t - z)^2 (t - w)^2} dt \right\} = 0, \tag{8.25}$$

where $A(t)$ has been defined in (8.19).

To prove the relation (8.25), we introduce a coordinates system (σ, τ) with its origin at t_1, and with the σ-axis inclined at $-45°$ with the x-axis. Since l is monotone nonincreasing in the first quadrant, geometrical consideration shows that if $t = x + iy$ and $\sigma + i\tau$ represent the same point on l and if $|\sigma| \leqq 2\varepsilon$ then

$$|t - z| \geqq \varepsilon/2^{1/2}, \quad |t - w| \geqq \varepsilon/2^{1/2}, \quad |t - t_1| \leqq 2^{1/2}\sigma, \quad |dt| \leqq 2^{1/2} d\sigma,$$

while if $|\sigma| \geqq \varepsilon^{1/2}$ then

$$|t - z| \geqq \sigma, \quad |t - w| \geqq \sigma, \quad |t - t_1| \leqq 2^{1/2}\sigma, \quad |dt| \leqq 2^{1/2} d\sigma.$$

Therefore, the integral (8.25) is in absolute value smaller than a fixed constant times

$$\begin{aligned} &\varepsilon^{-3} \int_{-2\varepsilon}^{2\varepsilon} \left| \frac{A(t) - A(t_1)}{t - t_1} - U_{\bar{t}_1}\bar{t}_1'^2 \right| \sigma^2 d\sigma \\ &\qquad + \varepsilon \left\{ \int_{-\varepsilon^{1/2}}^{-\varepsilon} + \int_{\varepsilon}^{\varepsilon^{1/2}} \right\} \left| \frac{A(t) - A(t_1)}{t - t_1} - U_{\bar{t}_1}\bar{t}_1'^2 \right| \frac{d\sigma}{\sigma^2} \\ &\qquad + \varepsilon \int \left| \frac{A(t) - A(t_1)}{t - t_1} - U_{\bar{t}_1}\bar{t}_1'^2 \right| \frac{d\sigma}{\sigma^2} = I_1 + I_2 + I_3, \end{aligned}$$

where the last integral I_3 is extended over the projection of l on the σ-axis, outside the interval, $-\varepsilon^{1/2} \leqq \sigma \leqq \varepsilon^{1/2}$. The first two integrals, I_1 and I_2, approach zero as $\varepsilon \to 0$ because

$$\lim_{t \to t_1} \left| \frac{A(t) - A(t_1)}{t - t_1} - U_{\bar{t}_1}(t_1, \bar{t}_1)\bar{t}_1'^2 \right| = 0$$

by (8.21) and by the choice of t_1. The last integral $I_3 \to 0$ as $\varepsilon \to 0$ because the difference quotient,

$$\left| \frac{A(t) - A(t_1)}{t - t_1} \right|,$$

is bounded. Hence (8.24) is proved.

In a completely similar way, we also obtain the jump conditions,

$$\lim_{z \to t_1} \left\{ \int_l \frac{U_t(t,\bar{t})dt}{t - z} - \int_l \frac{U_t(t,\bar{t})dt}{t - w} \right\} = 2\pi i U_{t_1}(t_1, \bar{t}_1), \tag{8.26}$$

for almost all t_1 on l where the right-hand side is uniquely defined, where the points z in E_0 and w in P_0 are also related by the equations in (8.22) and (8.23).

Finally, we wish to establish another jump condition; namely,

$$\lim_{z \to t_1} \left\{ \int_l \frac{U(t,\bar{t})}{(t - z)^2} dt - \frac{U(t,\bar{t})}{(t - w)^2} dt \right\} = 2\pi i U_{t_1}(t_1, \bar{t}_1), \tag{8.27}$$

wherever U_{t_1} is uniquely defined. Here, the points z in E_0 and w in P_0 are also related by equations in (8.22) and (8.23). The improper integral in (8.27) can be written as

$$\lim_{z \to t_1} \left\{ \int_l \frac{U(t,\bar{t})}{t - t_1} \frac{(t - t_1)^2 (z - w)^2}{(t - w)^2 (t - z)^2} dt \right\}.$$

From this we can see that it does converge as $z \to t_1$. Indeed upon integration by parts, (8.27) goes over to

$$\lim_{z \to t_1} \left\{ \int \frac{U_t(t,\bar{t})}{t - z} dt - \int_l \frac{U_t(t,\bar{t})}{t - w} dt \right\} = 2\pi i U_{t_1}(t, \bar{t}). \tag{8.28}$$

But this is precisely the same as that in (8.26). Therefore, we can prove its validity in the same way as has been done for (8.24).

Our equation (8.17) obtained by purely formal differentiation is now completely justified by the results in (8.24), (8.26) and (8.27). It shows that for fixed t_0 on l, the function $\hat{U}(\zeta, \bar{t}_0)$ is analytic in E_0 and continuous in $E_0 + l$ and that for almost all t on l, $\hat{U}_t(t, \bar{t}_0)$ has the boundary values as given by the right-hand side of (8.17). Since for all t on l,

$$\bar{t} = \bar{t}_0 + \int_{t_0}^{t} \bar{z}'^2 \, dz, \tag{8.29}$$

where the integral is, of course, evaluated along l, it also means that the complex-valued function $\bar{t}'$ must satisfy the Volterra integral equation of the second kind,

$$\bar{t}'^2 = \frac{2\bar{t}_0}{t} + \frac{\hat{U}_t(t,\bar{t}_0)}{\pi\mu\theta t} + \int_{t_0}^{t} \frac{2}{t}\,\bar{z}'^2 dz, \tag{8.30}$$

for almost all t on l where the integral is evaluated along l and where we have substituted the expressions in (8.2) and (8.3) for evaluating $U_t(t,\bar{t})$ and $U_{\bar{t}}(t,\bar{t})$ respectively.

We wish to construct a function analytic in E_0 with $\bar{t}'$ as its boundary values almost everywhere along l. To do this, we introduce the following Volterra integral equation of the second kind,

$$f(z) = \frac{2\bar{t}_0}{z} + \frac{\hat{U}_z(z,\bar{t}_0)}{\pi\mu\theta z} + \int_{t_0}^{z} \frac{2}{z} f(\zeta)\, d\zeta, \tag{8.31}$$

for the determination of the unknown analytic function $f(z)$ in $E_0 + l$. This was obtained by replacing the variable t on l by the variable z in $E_0 + l$ and by replacing $\bar{t}'^2$ by the unknown function $f(z)$. This integral equation has a sense for analytic function $f(z)$, since by Cauchy's integral theorem the integral on the right is independent of path. Moreover, we may expect the solution to be analytic because $\hat{U}_z(z,\bar{t}_0)$ is analytic in E_0 and according to the results in §5 the variable z in (8.31) will never vanish in a sufficiently small neighborhood for l. Indeed, such an analytic function $f(z)$ can be constructed by the iteration process and its uniqueness can be proved in a way similar to that in [14].

It follows from the uniqueness theorem for the solution of (8.31) that $f(z)$ must agree with the known function $\bar{t}'^2$ almost everywhere along l because it satisfies the equation (8.30) along l. Now, the function $g(z)$ defined by the line integral in $E_0 + l$,

$$g(z) = \bar{t}_0 + \int_{t_0}^{z} f(\zeta)\, d\zeta,$$

is analytic in E_0 and it must be continuous in $E_0 + l$. Furthermore, for all t on l,

$$g(t) = \bar{t}_0 + \int_{t_0}^{t} f(\zeta) d\zeta = \bar{t}_0 + \int_{t_0}^{t} \bar{\zeta}'^2 = \bar{t} \tag{8.32}$$

in view of (8.29). Thus, if we define the analytic functions in $E_0 + l$ by the formulas,

$$G(z) = z + g(z), \qquad H(z) = z - g(z), \tag{8.33}$$

then because of (8.32) we find that G and H assume the continuous boundary values on l,

$$G(t) = t + \bar{t} = 2\,\mathrm{Re}\{t\}, \qquad H(t) = t - \bar{t} = 2\,\mathrm{Im}\{t\}.$$

Therefore, G maps l onto a horizontal line segment and H maps l onto a vertical line segment. Denote by $z = F(w)$ a conformal mapping of the upper half w-plane onto E_0. Since l is a monotone curve, the mapping F can be extended continuously to the real-axis of the w-plane. It follows from the Schwarz principle of reflection that the analytic functions, $G(F(w))$, $H(F(w))$, of w can be continued analytically across those segments of the real axis of the w-plane corresponding to l. Hence, $G(z)$ and $H(z)$ can be continued analytically across the elastic-plastic boundary l. Consequently, we have from (8.33) that in a complete neighborhood of those segments of the real axis of the w-plane corresponding to l, the function,

$$z = \frac{1}{2}\{G(F(w)) + H(F(w))\},$$

is analytic in w. In particular, the function,

$$t = \frac{1}{2}\{G(F(w)) + H(F(w))\},$$

which defines the elastic-plastic boundary l, is analytic in those segments of the real axis of the w-plane corresponding to l. This completes the proof that the elastic-plastic boundary consists of analytic arcs.

Once the elastic-plastic boundary l is known to be analytic, it is well known that the harmonic function $u(x, y)$ has all derivatives continuous up to the boundary l. However, in the present case, it can be shown directly that u is analytic on l by the Cauchy-Kowalewski power series method [11]. Hence, from the defining relation (8.1), we conclude that ψ is analytic on l. In particular $\partial\psi/\partial n = \partial\Psi/\partial n$ on l and ψ is smooth on Q.

9. **Solution of elastic-plastic torsion problem.** We have established the existence of the extremal ψ which is continuous in $\bar{Q}$ and continuously differentiable in Q. The square Q is partitioned by the analytic elastic-plastic arcs into a plastic portion P where $\psi = \Psi$ and an elastic region R where $\psi < \Psi$. The plastic portion P consists of four simply connected closed regions each of which does not intersect the diagonals of Q. The elastic region E is simply connected and contains the diagonals of Q. Moreover ψ is analytic in E and satisfies the Poisson equation (2.2) there.

In a coordinate system with origin at the center of Q and with coordinate axes coinciding with the diagonals of Q, the three functions, Ψ, ψ and $\Psi - \psi$, are all monotone nonincreasing in $|x|$ and $|y|$, it follows that

$$|\psi_x| \leqq |\Psi_x|, \qquad |\psi_y| \leqq |\Psi_y|$$

in $\bar{Q}$ except on the diagonals of Q where grad Ψ is not uniquely defined. But

$\operatorname{grad}\psi$ is continuous in Q and $|\operatorname{grad}\Psi|^2 = k^2$ in $\bar{Q}$ except on the diagonals of $\bar{Q}$, so we conclude that

$$|\operatorname{grad}\psi|^2 \leqq k^2 \text{ in } \bar{Q}. \tag{9.1}$$

In particular, we have the important inequality,

$$|\operatorname{grad}\psi|^2 \leqq k^2 \text{ on } \partial E. \tag{9.2}$$

Since ψ is analytic in E, we may appeal to direct computation to obtain the inequality,

$$\Delta(|\operatorname{grad}\psi|^2) = 2(\psi_{xx}{}^2 + 2\psi_{xy}{}^2 + \psi_{yy}{}^2) \geqq 0 \text{ in } E. \tag{9.3}$$

This shows that $|\operatorname{grad}\psi|^2$ is subharmonic in E. Hence we conclude from the strong maximum principle [15] and the inequality (9.2) that the following strict inequality must hold,

$$|\operatorname{grad}\psi|^2 < k^2 \text{ in } E. \tag{9.4}$$

For if $|\operatorname{grad}\psi|^2$ is identically equal to k^2 in E, then it follows from (9.3) that

$$\psi_{xx} = \psi_{xy} = \psi_{yy} = 0 \text{ in } E.$$

which contradicts that ψ is a solution of the Poisson equation (2.2) there. On the other hand, $\psi = \Psi$ in the plastic portion P. If P is nonempty, it is obvious that

$$|\operatorname{grad}\psi|^2 = |\operatorname{grad}\Psi|^2 = k^2 \text{ in } P. \tag{9.5}$$

The strict inequality (9.4) and the equality (9.5) indicate that our definitions in (5.1) for the elastic region E and the plastic portion P are completely equivalent to those used in the theory of plasticity. Thus, we have proved that the extremal fulfills all the requirements listed in [1], [2] and hence it solves the elastic-plastic torsion problem.

10. **Uniqueness and continuous dependence upon parameter.** We have established the existence of elastic-plastic torsion by using the minimum problem (2.1). We wish to show the uniqueness of a smooth solution and its continuous and monotone dependence upon the applied torque or equivalently upon the angle of twist per unit length.

To prove the uniqueness of the continuous stress distribution, let ψ_1, ψ_2 be two smooth solutions of the minimum problem. Consider the function, $\hat{\psi} = (\psi_2 + \psi_1)/2$. It is continuous and is less than or equal to Ψ in $\bar{Q}$ and it possesses finite Dirichlet integral over Q. Hence it is an admissible function of our variational problem. Accordingly,

$$\mathrm{I}[\hat{\psi}] \geqq \frac{1}{2}\{I[\psi_1] + I[\psi_2]\}. \tag{10.1}$$

On the other hand, direct computation shows that

$$= \iint_Q \left\{\frac{1}{4}[(\nabla\psi_1)^2 + 2(\nabla\psi_1 \cdot \nabla\psi_2) + (\nabla\psi_2)^2] - 2\mu\theta(\psi_1 + \psi_2)\right\} dxdy$$

$$(10.2)\quad I[\hat{\psi}] \leqq \iint_Q \left\{\frac{1}{2}[(\nabla\psi_1)^2 - 4\mu\theta\psi_1] + \frac{1}{2}[(\nabla\psi_2)^2 - 4\mu\theta\psi_2]\right\} dxdy$$

$$\leqq \frac{1}{2}\{I[\psi_1] + I[\psi_2]\}.$$

Thus we have from (10.1) and (10.2) that

$$I[\hat{\psi}] = \frac{1}{2}\{I[\psi_1] + I[\psi_2]\}.$$

For this to be the case, it is necessary that the Cauchy inequality in (10.2) is actually an equality. That is,

$$[\nabla(\psi_2 - \psi_1)]^2 = (\nabla\psi_2)^2 + (\nabla\psi_1)^2 - 2\nabla\psi_1 \cdot \nabla\psi_2 = 0$$

identically in Q. But $\psi_1 = \psi_2$ on ∂Q, it is immediate that ψ_1 is equal to ψ_2 in $\bar{Q}$. The uniqueness of the smooth solution is established.

Let $\psi(x,y;\theta_1)$, $\psi(x,y;\theta_2)$ be the solutions of the minimum problem (2.1) for $\theta = \theta_1$, θ_2 respectively with the same constant μ. We shall first establish the following weaker results; namely, if $\theta_2 > \theta_1$ then

$$(10.3)\qquad \psi(x,y;\theta_2) \geqq \psi(x,y;\theta_1) \text{ in } \bar{Q}.$$

To simplify notations, we shall write

$$\psi_1(x,y) = \psi(x,y;\theta_1), \quad \psi_2(x,y) = \psi(x,y;\theta_2).$$

Let E_1, E_2, P_1, P_2 be respectively the elastic regions and the plastic portions of Q associated with the stress functions ψ_1 and ψ_2. Then we have the following decompositions of Q:

$$(10.4)\qquad \begin{aligned} Q &= E_1 + P_1 = E_2 + P_2, \\ Q &= E_1 \cap E_2 + E_1 \cap P_2 + P_1 \cap E_2 + P_1 \cap P_2. \end{aligned}$$

It may be noted that the plus signs on the right of (10.4) mean more than just disjoint unions of sets. Furthermore, as solutions of the minimum problem, it follows from the uniqueness theorem that both ψ_1 and ψ_2 must have all the properties established in §§5–9. In particular we, have

$$(10.5)\qquad \begin{aligned} &|\operatorname{grad}\psi_1|^2 < k^2 \text{ in } E_1, \quad |\operatorname{grad}\psi_1|^2 = k^2 \text{ in } P_1; \\ &|\operatorname{grad}\psi_2|^2 < k^2 \text{ in } E_2, \quad |\operatorname{grad}\psi_2|^2 = k^2 \text{ in } P_2. \end{aligned}$$

Now consider the function, $\hat{\psi} = \psi_2 - \psi_1$. It is (uniformly) continuous in $\bar{Q}$ and continuously differentiable in Q. We assert that $\hat{\psi}$ is nonnegative in $\bar{Q}$. For if this were not the case, it must achieve a negative interior minimum at some q in Q. Clearly, q does not belong to $P_1 \cap P_2$ because $\hat{\psi}$ vanishes identically there. If q belongs to the set $E_1 \cap P_2$, then we have, in view of (10.5), the absurd inequality,

$$k^2 > |\operatorname{grad} \psi_1(q)|^2 = |\operatorname{grad} \psi_2(q)|^2 = k^2;$$

if q belongs to the set $E_2 \cap P_1$, then we have again an absurd inequality

$$k^2 > |\operatorname{grad} \psi_2(q)|^2 = |\operatorname{grad} \psi_1(q)|^2 = k^2.$$

Thus, q can only belong to the open set $E_1 \cap E_2$. But for $\theta_2 > \theta_1$,

$$\Delta\hat{\psi} = -2\mu(\theta_2 - \theta_1) < 0 \text{ in } E_1 \cap E_2.$$

Therefore, q cannot belong to the set $E_1 \cap E_2$ either. For if q were in $E_1 \cap E_2$, then necessarily $\Delta\hat{\psi}(q) \geqq 0$. Since there is no place for q in Q as listed in (10.4), we conclude that $\hat{\psi}$ possesses no negative interior minimum in Q. That is, the inequality (10.3) must hold. We proceed to refine this inequality and to explore. its physical significance.

Consider the set, $P_1 \cap E_2$. We assert that it is empty. For if there were a point q in this set, then it follows from the very definitions for P_1 and E_2 that

$$\psi_1(q) = \Psi(q) > \psi_2(q).$$

But this contradicts the inequality (10.3) and the assertion follows. Now from the first equation in (10.4) we have

$$Q \cap P_1 = E_1 \cap P_1 + P_1 \cap P_1 = E_2 \cap P_1 + P_2 \cap P_1.$$

Since both the sets $E_1 \cap P_1$ and $E_2 \cap P_1$ are empty, this equation simply becomes $P_1 = P_2 \cap P_1$. It means that

$$P_1 \subset P_2, \; E_1 \supset E_2. \tag{10.6}$$

That is, *the plastic portion of $\bar{Q}$ grows as the angle of twist per unit length increases.* Indeed, this is what has been conjected in the literature [2]. We proceed to show how the plastic portion grows with increasing angle of twist.

Let l_1 and l_2 be the elastic-plastic boundaries corresponding to the angle of twist θ_1 and θ_2 respectively. We refine the results in (10.6) by proving that *l_1 and l_2 have no point of contact in $Q = \bar{Q} - \partial Q$.* For if l_1 and l_2 should have a point q in common which lies in Q, then (10.6) implies that the analytic arcs l_1 and l_2 must have a common tangent at the point q. But considering the function, $\hat{\psi} = \psi_2 - \psi_1$, in E_2, (10.6) also implies that

$$\Delta\hat{\psi} = -2\mu(\theta_2 - \theta_1) < 0 \text{ in } E_2, \quad \hat{\psi} \geqq 0 \text{ on } \partial E_2. \tag{10.7}$$

It follows that $\hat{\psi}$ cannot achieve its minimum in the region E_2 and hence $\hat{\psi}(q) = 0$ is the minimum of $\hat{\psi}$ in $\bar{E}_2$. Now, the strong maximum principle [16] asserts that the inward normal derivative at q,

$$\frac{d}{dn}\hat{\psi}(q) > 0.$$

But this contradicts that

$$\frac{d}{dn}\psi_2(q) = \frac{d}{dn}\Psi(q) = \frac{d}{dn}\psi_1(q),$$

which is necessary for ψ_1 and ψ_2 being smooth in Q. Thus our assumption that l_1 and l_2 have a common point q in Q is false and the contention is proved. What has been proved means that *if $\theta_2 > \theta_1$ then for every interior point q of l_2 we can draw a disc around q such that it is contained in E_1*. Thus the set, $E_1 - E_2$, has positive area. Furthermore, the differential inequality (10.7) together with the inequality for the boundary values there implies that $\psi_2 > \psi_1$ in E_2. Since

$$\hat{\psi} = \psi_2 - \psi_1 = \Psi - \psi_1 > 0 \text{ in } E_1 - E_2,$$

we can now state the inequality (10.3) more precisely as follows; namely, if $\theta_2 > \theta_1$ then

$$\psi_2 > \psi_1 \text{ in } E_1, \qquad \psi_2 = \psi_1 \text{ in } P_1. \tag{10.3'}$$

From this we reach the conclusion that $\theta_2 > \theta_1$ if, and only if,

$$\iint_Q \psi_2(x,y)\,dxdy > \iint_Q \psi_1(x,y)\,dxdy.$$

That is, *the angle of twist per unit length is a continuous strictly monotone increasing function of the applied torque.*

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