DILATIONS ON INVERTIBLE SPACES

BY

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Abstract. This paper primarily concerns certain groups of homeomorphisms which are associated in a natural way with a variety of spaces, which satisfy a set of axiomatic conditions put forth in §1.

Let us suppose that $X$ is a space of the type in question and that $G$ is an appropriate group of homeomorphisms of $X$ onto itself. In §2 we demonstrate the existence of a nonvoid subcollection $\mathcal{D}$, the "topological dilations," of $G$ which is characterized in Theorem 1 in the following fashion: suppose $f \in \mathcal{D}$ and $g \in G$, then $g \in \mathcal{D}$ if and only if $f$ is a $G$-conjugate of $g$, that is if and only if there exists an element $h$ of $G$ such that $f = hgh^{-1}$.

We proceed then to show in §3 that if $f$ and $g$ are nonidentity elements of $G$, then we may find $\delta, r \in G$ such that the product $(rgr^{-1})(\delta f \delta^{-1}) \in \mathcal{D}$. We then combine this fact with the characterization of $\mathcal{D}$ mentioned above to conclude that each element of $\mathcal{D}$ is a "universal" element of $G$ in the sense that if $d \in \mathcal{D}$, then any element $g$ of $G$ may be represented as the product of two $G$-conjugates of $d$. Furthermore we conclude that if $g$ is not the identity element of $G$, then $g$ can be represented as the product of three $G$-conjugates of any nonidentity element of $G$.

Finally, we apply the conclusions to groups of homeomorphisms of certain spaces: for example spheres, cells, the Cantor set, etc.

1. Definitions, notation, and axioms. If $X$ is a topological space, $A$ and $B$ are subsets of $X$, and $f$ is a mapping of $X$ to itself, then we write $'A \subseteq B'$ for $'A$ is a subset of $B'$, and $'A \subset B'$ for $'A \subseteq B$ and $A \neq B'$. The mapping $f$ is said to be supported on $A$ if $f(x) = x$ whenever $x \in \sim A$. $A$ will be called a perfect subset of $X$ if $A \neq \emptyset$ and no open subset of $X$ contains exactly one point of $A$. Throughout this paper all mappings under consideration will be homeomorphisms of some space $X$ onto itself.

Henceforth we suppose that we have given:

(a) $X$: a regular, first countable, Hausdorff space.
(b) $X'$: a perfect subspace of $X$.
(c) $(\mathcal{U}, \mathcal{X})$: $\mathcal{U}$ is a collection of open subsets of $X$ each having a nonvoid

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intersection with \(X'\), and each point of \(X'\) has a neighborhood basis contained in \(\mathcal{U}\). \(\mathcal{X}\) will denote the collection of the closures of the elements of \(\mathcal{U}\).

(d) \(G(X,X')\): a nonvoid collection of homeomorphisms of \(X\) onto itself, each carrying \(X'\) onto \(X'\). \(G\) will denote the collection of all homeomorphisms of \(X\) onto \(X\) each carrying \(X'\) onto \(X'\) and each supported on some element of \(\mathcal{X}\). \(G^*\) will denote the group generated by \(G\).

If \(X, X', \mathcal{X}, G(X,X')\) are such as to satisfy the following axioms, then \((X, X', \mathcal{X}, G(X,X'))\) will be called an \(A\)-quadruple. \(^2\)

1. If \(K \in \mathcal{X}\), then \(\text{Cl}(\sim K) \in \mathcal{X}\).
2. If \(K \in \mathcal{X}\), and if \(g \in G(X,X')\), then \(g(K) \in \mathcal{X}\).
3. If \(g_1, g_2 \in G(X,X')\), and if \(g \in G\), then \(gg_1, g_1g \in G(X,X')\) and \(g_2g_1 \in G^*\).
4. If \(K_1, K_2, K_3 \in \mathcal{X}\), with \(K_1 \subset K_3 \supset K_2\), then there exists an element \(g \in G\), supported on \(K_3\), with \((gK_1) = K_2\).\(^3\)
5. If \(g \in G^*\), and if \(K_1, K_2 \in \mathcal{X}\) with \(K_1 \subset K_2\) and \(g(K_1) = K_1\), then there exists an element \(g_1 \in G\), supported on \(K_2\), such that \(g_1 | K_1 = g | K_1\).

Axioms (1) and (2) will be frequently used and usually without being cited. Note that in the case where \(X = X'\) Axioms (1) and (4) assert a sort of invertibility for \(X\).

Axiom (3) implies immediately that if \(g \in G^*\), and if \(g_1 \in G(X,X')\), then \(gg_1, g_1g \in G(X,X')\). In view of this (3) also implies that if \(g \in G(X,X')\), then \(g^{-1} \in G^*\); hence \(g^{-2} \in G^*\), and therefore \(g^{-1} = gg^{-2} \in G(X,X')\).

It now follows that if \(g \in G^*\) and \(K \in \mathcal{X}\), then \(g(K) \in \mathcal{X}\). For since \(G(X,X') \neq \emptyset\), there exists \(g_1 \in G(X,X')\). Then \(gg_1^{-1} \in G(X,X')\), and \(g_1(K) \in \mathcal{X}\) by Axiom (2). Hence, also by Axiom (2), \(g(K) = g_1^{-1}(g_1(K)) \in \mathcal{X}\).

Also Axiom (3) implies that if \(g \in G(X,X')\) and \(\delta \in G^*\), then \(\delta g \delta^{-1} \in G(X,X')\), i.e., the conjugate of any element of \(G(X,X')\) by any element of \(G^*\) is an element of \(G(X,X')\).

It follows also that the set \(G(X,X')\) and \(G^*\) are either equal or disjoint; or, in other words, \(G(X,X') = G^*\) if and only if \(e \in G(X,X')\).

Note that since each of \(G(X,X')\) and \(G^*\) consists solely of homeomorphisms which carry \(X'\) onto \(X'\), any homeomorphism which we may subsequently construct must satisfy this condition if it is to be in either \(G(X,X')\) or \(G^*\).

Henceforth we shall assume that \((X, X', \mathcal{X}, G(X,X'))\) is an \(A\)-quadruple.

By way of example, while reading the arguments to follow, one might think of \(X\) as the \(n\)-cell, \(n > 1\), and \(X'\) as its boundary.

**Definition 1.** Suppose \(p \in X'\) and \(\mathcal{G} = \{C_i\}_{i = t_0}^\infty \subset \mathcal{X}\). Then \(\mathcal{G}\) is called a null sequence with respect to \(p\) or, more briefly, \(\mathcal{G}\) is said to be null for \(p\), if:

(a) \(C_{i+1} \subset C_i^0, \ i \geq t_0, \) and

\(^2\) This definition is due to R. D. Anderson [1] except that we omit his axiom two. Without using the concept of topological dilation he obtained previously Corollaries 3 and 4 to Theorem 2.

\(^3\) Do not forget that "\(\subset\)" denotes proper containment.
(b) \(\{C_{i}\}_{i=-\infty}^{t_0}\) is a neighborhood basis for \(p\).

Of course if \(\{C_{i}\}_{i=t_0}^{\infty}\) is null for \(p\), then \(p = \bigcap \{C_{i}\}_{i=t_0}\). Null sequences indexed from \(t_0\) to \(-\infty\) may be defined in a comparable fashion.

**Definition 2.** Suppose \(p\) and \(q\) are distinct points of \(X'\), and suppose \(\mathcal{G} = \{C_{i}\}_{i=-\infty}^{\infty} \subseteq X'\). Then \(\mathcal{G}\) is called a dilation structure with respect to the ordered pair \((p, q)\) provided:

(a) \(\{C_{i}\}_{i=-\infty}^{0}\) is a null sequence for \(p\), and

(b) \(\{C_{i}\}_{i=0}^{\infty}\) is a null sequence with respect to \(q\).

**Definition 3.** Suppose \(g \in G_x^*\) and suppose \(\mathcal{G} = \{C_{i}\}_{i=0}^{\infty}\) is a dilation structure with respect to \((p, q)\). If \(g(C_{i}) = C_{i+1}\), \(-\infty < i < \infty\), then \(g\) is called a topological dilation with carrier \(\mathcal{G}\); and \(\mathcal{G}\) is said to carry \(g\).

Note that if \(\{C_{i}\}_{i=0}^{\infty}\) is null for \(p\), and if \(g \in G_x^* \cup G(X, X')\), then \(\{g(C_{i})\}_{i=0}^{\infty}\) is null for \(g(p)\). Hence if \(g \in G_x^*\) is a topological dilation with carrier \(\mathcal{G} = \{C_{i}\}_{i=-\infty}^{\infty}\) with respect to \((p, q)\), and if \(\delta \in G_x^*\), then \(\delta g \delta^{-1}\) is a topological dilation carried by \(\mathcal{G}' = \{\delta(C_{i})\}_{i=-\infty}^{\infty}\) with respect to \((\delta(p), \delta(q))\).

2. Topological dilations. We begin this section by proving some lemmas which will be useful throughout the paper. Next we demonstrate the existence of topological dilations and, in fact, may conclude that \(G_x^*\) is richly supplied with them. We conclude \(\S 2\) with Theorem 1, which asserts any two topological dilations are conjugate in \(G_x^*\).

**Proposition A.** Suppose \(p\) and \(q\) are distinct points of \(X'\). Then there exists a dilation structure with respect to \((p, q)\).

**Proof.** Since \(X\) is first countable, regular, and Hausdorff, \(X'\) is perfect, and \(\mathcal{B}\) is a basis, it easily follows that we may find sequences of elements of \(\mathcal{B}\), \(\{C_{i}\}_{i=0}^{\infty}\) and \(\{D_{i}\}_{i=1}^{\infty}\), null for \(p\) and \(q\) respectively, with \(C_{0} \cap D_{1} = \emptyset\) and \(C_{0} \cup D_{1} \neq X\). Setting \(C_{i} = \text{Cl}(\sim D_{i})\) for \(i \geq 1\), we see that \(\mathcal{G} = \{C_{i}\}_{i=-\infty}^{\infty}\) is the desired dilation structure.

**Proposition B.** Suppose \(K_{1}, K_{2} \in \mathcal{B}\) and suppose \(K_{1} \subset K_{2}^{0}\). Then there exists \(K \in \mathcal{B}\) with \(K_{1} \subset K \subset K_{2}^{0}\).

**Proof.** Select \(x \in K_{1} \cap X'\) and \(E_{1} \in \mathcal{B}\) with \(x \in E_{1}^{0}\) and \(E_{1} \subset K_{1}^{0}\). This can be done since \(X\) is regular. Now select \(E_{2} \in \mathcal{B}\), with \(x \in E_{2}^{0}\), \(E_{2} \subset E_{1}^{0}\). Using Axiom (4), we obtain \(g \in G_x\), supported on \(K_{2}\), with \(g(K_{1}) = E_{2}\). Set \(K = g^{-1}(E_{1})\).

**Lemma 1.** Suppose \(K \in \mathcal{B}\) and \(x_{0}, y_{0} \in (K_{1}^{0} \cap X')\); suppose also that \(\mathcal{G} = \{C_{i}\}_{i=0}^{\infty}, D = \{D_{i}\}_{i=1}^{\infty} \subseteq \mathcal{B}\) are null for \(x_{0}, y_{0}\) respectively, with \(C_{1} \subset K_{1}^{0} \supset D_{1}\). Then there is a homeomorphism \(h \in G_x\), supported on \(K\), such that \(h(C_{i}) = D_{i}\) for \(i \geq 1\).

**Proof.** By Axiom (4) there exists \(h_{1} \in G_x\), supported on \(K\), such that \(h_{1}(C_{0}) = D_{1}\). For \(i \geq 2\) we define inductively, by making repeated use of Axiom (4), \(h_{i} \in G_x\), supported on \(D_{i-1}\), such that \(h_{i}h_{i-1} \cdots h_{1}(C_{0}) = D_{i}\).
Set \( h = \prod_{i=1}^{\infty} h_i \). If \( x \neq x_0 \), then for some \( j \geq 1, x \notin C_j \). Then \( \prod_{i=1}^{j} h_i(x) \notin D_j = \prod_{i=1}^{j} h_i(C_j) \). Since for \( i > j \), \( h_i \) is supported on \( D_j \), it follows that \( h(x) = \prod_{i=1}^{j} h_i(x) \). From this, since \( h_i \in G_x \) and \( \mathcal{E} \) and \( \mathcal{D} \) are null for \( x_0 \) and \( y_0 \) respectively, it follows that \( h \) is a homeomorphism supported on \( K \), \( h(x_0) = y_0 \), and \( h \) takes \( X' \) onto \( X' \). Hence \( h \in G_x \). As a corollary we obtain Lemma 2, which resembles Anderson's telescoping Lemma [1].

**Lemma 2.** Suppose \( K_0, K_1 \in \mathcal{X} \) with \( K_1 \subseteq K_0 \) and \( g \in G(X, X') \) with \( g(K_0) \subseteq K_1 \). Suppose also that \( x_0 \in K_0 \cap X' \), \( \{D_i\}_{i=1}^{\infty} \) is null for \( x_0 \), \( D_i \subseteq K_0 \). Then there exists \( h \in G_x \), supported on \( X, h(x_0) = y_0 \), such that \( h(x) = \prod_{i=1}^{d} h_i(x) \). Hence \( h \in G_x \). As a corollary we obtain Lemma 2, which resembles Anderson's telescoping Lemma [1].

**Proof.** Set \( C_i = g(K_0) \) and \( C_i = g(D_{i-1}) \) for \( i \geq 2 \). Then \( \{C_i\}_{i=1}^{\infty} \) is null for \( g(x_0) \). Hence Lemma 1 asserts the existence of \( h \in G_x \), supported on \( K_1 \), such that \( h(C_i) = D_i \), for \( i \geq 1 \). Hence \( (hg)'(K_0) = D_i \), for \( i \geq 1 \).

**Lemma 3.** Any dilation structure carries a topological dilation.

**Proof.** Suppose \( \mathcal{D} = \{B_i\}_{i=-\infty}^{\infty} \) is a dilation structure with respect to \((p, q)\). By Proposition B there is an element \( K_1 \in \mathcal{X} \) such that \( B_0 \subseteq K_0 \), \( K_1 \subseteq B_0 \). We now apply Lemma 1 with \( K_1 \) as \( K, B_{-1} = C_i \), and \( B_{-i+1} \) as \( D_i \) for \( i \geq 1 \), and \( x_0 = p = y_0 \) to obtain \( h_1 \in G_x \), supported on \( K_1 \), with \( h_1(B_{-i}) = B_{-i+1} \), for \( i \geq 1 \).

Since \( B_0 \subset h_1(B_0) \), we may find \( K_2 \in \mathcal{X} \) with \( B_0 \subseteq K_2 \), \( K_2 \subseteq h_1(B_0) \). Then \( \text{Cl}(\sim h_1(B_0)) \subseteq \sim K_2 \), \( \text{Cl}(\sim B_0) \subseteq \sim K_2 \), \( i \geq 1 \). So we again apply Lemma 1 to obtain \( h_2 \in G_x \), supported on \( \text{Cl}(\sim K_2) \), with \( h_2(\text{Cl}(\sim h_1(B_0))) = \text{Cl}(\sim B_1) \), and \( h_2(\text{Cl}(\sim B_i)) = \text{Cl}(\sim B_{i+1}) \), \( i \geq 1 \).

\( h = h_2 h_1 \in G_x \) is the desired topological dilation.

**Corollary.** There exists an element \( g \in G_x \) such that \( g \) is a topological dilation.

**Proof.** Since \( X' \) is perfect, \( X' \) contains two distinct points. The corollary now follows from Proposition A and Lemma 3.

We remarked earlier that if \( g \in G_x \) is an arbitrary topological dilation, then any conjugate of \( g \) by an element of \( G_x \) is also a topological dilation. Theorem 1 asserts that the converse is true: that any topological dilation is a conjugate of \( g \).

**Theorem 1.** Suppose \( g_1 \) and \( g_2 \) are topological dilations. Then there is an element \( r \) of \( G_x \) such that \( g_1 = rg_2r^{-1} \).

**Proof.** Let \( \mathcal{E} = \{C_i\}_{i=-\infty}^{\infty} \) be a dilation structure with respect to \((p, q)\); and let \( \mathcal{D}' = \{D_i\}_{i=-\infty}^{\infty} \) be a dilation structure with respect to \((p', q')\); with \( \mathcal{E} \) and \( \mathcal{D} \) carrying \( g_2 \) and \( g_1 \) respectively.

We may select \( a \in \sim C_0 \cap X' \) \( a \neq p' \). There is an integer \( i_0 \leq 0 \) such that \( a \notin D_{i_0}' \). Hence we may find \( K_1 \in \mathcal{X} \) with \( a \in K_0 \), \( K_1 \cap (C_0 \cup D_{-i_0}) = \emptyset \), and \( \sim K_1 \neq (C_0 \cup D_{-i_0}) \). Set \( D_i = D_{i+i_0} \), \( -\infty < i < \infty \); then \( \mathcal{D} = \{D_i\}_{i=-\infty}^{\infty} \) carries \( g_1 \), and \( D_0 = D_{i_0} \).
Since \( C_0 \cup D_0 \subseteq \sim K_1 \), we may invoke Lemma 1 to obtain \( \delta_1 \in G_x \), supported on \( \text{Cl}(\sim K_1) \), with \( \delta_1(C_0) = D_0, \ i \leq 0 \).

\( \{\delta_1(\text{Cl}(\sim C_0))\}_{i=1}^{\infty} \) is null for \( \delta_1(q_0) \), and \( \delta_1(\text{Cl}(\sim C_0)) = \text{Cl}(\sim D_0) \). Hence we may again apply Lemma 1, this time to the sequences \( \{\delta_1(\text{Cl}(\sim C_0))\}_{i=1}^{\infty} \) and \( \{\text{Cl}(\sim D_0)\}_{i=0}^{\infty} \), to obtain \( \delta_2 \in G_x \), supported on \( \text{Cl}(\sim D_0) \), with \( \delta_2 \delta_1(\text{Cl}(\sim C_0)) = \text{Cl}(\sim D_0), \ i \geq 1 \).

Set \( \delta = \delta_2 \delta_1 \in G_x \), \( \delta(C_0) = D_0, \ -\infty < i < \infty \), and \( \delta(p) = p', \delta(q) = q' \). Hence \( \delta \) carries the topological dilation \( g = \delta g_2 \delta^{-1} \).

We may think of the construction of \( \delta \) as the first step in the conjugation procedure to change \( g_2 \) into \( g_1 \), since it yields the topological dilation \( g \) which is carried by the same structure that carries \( g_1 \). Of course pointwise \( g \) need not equal \( g_1 \).

To achieve pointwise equality a further conjugation is necessary, which in the event that \( D_0 \) is open and closed is quite simple. We proceed to that case:

Set \( B_i = (\text{boundary of } D_i) = D_i - D_0 \). Since \( g_i(B_0) = B_0, \ -\infty < i < \infty \), \( B_i = \emptyset \) if and only if \( B_0 = \emptyset \). If \( B_0 = \emptyset \), define \( \alpha \) as follows:

\[
\alpha(p') = p', \quad \alpha(q') = q';
\]

if \( x \in D_{i+1} - D_i \), \( \alpha(x) = g_i^{-1}(x), \ -\infty < i < \infty \).

Then \( \alpha \) is a homeomorphism of \( X \) onto itself taking \( X' \) onto \( X' \). Since \( \alpha \) takes \( D_0 \) onto itself, and since \( B_0 = \emptyset \), if we define \( \alpha_1 = \alpha \) on \( D_0 \) and \( \alpha_1 = e \) on \( \sim D_0 \); and \( \alpha_2 = \alpha \) on \( \sim D_0 \) and \( \alpha_2 = e \) on \( D_0 \), then \( \alpha_1, \alpha_2 \in G_x \) and \( \alpha = \alpha_2 \alpha_1 \in G_x^* \). Since \( g_1 = g \alpha^{-1} \), if we set \( r = \alpha \delta, \) then \( g_1 = rg_2 r^{-1} \).

It is in the remaining case that Axiom (5) finds its only employment. Thus we might observe now, if the reader hasn’t already done so, that in case \( \mathcal{K} \) consists of sets which are open and closed in \( X \), Axiom (5) may be omitted.

Now suppose \( B_0 \neq \emptyset \).

By Proposition B there exists \( K_i \in \mathcal{K} \) with \( D_0 \subseteq K_i \), \( K_i \subseteq D_i \). Set \( \alpha_1 = g_1 g_2^{-1} \in G_x \); \( \alpha_1 \) takes \( D_1 \) onto \( D_i \), \( -\infty < i < \infty \). By Axiom (5) there exists \( \tilde{a}_1 \in G_x \), supported on \( \text{Cl}(\sim K_1) \) such that \( \tilde{a}_1 \mid \text{Cl}(\sim D_1) = \alpha_1 \). In particular, \( \tilde{a}_1 \) takes \( D_i \) onto \( D_1 \) and \( \tilde{a}_1 \mid B_1 = \alpha_1 \).

Proceeding inductively, for \( i \geq 2 \) we may select \( K_i \in \mathcal{K} \) with \( D_i \subseteq K_0 \), \( K_i \subseteq D_i \). Set

\[
\alpha_i = g_1^{i} g_2^{-i}(\tilde{a}_{i-1} \cdots \tilde{a}_{i-2} \cdots \tilde{a}_1)^{-1}.
\]

\( \alpha_i \in G_x^* \) and \( \alpha_i \) takes \( D_j \) onto \( D_j, \ -\infty < j < \infty \). By Axiom (5) there is \( \tilde{a}_i \in G_x^* \), supported on \( \text{Cl}(\sim K_i) \), such that \( \tilde{a}_i \mid \text{Cl}(\sim D_j) = \alpha_i \). In particular, \( \tilde{a}_i \) takes \( D_j \) onto \( D_j \) and \( \tilde{a}_i \mid B_j = \alpha_i \).

Set \( \alpha' = \prod_{i=1}^{\infty} \tilde{a}_i \). \( \alpha' \) is a homeomorphism, supported on \( \text{Cl}(\sim K_1) \), and taking \( X' \) onto \( X' \) and \( X \) onto \( X' \); hence \( \alpha' \in G_x \).

Observe that \( \alpha' \mid B_i = g_1^{i} g_2^{-i}, \ i \geq 0 \); and \( \alpha' \) takes \( D_i \) onto \( D_j, \ -\infty < i < \infty \).

Now take \( K_{-1} \in \mathcal{K} \) with \( D_{-1} \subseteq K_{0}, \ K_{-1} \subseteq D_0 \). Set \( \alpha_{-1} = g_1^{-1} g_1 \). \( \alpha_{-1} \in G_x^* \),
and \( \alpha_{-1} \) takes \( D_i \) onto \( D_{i+1} \), \( -\infty < i < \infty \). As before, from Axiom (5) we obtain \( \tilde{\alpha}_{-1} \in G_{\mathcal{X}} \), supported on \( K_{-1} \), with \( \tilde{\alpha}_{-1} \mid D_{-1} = \alpha_{-1} \). In particular, \( \tilde{\alpha}_{-1} \) takes \( D_i \) onto \( D_{i+1} \), and \( \tilde{\alpha}_{-1} \mid B_{-1} = \alpha_{-1} \).

Again proceeding inductively for \( i \geq 2 \), select \( K_{-i} \in \mathcal{K} \) with \( D_{-i} \cap K_{-i} = \emptyset \). Set \( \alpha_{-1} = g_{-1}^{-1} g \big( \tilde{\alpha}_{-i+1} \cdot \tilde{\alpha}_{-i+2} \cdots \tilde{\alpha}_{-1} \big)^{-1} \).

Then \( \alpha_{-1} \) takes \( D_j \) onto \( D_{j+1} \), \( -\infty < j < \infty \), and \( \alpha_{-1} \in G_{\mathcal{X}}^* \). We use Axiom (5) to obtain \( \tilde{\alpha}_{-1} \in G_{\mathcal{X}}^* \), supported on \( K_{-1} \), with \( \tilde{\alpha}_{-1} \mid D_{-1} = \alpha_{-1} \). And \( \tilde{\alpha}_{-i} \) takes \( D_j \) onto \( D_{j+1} \), and \( \tilde{\alpha}_{-i} \mid B_{-i} = \alpha_{-i} \).

Set \( \tilde{\alpha} = \prod_{i=1}^{\infty} \tilde{\alpha}_{-i} \in G_{\mathcal{X}}^* \), supported on \( K_{-1} \), takes \( D_j \) onto \( D_{j+1} \), \( -\infty < i < \infty \), and \( \tilde{\alpha} \mid B_{-i} = g_{-1}^{-1} g_i \), \( i \geq 0 \).

Now set \( \alpha = \tilde{\alpha} \cdot \alpha' \in G_{\mathcal{X}}^* \). Since \( K_{-1} \cap Cl(\sim K_1) = \emptyset \), \( \alpha \) takes \( D_i \) onto \( D_i \) and \( \alpha \mid B_i = g_i g_i^{-1} \), \( -\infty < i < \infty \).

Setting \( \tilde{g} = x g x^{-1} \), we observe that \( \tilde{g} \mid B_i = g_i \), \( -\infty < i < \infty \).

Finally, we define \( \beta \) as follows:

\[
\beta(p') = p', \quad \beta(q') = q';
\]

if \( x \in D_i = D_j \), \( \beta(x) = g_i^{-1} \tilde{g}^{-1}(x) \), \( -\infty < i < \infty \).

\( \beta \) is a homeomorphism of \( X \) onto itself, carrying \( X' \) onto \( X' \). Also \( \beta \mid B_0 = e \), \( -\infty < i < \infty \). And since, in addition, \( \beta \) takes \( D_0 \) onto \( D_0 \), if we define \( \beta_1 = \beta_0 \), and \( \beta_2 = \beta_0 \), then \( \beta_1, \beta_2 \in G_{\mathcal{X}}^* \); hence \( \beta = \beta_1 \beta_2 \in G_{\mathcal{X}}^* \). And since \( g_i = \beta \tilde{g} \beta^{-1} \), if we set \( r = \beta \alpha \delta \in G_{\mathcal{X}}^* \), we have \( g_i = r g_2 r^{-1} \).

3. The principal theorem; its corollaries. In this section we prove the principal theorem of the paper and deduce some corollaries.

**Theorem 2.** Suppose \( f \) and \( g \) are elements of \( G(X, X') \) neither of which is the identity on \( X' \). Then there exists elements \( h_1 \) and \( h_2 \) of \( G_{\mathcal{X}}^* \) such that \( (h_2 gh_2^{-1}) \) is a topological dilation. Furthermore \( h_1 \) and \( h_2 \) may be chosen as topological dilations.

**Proof.** We distinguish two cases: Case 1. At least one of \( f \) and \( g \), say \( f \), is not of period two on \( X' \). Case 2. Both \( f \) and \( g \) are of period two on \( X' \).

**Case 1.** Since \( f \) is not of period two on \( X' \), neither \( f \) nor \( g \) is the identity on \( X' \), and \( X' \) is perfect, we may find elements \( C_1, D_1, A_0, B_0 \) of \( G_\mathcal{X}^* \) such that the sets \( \{C_1, D_1, A_0, B_0, f(A_0), f^{-1}(A_0), g(B_0)\} \) are mutually disjoint and such that the sets \( \{C_1, D_1, A_0, B_0, f(A_0), f^{-1}(A_0), g^{-1}(B_0)\} \) are mutually disjoint.

Select \( c \in C_1 \cap X', d \in D_1 \cap X', a \in A_0 \cap X', \) and \( b \in B_0 \cap X' \). We now take sequences of elements of \( \mathcal{X} - \{C_1, D_1, A_0, B_0\} \) and \( \{A_i\}_{i=1}^\infty, \{B_i\}_{i=1}^\infty \) null for \( c, d, a, \) and \( b \) respectively.

Set \( A_i = Cl(\sim C_i), B_i = Cl(\sim D_i), i \geq 1 \). Then \( \mathcal{A} = \{A_i\}_{i=-\infty}^{\infty} \) and \( \mathcal{B} = \{B_i\}_{i=-\infty}^{\infty} \).
are dilation structures with respect to \((a, c)\) and \((b, d)\) respectively. By Lemma 3 \(\mathcal{S}\) and \(\mathcal{B}\) respectively carry topological dilations \(\delta\) and \(r\).

Set \(\beta = (rg^{-1})(\delta f^{-1})\), \(\beta \in G_x^*\) by Axiom (3).

Now a remark about what is to follow: It is true that \(\beta(A_1) \subseteq A_1^0\) and \(\beta^{-1}(C_1) \subseteq C_1^0\), \((A_1 = \text{Cl}(\sim C_1))\). Hence \(\beta^i(A_1) \subseteq \beta^{i-1}(A_1^0)\) for \(i \geq 1\), and \(\beta^{-i}(C_1) \subseteq \beta^{-i+1}(C_1^0)\), \(i \geq 1\). Therefore if \(\{\beta^i(A_1)\}_{i=0}^\infty\) were null for a point of \(X'\) and if \(\{\beta^{-i}(C_1)\}_{i=0}^\infty\) were null for a point of \(X'\), then it is apparent that \(\beta\) would then be a topological dilation. Therefore, in order to achieve this we are going to multiply \(r\) and \(\delta\) by appropriate elements of \(G_x^*\).

Consider \(\beta(A_1)\):

\[
\delta^{-1}(A_1) = A_0,
\]
\[
f(A_0) \subseteq A_0^0 - A_0,
\]
\[
\delta(A_0^0 - A_0) = A_2^0 - A_1 = C_1^0 - C_2.
\]

Hence

\[
\delta f \delta^{-1}(A_1) \subseteq C_1^0 - C_2,
\]
\[
r^{-1}(C_1) \subseteq B_0^0 - B_{-1},
\]
\[
g(B_0) \subseteq B_1^0 - B_0,
\]
\[
r(B_0^0 - B_0) = B_2^0 - B_1 = D_1^0 - D_2.
\]

Hence \(\beta(A_1) \subseteq D_1^0 - D_2\). Set \(K_1 = rg(B_0)\), \(\beta(A_1) \subseteq K_1^0\), \(K_1 \subseteq D_1^0 - D_2\). Select \(x_1 \in K_1^0 \cap X'\) and a sequence of elements of \(X\), \(\{T_i\}_{i=1}^\infty\), null for \(x_1\), with \(T_1 \subseteq K_1^0\). We now apply Lemma 2, with \(A_1\) as \(K_0\) and with \(\beta\) as \(g\), to obtain \(\alpha_1 \in G_x\), supported on \(K_1\), with \((\alpha_1 \beta)(A_1) = T_i\), \(i \geq 1\).

Now set \(\tilde{\beta} = (\delta f^{-1} \delta^{-1})(\alpha_1 rg^{-1}r^{-1} \alpha_1^{-1})\) and consider \(\tilde{\beta}(C_1)\).

Since \(K_1 \cap C_1 = \emptyset\), \(rg^{-1}r^{-1} \alpha_1^{-1}(C_1) = rg^{-1}r^{-1}(C_1)\),

\[
r^{-1}(C_1) \subseteq B_0^0,
\]
\[
g^{-1}(B_0) \subseteq B_1^0 - B_0,
\]
\[
r(B_1^0 - B_0) = B_2^0 - B_1 = D_1^0 - D_2.
\]

So \(rg^{-1}r^{-1}(C_1) \subseteq D_1^0 - D_2\). Since \(K_1 \subseteq D_1^0 - D_2\),

\[
\alpha_1 rg^{-1}r^{-1}(C_1) \subseteq D_1^0 - D_2 \subseteq D_1^0,
\]
\[
\delta^{-1}(D_1) \subseteq A_0^0; \text{ and } f^{-1}(A_0) \subseteq A_1^0 - A_0,
\]
\[
\delta(A_1^0 - A_0) = A_2^0 - A_1 = C_1^0 - C_2.
\]

Hence \(\tilde{\beta}(C_1) \subseteq C_1^0 - C_2\).

Set \(K_2 = \delta f^{-1}(A_0)\). Then \(\tilde{\beta}(C_1) \subseteq K_2^0 \cap C_1^0 - C_2\). Note that \(K_2 \cap \delta f(A_0) = \emptyset\).
Select \( x_2 \in K_2^0 \cap X' \) and a sequence of elements of \( \mathcal{X} \), \( \{ T_i \}_{i=0}^{-\infty} \) null for \( x_2 \), with \( T_{-1} \cap K_2^0 \). We again apply Lemma 2, this time with \( C_1 \) as \( K_0 \) and with \( \beta \) as \( g \), to obtain \( \alpha_2 \in G_{\mathcal{X}} \), supported on \( K_2 \), with \((\alpha_2 \beta)(C_i) = T_{-i}, i \geq 1\).

Set \( d = (\alpha_1 r g^{-1} \alpha_1^{-1})(\alpha_2 \delta \delta^{-1} \alpha_2^{-1}) \) and observe that if \( d'(A_i) = T_i \) and \( d^{-i}(C_i) = T_{-i} \) for \( i \geq 1 \), then \( d \) is a topological dilation carried by the dilation structure \( \{ T_i \}_{i=0}^{-\infty} \) with respect to \((x_2, x_1)\), where \( T'_i = T_i, i \leq -1, T_0 = C_1 \), \( T'_i = \text{Cl}(\sim T_i), i \geq 1 \).

Consider \( d(A_1) \):

Since \( K_2 \cap A_1 = \emptyset \), \( \delta f \delta^{-1} \alpha_2^{-1}(A_1) = \delta f \delta^{-1}(A_1) = \delta f(A_0) \). Since \( K_2 \cap \delta f(A_0) = \emptyset \), \( \alpha_2 \delta f(A_0) = \delta f(A_0) \subset C_1 \). But \( K_1 \cap C_1 = \emptyset \); so \( \alpha_2 \delta f(A_0) = \delta f(A_0) \).

Hence \( d(A_1) = \alpha_1 r g^{-1} \delta f \delta^{-1}(A_1) = \alpha_1 \beta(A_1) = T_1 \). Since for \( i \geq 1 \), \( T \subset A_1 \), \( d(T_i) = \alpha_1 \beta(T_i) = T_{i+1} \). Therefore \( d^{-i}(A_1) = T_{-i}, i \geq 1 \).

Now consider \( d^{-1}(C_i) = \alpha_2 \delta f^{-1} \delta^{-1} \alpha_2^{-1} \alpha_1 r g^{-1} \alpha_1^{-1}(C_i) \):

As seen earlier, \( \alpha_1 r g^{-1} \alpha_1^{-1}(C_i) \subset D_0 \). Since \( K_2 \cap D_1 = \emptyset \),

\[ \alpha_2^{-1} \alpha_1 r g^{-1} \alpha_1^{-1}(C_i) = \alpha_1 r g^{-1} \alpha_1^{-1}(C_i). \]

Hence \( d^{-1}(C_i) = \alpha_2 \delta f^{-1} \delta^{-1} \alpha_1 r g^{-1} \alpha_1^{-1}(C_i) = \alpha_1 \beta(C_i) = T_{-1} \). And since for \( i \geq 1 \), \( T_{-i} \subset C_1 \), \( d^{-1}(T_{-i}) = \alpha_2 \beta(T_{-i}) = T_{-i-1} \). Hence \( d^{-i}(C_i) = T_{-i} \), and hence \( d \) is a topological dilation.

Furthermore, since \( \alpha_1 \) is supported on \( K_1 \subset D_0 - D_2 \) and \( \alpha_2 \) is supported on \( K_2 \cap C_0 - C_2 \), \( h_1 = \alpha_2 \delta \) and \( h_2 = \alpha_1 r \) are topological dilations carried by \( \mathcal{A} \) and \( \mathcal{B} \) respectively; and \( d = (h_2 g h_2^{-1})(h_1 f h_1^{-1}) \).

In the above we had set \( K_2 = \delta f^{-1}(A_0) \) and had then remarked that \( K_2 \subset \delta f(A_0) = \emptyset \). It was this that enabled us to construct \( \alpha_1 \) independently of \( \alpha_2 \). Of course the construction of \( \alpha_2 \) was dependent on \( \alpha_1 \); however we defined \( K_1 = r g(B_0) \) and had we been able to assert that \( K_1 \cap r g^{-1}(B_0) = \emptyset \) we could have constructed \( \alpha_1 \) and \( \alpha_2 \) independently of each other. But, since in the event that \( g \) was of period two on \( X' \), \( K_1 \cap r g^{-1}(B_0) \neq \emptyset \), no such assertion was possible. Had \( f \) also been of period two on \( X' \) then neither \( \alpha_1 \) nor \( \alpha_2 \) could have been constructed independently of each other. For this reason Case 2 requires a slightly modified procedure.

**Case 2.** Since neither \( f \) nor \( g \) is the identity on \( X' \), we may select elements of \( \mathcal{X} \), \( C_0, D_0, A_0, \) and \( *B_0 \), such that the sets \( \{ C_0, D_0, A_0, *B_0, f(A_0), g(*B_0) \} \) are mutually disjoint and the sets \( \{ C_0, D_0, A_0, *B_0, f^{-1}(A_0), g^{-1}(B_0) \} \) are mutually disjoint.

Now select \( c \in C_0 \cap X' \), \( d \in D_0 \cap X' \), \( a \in A_0 \cap X' \), and \( b \in *B_0 \cap X' \). As before, we may select null sequences \( \{ C_i \}_{i=0}^{\infty}, \{ D_i \}_{i=0}^{\infty}, \{ A_i \}_{i=0}^{\infty}, \) and \( \{ B_i \}_{i=0}^{\infty} \) for \( c, d, a, \) and \( b \) respectively with \( B_0 \subset *B_0 \).

We now set \( A_i = \text{Cl}(\sim C_i), B_i = \text{Cl}(\sim D_i) \) for \( i \geq 1 \) and observe that \( \mathcal{A} = \{ A_i \}_{i=-\infty}^{\infty} \) and \( \mathcal{B} = \{ B_i \}_{i=-\infty}^{\infty} \) are dilations with respect to \((a, c)\) and \((b, d)\) respectively and hence, by Lemma 3, carry topological dilations \( \delta_1 \) and \( r_1 \) respectively.
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Since \( g^{-1}(N) \subseteq g(B_0^0) \), we may select \( N \in \mathcal{X} \) with \( r_1^{-1}(c) \in N \subseteq r_1^{-1}(C_0^0) \) and such that \( g^{-1}(N) \subseteq g(B_0^0) \). Since \( C_1 \subseteq C_0^0 = r_1(N) \), we may invoke Axiom (4) to obtain \( \alpha_1 \in G_{\mathcal{X}} \), supported on \( C_0 \), with \( \alpha_1 r_1(N) = C_1 \).

Set \( r = \alpha_1 r_1 \) and observe that \( r \) is a topological dilation carried by \( \mathcal{A} \) and \( r(N) = C_1 \).

Employing an identical procedure, we can obtain \( K_1 \in \mathcal{X} \), and \( \alpha_2 \in G_{\mathcal{X}} \) such that \( \alpha_2 \) is supported on \( D_0 \), \( K_1 \subseteq A_0^0 - A_{-1}, f^{-1}(K_1) \subseteq f(A_0^0) \), and \( \alpha_2 \delta_1(K_1) = D_1 \).

Set \( \delta = \alpha_2 \delta_1 \) and note that \( \delta \) is a topological dilation carried by \( \mathcal{A} \) and \( \delta(K_1) = D_1 \).

Now \( rg(b) \in rg(*)B_0 \subseteq D_1^0 - D_2 \). Hence \( \delta^{-1}rg(b) \in K_1^0 \). Set \( x_0 = \delta^{-1}rg(b) \). Since \( g^{-1}r^{-1}d^{-1}(x_0) = gr^{-1}d^{-1}f(x_0) \), we may select \( \{K_i\}_{i=2}^\infty \), null for \( x_0 \), with \( K_2 \subseteq K_1^0 \), such that \( g^{-1}r^{-1}d^{-1}(K_i) \subseteq gr^{-1}d^{-1}f(K_{i-1})^0 \) for \( i \geq 2 \) and with \( \delta(K_2) \subseteq rg(B_0^0) \).

Set \( E_i = r^{-1}d^{-1}(K_i), i \geq 2 \). Then \( r(E_i) = \delta^{-1}(K_i) \) and \( g^{-1}(E_i) \subseteq \delta^{-1}(K_{i-1})^0 \).

Now define \( \pi \in G_{\mathcal{X}}^\oplus \) as: \( \pi = r g^{-1}d^{-1}f \delta^{-1} \).

Consider \( \pi(A_1) : \delta^{-1}(A_1) = A_0 \), so \( \delta \delta^{-1}(A_1) = \delta f(A_0) \). Since \( r^{-1}d^{-1}(A_0) \subseteq B_0^0 \), \( \pi(A_1) = r g^{-1}d^{-1}f(A_0) \subseteq r g^{-1}d^{-1}f(B_0^0) \).

Since \( \delta(K_1) \subseteq A_1^0 \), the sequence:

\[
rg(B_0) \supset \pi(A_1) \supset \pi(\delta(K_1)) \supset ... \supset \pi(\delta(\delta(K_i))) \supset ...
\]

is null for \( \pi(\delta(x_0)) \). Also the sequence \( \{\delta(K_i)\}_{i=2}^\infty \) is null for \( \delta(x_0) \). Finally we have \( \delta(K_2) \cup rg(B_0) \subseteq rg(*)B_0^0 \subseteq D_1^0 - D_2 \).

Hence by Lemma 1 there exists \( \alpha_3 \in G_{\mathcal{X}} \), supported on \( rg(*)B_0 \), such that \( \alpha_3 r g(B_0^0) = \delta(K_2) \), \( \alpha_3 \pi(A_1) = \delta(K_3) \), and \( \alpha_3 \pi(\delta(K_i)) = \delta(K_{i+3}) \), \( i \geq 1 \).

\( \alpha_3 r \) is a topological dilation carried by \( \mathcal{A} \).

We shall now show that \( d = \alpha_3 r g^{-1} \alpha_3^{-1} \delta f \delta^{-1} \) is a topological dilation.

Consider \( d(A_1) \):

Since \( \delta \delta^{-1}(A_1) \subseteq C_1 \), and \( \alpha_3 \) is supported on \( D_1 \), \( \alpha_3^{-1} \delta f \delta^{-1}(A_1) = \delta f \delta^{-1}(A_1) \).

Hence \( d(A_1) = \alpha_3 r g^{-1} \delta f \delta^{-1}(A_1) = \alpha_3 \pi(A_1) = \delta(K_3) \).

And since \( \delta(K_i) \subseteq A_i^0, i \geq 1 \), \( \alpha_3^{-1} \delta f \delta^{-1}(\delta(K_i)) = \delta f \delta^{-1}(\delta(K_i)) \).

Hence \( d(\delta(K_i)) = \alpha_3 \pi(\delta(K_i)) = \delta(K_{i+3}), i \geq 1 \). Therefore \( \{d(A_1)\}_{i=0}^\infty \) is null.

We now consider \( d^{-1}(C_1) = \delta f^{-1} \delta^{-1} \alpha_3 r g^{-1} \delta^{-1} \alpha_3^{-1} (C_1) \): Since \( \alpha_3 \) is the identity on \( C_1 \), \( rg^{-1} \delta^{-1} \alpha_3^{-1}(C_1) = rg^{-1} \delta^{-1}(C_1) \). But \( g^{-1}r^{-1}(C_1) \subseteq g(B_0^0) \), so \( rg^{-1}r^{-1}(C_1) \subseteq rg(B_0^0) \). Hence \( \alpha_3 r g^{-1}r^{-1}(C_1) \subseteq \alpha_3 r g(B_0^0) = \delta(K_2) \).

Therefore \( d^{-1}(C_1) = \delta f^{-1} \delta^{-1}(\delta(K_2)) = \delta f^{-1}(K_2) \subseteq C_1^0 \).

We now consider \( d^{-1}(\delta f^{-1}(K_i)), i \geq 2 \): Since \( \alpha_3 \) is the identity on \( \delta f^{-1}(K_i) \subseteq C_1 \), \( r^{-1} \alpha_3^{-1} f^{-1}(K_i) = r^{-1} \delta f^{-1}(K_i) = E_i \). Since \( g^{-1}(E_i) \subseteq rg^{-1} \delta f(K_{i-1}) \) and \( rg^{-1}(E_i) \subseteq rg^{-1} \delta f(K_{i-1}) = r g^{-1} \delta f \delta^{-1}(\delta(K_{i-1})) = \pi(\delta(K_{i-1})) \).

Hence \( d^{-1}(\delta f^{-1}(K_i)) \subseteq \delta f^{-1} \delta^{-1} \alpha_3 \pi(\delta(K_{i-1})) = \delta f^{-1} \delta^{-1}(\delta(K_{i+2})) = \delta f^{-1}(K_{i+2}), i \geq 2 \).
So we see that $d^{-1}(C_1) \subset \delta f^{-1}(K_2)$, $d^{-2}(C_1) \subset d^{-1}(\delta f^{-1}(K_2)) \subset \delta f^{-1}(K_4)$, etc. In general, $d^{-i}(C_1) \subset \delta f^{-1}(K_{2i})$, $i \geq 1$.

From this, together with the fact that $d^{-i}(C_1) \subset d^{-i+1}(C_i^0)$, $i \geq 1$, we conclude that $\{d^{-i}(C_1)\}_{i=0}^\infty$ is null for $\delta f^{-1}(x_0)$. If we set $h_1 = \delta$ and $h_2 = \alpha_3 x$, then $d = (h_2 h_2^{-1})(h_1 f h_1^{-1})$.

Let $d \in G^*_x$ be an arbitrary but fixed topological dilation:

**Corollary 1.** If $f, g \in G(X, X')$, neither $f$ nor $g$ being the identity on $X'$, then $d$ is the product of a conjugate of $f$ by a conjugate of $g$, the conjugating homeomorphisms being elements of $G^*_x$.

**Proof.** By Theorem 2 there exist $h_1, h_2 \in G^*_x$ such that $d = (h_2 g h_2^{-1})(h_1 f h_1^{-1})$, where $d$ is a topological dilation. But by Theorem 1 there exists $h_3 \in G^*_x$ such that $h_3 d h_3^{-1} = d$. Hence $h_3 d h_3^{-1} = (h_2 g h_2^{-1})(h_1 f h_1^{-1})$. Therefore

$$d = h_3^{-1}(h_2 g h_2^{-1})(h_1 f h_1^{-1}) h_3 = (h_3^{-1} h_2 g h_2^{-1}) h_3 (h_3^{-1} h_1 f h_1^{-1} h_3).$$

**Corollary 2.** If $f \in G(X, X')$ and either $f \mid X' \neq e$ or $f = e$ and if $d \in G(X, X')$, then $f$ is a product of two conjugates of $d$, the conjugating homeomorphisms being elements of $G^*_x$.

**Proof.** If $f \mid X' \neq e$, then by Corollary 1, with $d^{-1}$ as $g$, we have

$$d = (h_2 d h_2^{-1})(h_1 f h_1^{-1})$$

for some $h_2, h_1 \in G^*_x$. Hence $(h_2 d h_2^{-1}) d = h_1 f h_1^{-1}$; or $h_1^{-1}(h_2 d h_2^{-1}) d h_1 = f$. Therefore $(h_1^{-1} h_2 h_2^{-1} h_1)(h_1^{-1} d h_1) = f$.

If $f = e$, then notice that $d^{-1}$ is a topological dilation, and hence, for some $h_1 \in G^*_x$, $h_1 d h_1^{-1} = d^{-1}$. Then $d(h_1 d h_1^{-1}) = dd^{-1} = e = f$.

**Corollary 3.** Suppose $f, g, h, k$ are elements of $G(X, X')$, none of which are the identity on $X'$. Then $f$ is the product of a conjugate of $g$ by a conjugate of $h$ by a conjugate of $k$, the conjugating elements being in $G^*_x$.

**Proof.** By Corollary 1 $(h_1 f h_1^{-1})(h_2 g h_2^{-1}) = d = (h_3 k h_3^{-1})(h_4 h h_4^{-1})$ for some $h_1, h_2, h_3, h_4 \in G^*_x$. Then $h_1 f h_1^{-1} = (h_3 k h_3^{-1})(h_4 h h_4^{-1})(h_2 g h_2^{-1})$. Hence $f = (h_1^{-1} h_3 k h_3^{-1} h_1)(h_1^{-1} h_4 h h_4^{-1} h_1)(h_1^{-1} h_2 g h_2^{-1} h_1)$.

**Corollary 4.** Suppose $f, g \in G(X, X')$ and $f \mid X' \neq e \neq g \mid X'$. Then $f$ is the product of three conjugates of $g$, the conjugating elements being in $G^*_x$.

**Proof.** Corollary 3 with $h = k = g$.

4. **Examples.** We now make mention of a few examples of $A$-quadruples. In each instance, of course, the existence of topological dilations is guaranteed by the corollary to Lemma 3. However in the examples to follow a canonical topological dilation $d$, suggests itself and in some instances will be given. Then, by Theorem
1, the collection of topological dilations coincides with the collection of $G^*_{\mathcal{H}}$-conjugates of $d$.

(1) Let $X = X' = S_n$, the $n$-sphere for $n \geq 1$. Let $G(X, X') = G^*_{\mathcal{H}}$ be the group of stable(4) homeomorphisms of $S_n$ onto itself. $\mathcal{H}$ is the collection of images of a geometric $n$-cell under the elements of $G^*_{\mathcal{H}}$. That Axioms (1), (2), and (3) are satisfied in this case is evident. The proof that Axioms (4) and (5) obtain may be found in [3]. Let $p \in S_n$ and let $q$ be the point of $S_n$ antipodal to $p$. We may take as $d$ an ordinary radial expansion homeomorphism of $S_n$ onto itself which fixes only $p$ and $q$. (We think of $d$ as the natural extension to $S_n$ of $d': E_n \to E_n$ given by $d'(x) = r_0x$ for some fixed, positive $r_0 \neq 1$.)

(2) Take $X, X', \mathcal{H}, G^*_{\mathcal{H}}$, and $d$ as above. Let $G(X, X')$ be the set of all homeomorphisms which are the product of some geometric orientation-reversing involution, $h$, and an element of $G^*_{\mathcal{H}}$.

Since $d \not\in G(X, X')$, Corollary 2 is not applicable here. However $dh \in G(X, X')$ so we may substitute for Corollary 2 the following: if $f \in G(X, X')$, then $f$ is the product of a conjugate of $d$ by a conjugate of $dh$, the conjugating homeomorphisms, of course, being in $G^*_{\mathcal{H}}(5)$.

(3) Let $X = S_n, n \geq 1$. Let $X'$ be a countable dense subset of $X$, and let $G^*_{\mathcal{H}} = G(X, X')$ be those stable homeomorphisms which take $X'$ onto $X'$. $\mathcal{H}$ is the collection of images of a geometric $n$-cell under elements of $G^*_{\mathcal{H}}$.

(4) Let $X = D_n = \text{the } n\text{-cell, } n \geq 2$. Call $X'$ the boundary of $D_n$, and $G^*_{\mathcal{H}}$ is the set of all homeomorphisms of $D_n$ onto itself each of which is the identity on some neighborhood of some point of $X'$. $G(X, X') = G^*_{\mathcal{H}}$, and $\mathcal{H}$ is the collection of images of a geometric half-cell under the elements of $G^*_{\mathcal{H}}$. Axioms (1), (2), and (3) are obviously satisfied, and Axioms (4) and (5) follow from arguments like those given in [3].

(5) Let $X = X'$ be the rationals, the irrationals, or the Cantor set. Let $G^*_{\mathcal{H}} = G(X, X')$ be the collection of all homeomorphisms of $X$ onto itself. $\mathcal{H}$ is the collection of all nonvoid, proper, open and closed subsets of $X$.

If $X$ is the Cantor set, realized as the 'middle third' set in $[0,1]$, define $\{C_i\}_{i=-\infty}^{\infty}$ as follows: for $i \leq 0$,

$$C_i = \left[ \frac{2}{3^{-i+2}}, \frac{1}{3^{-i+1}} \right] \cap X,$$

and for $i \geq 1$,

$$C_i = \left[ \frac{3^i - 1}{3^i}, \frac{3^{i+1} - 2}{3^{i+1}} \right] \cap X.$$

Then $\bigcup_{i=-\infty}^{\infty} \{C_i\} \cup \{0\} \cup \{1\} = X$. Take $d$ as any homeomorphism of $X$ onto itself such that $d(0) = 0, d(1) = 1$, and $d(C_i) = C_{i+1}, -\infty < i < \infty$.

(4) $G^*_{\mathcal{H}}$ is the group generated by the collection of all those homeomorphisms of $S_n$ onto itself each of which is the identity on some nonvoid open set. The term 'stable' used to denote this group appears to be due to Brown and Gluck [3].

(5) If the annulus problem is solved affirmatively, then in (1) we may take $G^*_{\mathcal{H}} = G(X, X')$ to be the collection of all orientation-preserving homeomorphisms, and in (2) we may take $G(X, X')$ to be all the orientation-reversing homeomorphisms.
A few words about possible extensions of the preceding conclusions. Let $G$ denote the group of stable homeomorphisms of $E_n$. Set $d(x) = r_0x$, $x \in E_n$, $r_0 > 1$. By arguments similar in spirit to those of Theorems 1 and 2, one can show that if $f \in G_n$, then there exist stable homeomorphisms $\delta$ and $r$ such that $\delta d \delta^{-1} f = r dr^{-1}$. Therefore $f$ is the product of a stable conjugate of $d$ by a stable conjugate of $d^{-1}$.

It is impossible to obtain Corollary 4 for the group $G_n$, since it is not simple. However the following question might well be asked: suppose $f$, $g \in G_n$ with $f$ not the identity and $g$ not supported on a cell. Is $f$ the product of three conjugates of $g$? Or if three won't work, what is the smallest number which will? Three appears to suffice, if $g$ moves a tame ray off of itself.

Finally we remark that the number of conjugates of $g$ employed in Corollary 4 cannot, in general, be reduced: In each of the examples (1)–(5) above it is easy to find nonidentity elements $f$ and $g$ of $G(X, X')$ such that $f$ is not the product of two conjugates of $g$.

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