ON MILNOR'S INVARIANT FOR LINKS

BY

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1. Introduction. Let $C_n$ be the space consisting of $n$ disjoint (oriented) circles. By an (oriented) link of multiplicity $n$ in 3-space $R^3$ is meant a homeomorphic image of $C_n$ in $R^3$.

Two links $l$ and $l'$ of multiplicity $n$ and $n'$ in $R^3$ are said to be of the same type if there exists an orientation preserving homeomorphism of $R^3$ onto itself that maps $l$ onto $l'$ (and hence necessarily $n = n'$). A link is polygonal if it has a polygonal representative. Further, two links $l$ and $l'$ are said to be isotopic if there exists a continuous family $h_t$ of homeomorphisms of $C_n$ into $R^3$ with $h_0(C_n) = l$ and $h_1(C_n) = l'$. These two concepts are not necessarily equivalent. In fact, the group of a link $l$, $G = \pi_1(R^3 - l)$, is an invariant of the link type but is not an isotopy invariant of the link.

In 1952, K.T. Chen investigated the lower central series $\{G_q\}$ of the group $G$ of a polygonal link and proved that $G/G_q$ is an isotopy invariant for all $q \geq 1$. $G_q$ is defined inductively by $G_1 = G$, $G_{i+1} = [G,G_i]$, $i = 1, 2, \ldots$, where $[G,G_i]$ is the subgroup generated by all $aba^{-1}b^{-1}$ with $a \in G$, $b \in G_i$. Later J. Milnor generalized this result for the group $G = \pi_1(M - l)$, where $M$ is an arbitrary open orientable 3-manifold and $l$ is a link in $M$ which is not necessarily polygonal. In particular, for a link in 3-space $R^3$, Milnor defined a numerical invariant $\bar{\mu}(i_1 \cdots i_k)$, where $i_1 \cdots i_k$ is a sequence of positive integers between 1 and $n$. $\bar{\mu}$ will be called Milnor's invariant in this paper. $\bar{\mu}(ij)$, $i \neq j$, is the linking number of the $i$th component $l_i$ and $j$th component $l_j$ of $l$.

On the other hand, R.H. Fox defined the polynomial $\Delta(x_1, \cdots, x_n)$ with integral coefficients for a given link $l$ of multiplicity $n$ based on a presentation of the group $G$ of $l$ by means of his own free differential calculus [2], [3]. It is now called the Alexander polynomial of $l$. This is the natural generalization of the so called Alexander polynomial of a knot, (a knot being a link of multiplicity one). As is well known, the Alexander polynomial $\Delta(x_i, x_j)$ of a link $l_i \cup l_j$ of multiplicity two evaluated at $x_i = x_j = 1$ coincides up to sign with the linking number of $l_i$ and $l_j$ [7]. Therefore, we can write

$$|\Delta(x_i, x_j)|_{x_i = x_j = 1} = |\bar{\mu}(ij)|.$$

This immediate relation between $\Delta(x_i, x_j)$ and $\bar{\mu}$ is the only one obtained up to the present time.

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In this paper we shall establish some relations between the partial derivatives of $\Delta(x_1, \ldots, x_n)$ and $\bar{\mu}(i_1, \ldots, i_k)$ (Theorems 4.1-4.3). Since $\Delta(x_1, \ldots, x_n)$ is an invariant of $G/[G_2, G_2]$, it follows that for sequences treated in our theorems $\bar{\mu}(i_1, \ldots, i_k)$ depends on $G/[G_2, G_2]$ rather than on $G/G_{2^*}$.

Noting that $\bar{\mu}(i_1, \ldots, i_k)$ is defined by means of free differentiation, our relations seem to be quite natural. However, the delicate differences arise from the fact that while the usual partial differentiation is commutative, free differentiation is not. Fortunately, these differences do not cause any serious difficulties when links of multiplicity two are treated. Nevertheless the proof of Theorem 4.1 (§§6-8) is quite complicated. The form of Theorems 4.2 and 4.3 is chosen to avoid unnecessary complications. However the direction in which these relations can be generalized will be indicated.

2. The Alexander polynomial. Let $l = l_1 \cup l_2 \cup \cdots \cup l_n$ be an oriented polygonal link of multiplicity $n \geq 2$ in 3-space $R^3$. Let $G = \pi_1(R^3 - l)$ be the group of $l$ and let $\mathcal{P}$ be the Wirtinger presentation determined by a link projection:

$$\mathcal{P} = (x_{i,j}; r_{i,j}, 1 \leq i \leq n, 1 \leq j \leq \lambda_i),$$

where $x_{i,j}$ is represented by a loop going once around an arc of the $i$th component $l_i$ of $l$ in the positive direction and $r_{i,j} = u_{i,j}x_{i,j}u_{i,j}^{-1}x_{i,j+1}^{-1}(1)$, $u_{i,j} = x_{p,q}$ or $x_{p,q}^{-1}$.

Now consider a set of elements $s_{i,j}$ defined by $s_{i,j} = v_{i,j}x_{i,j}v_{i,j}^{-1}x_{i,j+1}^{-1}$, where $v_{i,j} = u_{i,j}u_{i,j-1} \cdots u_{i,1}$. Then it is an elementary matter to show that the set $\{r_{i,j}\}$ can be replaced by the set $\{s_{i,j}\}$ so that a new presentation $\mathcal{S}$ of $G$ is obtained [2]:

$$\mathcal{S} = (x_{i,j}; s_{i,j}, 1 \leq i \leq n, 1 \leq j \leq \lambda_i).$$

$\mathcal{S}$ will be called the standard presentation of $G$ with respect to $\mathcal{P}$.

Let $F$ be the free group generated by $x_{i,j}, 1 \leq i \leq n, 1 \leq j \leq \lambda_i$. Let $\phi$ be the canonical homomorphism of $F$ onto $G$. $\phi$ can be uniquely extended to the ring homomorphism of the integral group rings $\mathbb{Z}F$ onto $\mathbb{Z}G(\mathbb{Z})$.

Let $A_n$ be a free abelian group of rank $n$ with a basis $\{x_1, \ldots, x_n\}$. Then a homomorphism $\psi: G \to A_n$ defined by $x_{i,j}^\psi = x_{i,j}(3), 1 \leq i \leq n$ and $1 \leq j \leq \lambda_i$, can be uniquely extended to the ring homomorphism $\mathbb{Z}G \to \mathbb{Z}A_n$.

Let $M$ be the Jacobian matrix of $\mathcal{S}$ at $\psi \phi$ [3, II],

$$M = \left[ \frac{\partial s_{i,j}}{\partial x_{k,l}} \right]_{i,j,k,l}^\phi \phi$$

\begin{equation}
(2.1)
\frac{\partial s_{i,j}}{\partial x_{k,l}} = (1 - x_{i,j}) \left( \frac{\partial v_{i,j}}{\partial x_{k,l}} \right) + \delta_{i,j} \left( u_{i,j}^{-1} v_{i,j}^\psi - u_{i,j+1} \right),
\end{equation}

where $\delta_{p,q}$ is Kronecker’s delta.

(1) The second index is mod $\lambda_i$.

(2) Any group homomorphism: $G \to H$ determines uniquely the ring homomorphism of integral group rings: $\mathbb{Z}G \to \mathbb{Z}H$. These two homomorphisms always denoted by the same letter.

(3) $x_{i,j}^\psi$ denotes the image of $x_{i,j}$ under $\psi$ and $x_{i,j}^\psi = (x_{i,j})^\psi$. 

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$M$ is a square matrix of order $\sum_{i=1}^{n} \lambda_i$. Each row of $M$ corresponds to a relator $s_{i,j}$ and each column to a generator $x_{k,l}$ in $\mathcal{R}$. These will be called the $s_{i,j}$-rows and $x_{k,l}$-columns respectively.

Let $\tilde{M}(s_{i,j}: x_{k,l})$ denote the square matrix obtained from $M$ by deleting the $s_{i,j}$-row and $x_{k,l}$-column. Then $\text{g.c.d.}_{s_{i,j}: x_{k,l}} \{ \det \tilde{M}(s_{i,j}: x_{k,l}) \}$ is uniquely determined up to a factor $\pm x_1^{s_{1,j}} \cdots x_n^{s_{n,j}}$ [3,11]. It does not depend on the presentation of $G$ and is denoted by $\Delta(x_1, \ldots, x_n)$. $\Delta(x_1, \ldots, x_n)$ is, in fact, an invariant of link types and is called the Alexander polynomial of $l$. For properties of the Alexander polynomial, the reader should refer to [7].

Now let $\det \tilde{M}(s_{i,j}: x_{k,l})$ be a polynomial in $x_1, \ldots, x_n$ with a possibility of having negative exponents. Two such polynomials $f$ and $g$ are said to be equivalent, written $f \sim g$, if $f = x_1^{a_1} \cdots x_n^{a_n} g$ for some integers $a_1, \ldots, a_n$, where $\varepsilon = \pm 1$ or 0.

In the following, we will show that $\det \tilde{M}(s_{i,j}: x_{k,l})/(1 - x_k)$ may be specified as the Alexander polynomial of $l$.

First we will prove the following lemma.

**Lemma 2.1.**

$$
\det \tilde{M}(s_{i,j}: x_{k,l}) \sim P(i,j: k,l: p) \det \tilde{M}(s_{p,q}: x_{k,l}),
$$

where $P$ is a polynomial in $x_1, \ldots, x_n$ and $P = 1$ if $j = \lambda_i$.

**Proof.** As is well known ([2] or [6, Lemma 5]), given the Wirtinger presentation $\mathcal{R}$ of $G$, there are elements $w_{i,j}$ of $F$ such that

$$
\prod_{i=1}^{n} \prod_{j=1}^{\lambda_i} w_{i,j} f_{i,j} w_{i,j}^{-1} = 1 \quad \text{in} \ F.
$$

Since any relator $r_{i,j}$ is represented by means of relators $s_k,l$ in $\mathcal{R}$ as $r_{i,j} = u_{i,j} s_{i,j-1} u_{i,j}^{-1}$, it follows from (2.2) that $s_{p,q}$ is a consequence of the others. Thus $\frac{\partial s_{p,q}}{\partial x_{k,l}}$ is a linear combination of other $\frac{\partial s_{i,j}}{\partial x_{k,l}}$. Thus

$$
\det \tilde{M}(s_{i,j}: x_{k,l}) \sim P(i,j: k,l: p) \det \tilde{M}(s_{p,q}: x_{k,l}).
$$

It is obvious that $P = 1$ if $j = \lambda_i$.

From (2.1) and Lemma 2.1, we see that $\det \tilde{M}(s_{i,j}: x_{k,l}) \equiv 0 \mod 1 - x_k$.

We define $N(s_{i,j}: x_{k,l}) = \det M(s_{i,j}: x_{k,l})/(1 - x_k)$. $N$ is a polynomial in $x_1, \ldots, x_n$.

Now the fundamental formula [3, I.(2.3)] implies that $N(s_{i,j}: x_{k,l}) \sim N(s_{i,j}: x_{p,q})$. Further, from Lemma 2.1, it follows immediately that $N(s_{i,j}: x_{k,l}) \sim N(s_{1,1}: x_{k,l})$. Thus, g.c.d. $\{\det M(s_{i,j}: x_{k,l})\}$ $\sim N(s_{1,1}: x_{k,l})$. In other words, we have proved the following

**Lemma 2.2.** $\Delta(x_1, \ldots, x_n) \sim N(s_{1,1}: x_{n,1})$.

$N(s_{1,1}: x_{n,1})$ is specified, hereafter, as the Alexander polynomial of $l$ and is denoted by $\Delta(x_1, \ldots, x_n)$. Since each element of the $s_{n,1}$-row in $\tilde{M}(s_{1,1}: x_{n,1})$ has

(4) More generally $\tilde{M}(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q)$ denotes the matrix obtained from $M$ by deleting the $\alpha_1$-rows and $\beta_q$-columns.
the factor $1 - x_n$, the Alexander polynomial is the determinant obtained from $\tilde{M}$ by dividing the $s_{n,1,n}$-row by $1 - x_n$. In the case $n = 2$, our specification is justified by the following

(2.3) \quad \Delta(1,1) \text{ equals the linking number.}

The proof will be given in §4.

The Alexander polynomial of a subset of $l$ is discussed in detail by Torres in his paper [7]. We only mention without proof the following lemma.

Let $l_{i_1} \cup l_{i_2} \cup \cdots \cup l_{i_p}$ be a subset of $l$ and let $\Delta(x_{i_1}, x_{i_2}, \cdots, x_{i_p})$ be its Alexander polynomial. Let $M_{i,j}$ denote the submatrix of $M$ consisting of the $s_{i,1,r}, \cdots, s_{i,1,n}$-rows and the $x_{i,1,r}, \cdots, x_{i,1,n}$-columns. Then we have

\begin{equation}
\text{Lemma 2.3.}
\end{equation}

\[ \pm x_{i_1}^{s_{i_1}} \cdots x_{i_p}^{s_{i_p}} \Delta(x_{i_1}, \cdots, x_i) = \det \begin{vmatrix} \tilde{M}_{i_1,i_p}(s_{i_1,1,i_p}; x^{s_{i_1}}_{i_1,1,1})^{\Phi_{i_1}} \end{vmatrix}, \quad 1 \leq k, l \leq p, \]

where $\Phi$ is a homomorphism $JA_n \to JA_p$ defined by

\[ x_i^\Phi = 0 \quad \text{for } j \neq i_1, \cdots, i_p, \]

\[ x_i^\Phi = x_j \quad \text{for } j = i_1, \cdots, i_p. \]

3. Milnor's invariant. Let $\mathcal{F} = (x_{ij}; s_{ij})$ be the standard presentation of $G = \pi_1(R^3 - l)$ with respect to the Wirtinger presentation given in §2. Let $F_n$ be the free group generated by $x_1$, $x_2$, $x_n$, and let $A_n$ be the commutator quotient group of $F_n$, hence $A_n$ is the free abelian group of rank $n$ with a basis $\{x_1, \cdots, x_n\}$. Let $\phi$ be the canonical homomorphism of $F_n$ onto $A_n$. $\phi$ can be uniquely extended to the ring homomorphism $JF_n \to JA_n$.

Now, we shall define the homomorphisms $\theta_p: F \to F_n$ by induction on $p$ as follows [6].

\[ x_i^{\theta_p} = x_i \quad \text{for } 1 \leq i \leq n, 1 \leq j \leq \lambda_i, \]

(3.1)

\[ x_i^{\theta_p+1} = x_i, \]

\[ x_i^{\theta_p+1} = (v_{i,i} x_{i,1} v_{i,1}^{-1})^{\theta_p} \quad \text{for } 1 \leq i \leq n, 1 \leq j < \lambda_i. \]

$\theta_p$ will be extended to the ring homomorphism $JF \to JF_n$. The particular homomorphism $\phi \theta_1: JF \to JA_n$ will be denoted by $\rho$, i.e. $x_i^\rho = x_i$.

The trivializer $o: JG \to J$ is a homomorphism defined by $(\sum a_j x_i^{s_{i,j}} \cdots x_i^{s_{i,p}})^o = \sum a_j$.

Then we have the following commutative diagram:

\[ \begin{array}{ccc}
JF & \xrightarrow{\theta_p} & JF_n \\
\downarrow \phi & & \downarrow o \\
J & \xrightarrow{o} & JA_n \\
\end{array} \]

\[ JF \xrightarrow{\rho} JA_n \]

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Let \( \eta_k = v_k, \lambda_k \). It represents a parallel of the \( k \)-th component of \( l \). Then, following Milnor, we define an integer \( \mu(i_1 \cdots i_pk) \) for a sequence \( i_1 \cdots i_pk \) \((p \geq 1)\) of integers between 1 and \( n \) as follows [6].

\[
(3.2) \quad \mu(i_1 \cdots i_pk) = \left( \frac{\partial^n\eta_k^{\theta_p}}{\partial x_{i_1} \cdots \partial x_{i_p}} \right)^\circ.
\]

Let \( \Delta(i_1 \cdots i_s) = \gcd \mu(j_1 \cdots j_s) \), where \( j_1 \cdots j_s \) \((2 \leq s < r)\) is to range over all sequences obtained by cancelling at least one of the indices \( i_1, \ldots, i_r \) and permuting the remaining indices cyclically. Then Milnor proved that

\[
(3.3) \quad \bar{\mu}(i_1 \cdots i_pk) \equiv \mu(i_1 \cdots i_pk) \mod \Delta(i_1 \cdots i_pk)
\]

is an isotopy invariant of a link.

\( \bar{\mu} \) will be called Milnor's invariant in this paper. Further, let

\[
(3.4) \quad \mu^*(i_1 \cdots i_pk) \equiv \mu(i_1 \cdots i_pk) \mod \Delta^*(i_1 \cdots i_pk)
\]

is also an isotopy invariant of a link.

\( \mu^* \) will be called the weak Milnor invariant.

Remark 1. Milnor proved [6] that \( \mu^* \) and \( \bar{\mu} \) are isotopy invariants of (not necessarily polygonal) links. However, we consider only polygonal links in 3-space \( R^3 \), because we are concerned about relations with the Alexander polynomials which may not be defined for "wild" links.

Remark 2. As Milnor pointed out [6], \( \mu^*(i_1 \cdots i_s) \) is an invariant of the homotopy type of a link in the sense of [5] provided \( i_1, \ldots, i_s \) are mutually distinct.

Now many of the relations among \( \bar{\mu}(i_1 \cdots i_s) \) and \( \mu^*(i_1 \cdots i_s) \) for various sequences \( i_1 \cdots i_s \) are found in [6]. Those relations needed to prove our theorems will be collected in §5. The notation used for Milnor's invariant is justified by the following lemma which shows that \( \bar{\mu}(i_1 \cdots i_s) \) depends only on the sublink \( l_{i_1} \cup \cdots \cup l_{i_s} \).

Lemma 3.1. Let \( l' \) be a sublink of \( l \) multiplicity \( m \). We may assume without loss of generality that \( l = l_1 \cup \cdots \cup l_m, 2 \leq m < n \). Let \( j_1 \cdots j_pk \) be a sequence of integers between 1 and \( m \). Then two integers \( \mu(j_1 \cdots j_pk) \) and \( \mu'(j_1 \cdots j_pk) \) are defined by means of \( \mathcal{S} \), the standard presentation of \( G = \pi_1(R^3 - l) \), and \( \mathcal{S}' \), one such of \( G' = \pi_1(R^3 - l') \). Then

\[
(3.5) \quad \mu(j_1 \cdots j_pk) = \mu'(j_1 \cdots j_pk).
\]

The proof is straightforward.

4. Main theorems. Let \( f(x_1, \ldots, x_n) \) be an element of the integral group ring of the free group \( F_n \). As usual the free derivative of \( f(x_1, \ldots, x_n) \) with respect
to \( x_i \) is denoted by \( \partial f/\partial x_i \). On the other hand, since \( f(x_1, \cdots, x_n) \) is an element of the integral polynomial ring \( JA_n \), we can define the partial derivative of \( f \) with respect to \( x_p \), which will be denoted by \( df^p/\partial x_i \) throughout this paper. By the trivializer \( o \) of a function \( f(x_1, \cdots, x_n) \) is meant the homomorphism of \( JA_n \) into \( J \), the ring of integers, defined by \( \left[ f(x_1, \cdots, x_n) \right]^o = f(1, \cdots, 1) \).

The main theorem of this paper is Theorem 4.1 which establishes a relation between the Alexander polynomial and the weak Milnor invariant for a link of multiplicity two. (The Alexander polynomial is an element of \( JA_n \).)

For the sake of simplicity, hereafter, the particular sequence \( 1 \cdots 12 \cdots 2 \), where there are \( a \) 1's and \( b \) 2's, will be denoted by \([a, b]\).

**Theorem 4.1.** Let \( l \) be a link of multiplicity two and let \( \Delta(x, y) \) be the Alexander polynomial of \( l \), which is specified in \$2\). Then for all \( p, q \geq 0 \),

\[
\bar{\mu}([p + 1, q + 1]) \equiv (-1)^q \frac{1}{p!} \frac{1}{q!} \left[ \frac{d^{p+q}}{dx^p dy^q} \Delta(x, y) \right]^o \mod \Delta^*[([p + 1, q + 1])].
\]

(4.1)

First observe that Theorem 4.1 is true in the simplest case \( p = q = 0 \). In fact, by [3, I,(2.6)] or Lemma 5.1,

\[
\bar{\mu}(12) = \left( \frac{\partial \eta_{11}^o}{\partial x_1} \right)^o = \sum_{i,j} \left( \frac{\partial \eta_{i,j}^o}{\partial x_{i,j}} \right)^o = \sum_{i=1}^{1} \left( \frac{\partial \eta_{1,j}^o}{\partial x_{1,j}} \right)^o = [\Delta(x, y)]^o,
\]

which also proves (2.3). The proof of Theorem 4.1 will be done by induction on \( p + q \) in §§6–8.

If \( \Delta(x, y) \) is normalized by multiplying by \( x^ry^s \), Theorem 4.1 remains true. This is verified straightforwardly. Therefore our reason in specifying the Alexander polynomial in \$2\) is to ensure that \( \mu([p + 1, q + 1]) \) coincides with the derivative of \( \Delta(x, y) \) in Theorem 4.1 including the sign. If we disregard the sign of \( \Delta(x, y) \) and consider only the absolute value of \( \bar{\mu} \), we can choose \( \Delta(x, y) \) arbitrary.

In the case where the multiplicity of \( l \) is greater than two, it is not so easy to establish relations between them. However, the following theorems indicate in what direction Theorem 4.1 will be generalized.

**Theorem 4.2(5).**

\[
\pm \frac{1}{p!} \left[ \frac{d^p}{dx_1^p} \Delta(x_1, \cdots, x_n) \right]^o \equiv \bar{\mu}([r_1]2)\bar{\mu}([r_2]3) \cdots \bar{\mu}([r_n-1]1)n)
\]

(4.2)

\( \mod \text{g.c.d.} \{\Delta^*([r_1]2), \Delta^*([r_2]3), \cdots, \Delta^*([r_n-1]1)n\} \),

where \( r_1 + \cdots + r_n-1 = p, \ r_i \geq 0 \) and \( \bar{\mu}(i) = 0 \) for any \( i \).

(9) \([a]i\) stands for the sequence \( 1 \cdots 1_i \), where there are \( a \) 1's.
Corollary.

\[ \pm \frac{1}{(n-2)!} \left[ \frac{d^{n-2}}{dx_1^{n-2}} \Delta(x_1, \ldots, x_n) \right]^\circ = \bar{\mu}(12)\bar{\mu}(13) \cdots \bar{\mu}(1n). \]

Remark(6). It is not hard to show that

\[ \bar{\mu}([r]i) \equiv \binom{p}{r} \mod \Delta^*([r]i), \]

where \( p \) denotes the linking number of the first and \( i \)th components of a link. Therefore (4.2) may be stated independently of the Milnor invariants.

**Theorem 4.3.** Let \( \Delta(x, y, z) \) be the Alexander polynomial of \( l \) of multiplicity three. Then

\[
\pm \left[ \frac{d^3}{dx\,dy\,dz} \Delta(x, y, z) \right]^\circ = \mu(123)^2 \\
+ \bar{\mu}(112)\bar{\mu}(233) - \bar{\mu}(113)\bar{\mu}(223) - \bar{\mu}(122)\bar{\mu}(133) \mod \Delta^*(123).
\]

Every term in the right-hand side other than the first is a product of two Milnor's invariants of types considered in Theorem 4.1. Thus by Lemma 3.1 and (4.1), these invariants can be obtained from the Alexander polynomial of the corresponding link of two components. (See the above remark.) If \( \mu^*(i_1 \cdots i_p) = 0 \) for all sequences \( i_1 \cdots i_p \) of mutually distinct integers between 1 and \( n \), then the link is homotopically trivial [5]. Thus Theorem 4.3 implies immediately the following

**Corollary.** Let \( l = \cup_{i=1}^{3} l_i \) and let \( \Delta(x_i, x_j) \) denote the Alexander polynomial of \( l_i \cup l_j \). If \( \Delta(x_i, x_j) = 0, 1 \leq i, j \leq 3, i \neq j \) and \( \Delta(x_1, x_2, x_3) = 0 \), then \( l \) is homotopically trivial.

This corollary can not be generalized to the case \( n \geq 4 \). In fact, there exists an almost trivial [7], but not trivial link for which the Alexander polynomial vanishes. The link illustrated by Figure 7 in [5, p. 190] is one of such links.

**Example.** The Alexander polynomial \( \Delta(x, y) \) of the link \( l \) considered in [6, p. 301] is \( (-1)^{n-1}(1-x)^{2m-1}(1-y) \). Therefore

\[
\bar{\mu}([2m, 2]) \equiv (-1) \frac{1}{(2m-1)!} \left[ \frac{d^{2m}}{dx^{2m-1}dy} \Delta(x, y) \right]^\circ \\
= (-1)^m \mod \Delta^*([2m, 2]).
\]

In this case, \( \Delta^*([2m, 2]) = 0 \).

5. Preliminary lemmas. In this section, we collect some lemmas which will be used frequently in the following sections. Many of them are formulas appearing in [3], [4], [6] or easy consequences.

(6) The author acknowledges to the referee for pointing out this remark.

(7) For the definition, see [5, p. 189].
Lemma 5.1 (Chain Rule). Let $Y$ and $Z$ be free groups generated by $y_1, \ldots, y_n$ and $z_1, \ldots, z_m$ respectively, and let $\tau$ be a homomorphism $\tau : Y \to Z$. Then for any $f \in JY$,

$$\frac{\partial f^\tau}{\partial z_k} = \sum_{j=1}^{n} \left( \frac{\partial f}{\partial y_j} \right)^\tau \left( \frac{\partial y_j^\tau}{\partial z_k} \right), \quad 1 \leq k \leq m$$

[3,1.(2.6)].

In the rest of this section, we assume that $X$ is the free group generated by $x$ and $y$, and $\phi$ is the canonical homomorphism $JX \to J(X/[X, X])$.

Let $i_1 \cdots i_r$ and $j_1 \cdots j_s$ be two sequences. By a proper shuffle of these two sequences is meant one of the $(r + s)!/r!s!$ sequences obtained by intermeshing $i_1 \cdots i_r$ with $j_1 \cdots j_s$ [6, p. 294]. Let $S(a, b)$ denote the set of all proper shuffles of two sequences $1 \cdots 1$ ($a$ times) and $2 \cdots 2$ ($b$ times). For the sake of brevity,

$$\frac{\partial^{a+b}f}{\partial z_{i_1} \cdots \partial z_{i_{a+b}}}$$

is denoted by $\partial^{a+b}f / \partial \omega$, where $\omega = i_1 \cdots i_{a+b} \in S(a, b)$ and $z_{i_j} = x$ or $y$ according as $i_j = 1$ or $2$, and

$$\frac{d^{a+b}f}{dx^a dy^b}$$

is denoted by $D^{a,b}f$.

Lemma 5.2.

$$(S) \quad \left[ \sum_{\omega \in S(a, b)} \frac{\partial^{a+b}f}{\partial \omega} \right]^{n} = \frac{1}{a!} \frac{1}{b!} (D^{a,b}\phi)^n, \quad \text{for any } f \in JX.$$

This follows from [3,1.(3.9)] and [4, (3.3)].

Lemma 5.3.

$$(\Phi) \quad (D^{n,0}\phi)^n = n \left[ D^{n-1,0} \left( \frac{\partial f}{\partial x} \right) \phi \right]^n, \quad \text{for } f \in JX \text{ and } n \geq 1.$$

This follows from Lemma 5.2.

Lemma 5.4.

$$(\tilde{\Phi}) \quad (-1)^q \tilde{\mu}(p+1, q+1) \equiv \sum_{\omega \in S(a, b)} \tilde{\mu}(\omega) \mod \Delta^*([p+1, q+1])$$

[6,(26)].

Lemma 5.5. Let $M = (a_{i,j})$ be an $n \times (n+1)$ integral matrix. Then (see(4)) for $1 \leq i \leq n$,

$$\det \bar{M}(i+1) = (-1)^{i-1}a_{i,1} \sum_{k=2}^{n+1} \det \bar{M}(k-1:1,k)$$

$$+ \sum_{k=2}^{n+1} (-1)^{i+k}a_{i,j} \sum_{k=2, k\neq j}^{n+1} \varepsilon \det \bar{M}(k-1:j,k),$$

[6, (26)].
where \( \varepsilon = 1 \) or \(-1\) according as \( j < k \) or \( j > k \).

**Proof.** Let

\[
L_k = a_{i,1} \det \tilde{M}(k-1:1,k) - a_{i,2} \det \tilde{M}(k-1:2,k)
+ \cdots + (-1)^{k-2} a_{i,k-1} \det \tilde{M}(k-1:k-1,k)
+ (-1)^{k-1} a_{i,k+1} \det \tilde{M}(k-1:k+1,k)
+ \cdots + (-1)^n a_{i,n+1} \det \tilde{M}(k-1:n+1,k).
\]

Then the right-hand side of (5.5) is equal to \( \sum_{k=2}^{n+1} (-1)^{i-1} L_k \). Let \( M' \) be the \((n+1) \times (n+1)\) matrix obtained from \( M \) by adjoining a row which is identical to the \( i \)th row. Then it is evident that \((-1)^{n-1} L_k \) is the expansion of \( \tilde{M}'(k-1:k) \) by minors of the last row which is adjoined. Thus \( L_k = 0 \) if \( k \neq i+1 \) and \((-1)^{n-1} L_{i+1} = (-1)^i \det \tilde{M}(a_i+1) \). Therefore

\[
2(-1)^i L_k = \text{det} \tilde{M}(i+1), \quad \text{q.e.d.}
\]

6. **Proof of Theorem 4.1(1).** To simplify notation involved, throughout §§6–8, we will use \( x_j, y_k \) instead of \( x_{1,j}, x_{2,k} \); \( R_j, S_k \) instead of \( s_{1,j}, s_{2,k} \); \( A_j, B_k \) instead of \( v_{1,j}, v_{2,k} \); \( \xi, \eta \) instead of \( \eta_1, \eta_2 \); \( \alpha, \beta \) instead of \( \lambda_1, \lambda_2 \); and \( x, y \) instead of \( x_{1,i}, x_{2,j} \). Therefore \( F \) is the free group with free generators \( x_1, \ldots, x_n, y_1, \ldots, y_p \); \( F_2 \) is a free group with two free generators \( x, y \); \( A_2 \) is a free abelian group of rank two with a free basis \( \{x, y\} \); and \( \phi \) is a canonical homomorphism \( JF_2 \rightarrow JA_2 \). With these notations, we can write

\[
\mu([p+1,q+1]) = \left( \frac{\partial p+q+1}{\partial x^p, \partial y^q} \right) \phi.
\]

Further, to avoid unnecessary effort we introduce the following abbreviated symbols.

\[
X_i^{\phi_2}(p,q) = D^{p,q} \left( \frac{\partial x^{\phi_2}}{\partial x} \right),
Y_k^{\phi}(p,q) = D^{p,q} \left( \frac{\partial y^{\phi}}{\partial x} \right),
C'(p,q) = D^{p,q} C',
C'_2(p,q) = D^{p,q} \left( \frac{\partial C_i}{\partial z} \right),
H_z(p,q) = D^{p,q} \left( \frac{\partial \eta}{\partial z} \right),
\]

where \( C = A \) or \( B \) and \( z = x_j \) or \( y_k \).
Now the proof of Theorem 4.1 will be done as follows.

First we note that \((-1)^q \bar{\mu}(p+1, q+1)\) is the sum of \(\bar{\mu}(\omega 12)\), \(\omega \in S(p, q)\) (§5(5.4)). Thus from (5.2), it follows that \((-1)^q \bar{\mu}(p+1, q+1)\) can be represented as the sum of \(\sum H_\alpha(p, q)\) and a linear combination of \(H_\alpha(r, s)\) and \(H_\beta(r, s)\) with coefficients \(X_i\) and \(Y_k\) respectively. (§6, Lemma 6.1.) On the other hand,

\[
(1/p!) (1/q!) D^{p,q} \Delta(x, y)
\]
is the derivative of the determinant which is specified in §2. Then by using the inductive hypothesis, we will show in §7 that it is also represented as the sum of \(\sum_i H_\alpha(p, q)\) and a linear combination of \(H_\alpha(r, s)\) and \(H_\beta(r, s)\) with coefficients \(\Gamma_i\) and \(\Lambda_k\) respectively (§7 (7.3)). Hence, the final step of the proof of Theorem is to show the equalities between \(X\) and \(\Gamma\), and \(Y\) and \(\Lambda\). This will be done in §8.

Now the first lemma to be proved is

**Lemma 6.1** (8).

\[
(-1)^q \bar{\mu}(p+1, q+1) = \frac{1}{p!} \frac{1}{q!} \sum_{i=1}^{\alpha} H_i(p, q)^{q}
\]

\[+ \frac{1}{p!} \frac{1}{q!} \sum_{(0,0) \leq (r,s) < (p,q)} \left( \frac{p}{r} \right) \left( \frac{q}{s} \right) \left[ \sum_{k=2}^{\beta} H_k(r, s)X_k^q(p-r, q-s) + \sum_{k=2}^{\beta} H_k(r, s)Y_k^q(p-r, q-s) \mod \Delta^*(p+1, q+1) \right],
\]

where \(\theta = \theta_{p+q+1}\).

**Proof.** From (5.2), putting \(f = \partial \eta^\theta / \partial x\), and (5.4), it follows that

\[
(-1)^q \bar{\mu}(p+1, q+1) = \frac{1}{p!} \frac{1}{q!} \left[ D^{p,q} \left( \frac{\partial \eta^\theta}{\partial x} \right) \right]^q \mod \Delta^*(p+1, q+1).
\]

To obtain (6.1), we have only to apply the chain rule on \(\partial \eta^\theta / \partial x\) and the rule for differentiating product, noting that \(f^{\theta \phi} = f^\theta\) for any \(f \in JF\), q.e.d.

In the following lemma, the recursive formulas for \(X_i\) and \(Y_k\) will be given.

**Lemma 6.2.** For \(i, k \geq 1\), \(\lambda \geq 2\), \((p, q) \geq (0,0)\),

(6.2) (i) \([X_{i+1}^q(p, q)]^q = A^i(p, q)^q - p \sum_{j=1}^{\alpha} A_j^i(p-1, q)^q

- p \sum_{(0,1) \leq (r,s) \leq (p-1,q)} \left( \frac{p-1}{r} \right) \left( \frac{q}{s} \right) \left[ \sum_{j=1}^{\alpha-1} A_{j+1}(p-1-r, q-s)^q X_{j+1}(r, s)^q

+ \sum_{i=1}^{\beta-1} A_{i+1}(p-1-r, q-s)^q Y_{i+1}(r, s)^q \right].

(8) \((r, s) < (p, q)\) (or \((r, s) \leq (p, q))\) means that \(r \leq p\), \(s \leq q\) and \(r + s < p + q\)
(or \(r + s \leq p + q\)).
(6.2) (ii) \[ Y_{k+1}^q(p, q)^o = -q \sum_{j=1}^{n} B^k_x(p, q - 1)^o \]

\[-q \sum_{(0,1) \leq (r,s) \leq (p,q-1)} \binom{p}{r} \binom{q-1}{s} \left\{ \sum_{j=1}^{n-1} B^k_{xj+1}(p-r, q-1-s)^o X^q_{j+1}(r,s)^o \right. \]
\[+ \sum_{i=1}^{\beta-1} B^k_{yi+1}(p-r, q-1-s)^o Y^q_{i+1}(r,s)^o \} \]

All the proofs are straightforward, hence omitted.

7. Proof of Theorem 4.1 (II). In §2, the Alexander polynomial \( \Delta(x, y) \) is defined as the determinant \( N(R_2; y_i) \). Now let \( L \) be the determinant obtained from \( N(R_2; y_1) \) by adding the \( x_2, \ldots, x_\alpha \)-columns to \( x_1 \)-column. Since
\[ L = N(R_2; y_1), \]
we can write \( L = \Delta(x, y) \). Thus \( D^{p,q}\Delta(x, y) \) is obtained by differentiating each row of \( L \) w.r.t. \( x \) and/or \( y \) such that the total number of differentiations w.r.t. \( x \) and \( y \) are exactly \( p \) and \( q \) respectively.

Let us write (see (4))
\[ D_j(r,s) = [D^{r,s}\tilde{L}(S_p; x_j)]^o, \]
\[ E_k(r,s) = [D^{r,s}\tilde{L}(S_p; y_k)]^o. \]

Now by expanding \( D^{p,q}L \) by minors of the \( S_p \)-row, we have the following formula. (Note that \( D_1(0,0) = (-1)^{\alpha + \beta - 2} \))
\[ [D^{p,q}\Delta(x, y)]^o = \sum_{j=1}^{n} H_{x_j}(p,q)^o \]
\[+ (-1)^{\alpha + \beta - 2} \sum_{(0,0) \leq (r,s) \leq (p,q)} \binom{p}{r} \binom{q-1}{s} \left\{ \sum_{j=1}^{n} H_{x_j}(r,s)^o D_1(p-1, q-s) \right. \]
\[+ \sum_{l=2}^{\beta} (-1)^{l-1} H_{x_l}(r,s)^o D_1(p-r, q-s) \]
\[+ \sum_{k=2}^{\beta} (-1)^{s+k} H_{y_k}(r,s)^o E_k(p-r, q-s) \} \]

Since (4.1) is true for \((r,s) < (p,q)\) by the inductive hypothesis, it follows that
\[ \frac{1}{r!} \frac{1}{s!} [D^{r,s}\Delta(x, y)]^o \equiv (-1)^{r+s} r! \] \( \equiv 0 \mod \Delta^o([p+1,q+1]). \)

Thus \( \sum_{j=1}^{n} H_{x_j}(r,s)^o \) can be represented as a linear combination on \( H_{x_j}(t,u)^o \), \( H_{y_k}(t,u)^o \) and \( \sum_{j=1}^{n} H_{x_j}(t,u)^o \), where \( j, k \geq 2 \) and \((t,u) < (r,s)\). Hence, by using this fact repeatedly, we can conclude that \( \sum_{j=1}^{n} H_{x_j}(r,s)^o \) for \((r,s) < (p,q)\) may be
represented as a linear combination of $H_{x,t}(t,u)$ and $H_{y,k}(t,u)$, $j,k \geq 2$. (It should be noted that $\sum_{j=1}^{z} H_{x,j}(0,0) \equiv \mu(12) \equiv 0 \mod \Delta^*([p+1,q+1])$.) Therefore we obtain the following formula.

$$
\begin{align*}
\frac{1}{p!} \frac{1}{q!} [D^{p,q} \Delta(x,y)]^o & \equiv \frac{1}{p!} \frac{1}{q!} \sum_{j=1}^{z} H_{x}(p,q)^o \\
& \quad + \frac{1}{p!} \frac{1}{q!} \sum_{(0,0) \leq (r,s) \leq (p,q)} \binom{p}{r} \binom{q}{s} \left( \sum_{j=2}^{z} H_{x,j}(r,s)^o \Gamma_{j}^{p,q}(r,s) \right) \\
& \quad + \sum_{k=2}^{p} H_{y,k}(r,s)^o \Lambda_{k}^{p,q}(r,s)
\end{align*}
$$

(7.3)

mod $\Delta^*([p+1,q+1])$,

where $\Gamma_{j}^{p,q}(r,s)$ and $\Lambda_{k}^{p,q}(r,s)$ are the resulting coefficients of $H_{x,j}(r,s)^o$ and $H_{y,k}(r,s)^o$ respectively. In particular, we define $\Gamma_{j}^{p,q}(p,q) = 1$ and $\Lambda_{k}^{p,q}(p,q) = 0$.

The above process of obtaining (7.3) from (7.1) leads us to the following recursion formulas for $T_j$ and $A_k$. (The proofs are straightforward, hence omitted.)

**Lemma 7.1.** For $i,k \geq 2$, $(p,q) > (r,s) \geq (0,0)$,

(i) $\Gamma_i^{p,q}(r,s) = (-1)^{x+\beta+i-1} D_i(p-r,q-s)$

$$+ (-1)^{x+\beta-1} \sum_{(r,s) > (t,u) > (p,q)} \binom{p-r}{p-t} \binom{q-s}{q-u} \Gamma_i^{u}(r,s) D_i(p-t,q-u).$$

(7.4)

(ii) $\Lambda_k^{p,q}(r,s) = (-1)^{\beta+k} E_k(p-r,q-s)$

$$+ (-1)^{x+\beta-1} \sum_{(r,s) > (t,u) > (p,q)} \binom{p-r}{p-t} \binom{q-s}{q-u} \Lambda_k^{u}(r,s) D_i(p-t,q-u).$$

8. **Proof of Theorem 4.1 (Conclusion).** In this section all values will be considered mod $\Delta^*([p+1,q+1])$.

By comparing (6.1) and (7.3), we see that in order to prove Theorem 4.1 it is sufficient to show the following:

(8.1) For $i,k \geq 2$, $(p,q) \geq (r,s) > (0,0)$,

(i) $X_i^{p+q+1}(r,s)^o \equiv \Gamma_i^{p,q}(p-r,q-s),$

(ii) $Y_k^{p+q+1}(r,s)^o \equiv \Lambda_k^{p,q}(p-r,q-s).$

Now it is easy to check that (8.1) is true for $(p,q) < (1,1)$. Thus we may assume as the first inductive hypothesis that (8.1) is true for $(l,m) < (p,q)$. Further, since (8.1) is also true for $(r,s) < (1,1)$, we can assume as the second inductive hypothesis that (8.1) is true for $(t,u) < (r,s)$.
Now $X^q(r,s)^p, \quad \theta = \theta_{p+q+1}$, and $\Gamma^q_i(r,s)$ are finite sums of products of terms $A_{x_j+1}^{-1}(t,u)^q, A_{x_j+1}^i(t,u)^q, B_{x_j+1}^i(t,u)^q$ and $B_{x_j+1}^i(t,u)^q$. Let $X^q_i(r,s; \lambda)$ and $\Gamma^q_i(r,s; \lambda)$ denote the sums of “homogeneous” terms of degree $\lambda$. Similarly define $D_j(r,s; \lambda)$ and $E_k(r,s; \lambda)$. Then the proof of (8.1) is reduced to that of the following:

(8.2) For any $\lambda \geq 1$,

(i) $X^q_i(r,s; \lambda)^p \equiv \Gamma^q_i(p - r, q - s; \lambda),$

(ii) $Y^q_i(r,s; \lambda)^p \equiv \Lambda^p_i(r - q, q - s; \lambda)$.

(8.2) will be proved by induction on $\lambda$.

For the case $\lambda = 1$, (8.2)(i) is true, because

$$\Gamma^q_i(p - r, q - s; 1) = (-1)^{s+\beta+i-1}D_i(r, s; 1)$$

$$= A_{x_j}^{-1}(r, s) - r \sum_{j=1}^{\alpha} A_{x_j}^{-1}(r - 1, s)$$

$$= X^q_i(r, s; 1)^p.$$

Similarly (8.2)(ii) is true for $\lambda = 1$.

We may therefore assume as the third inductive hypothesis that (8.2) is true for any $\nu < \lambda$. In the following, we will prove only (8.2)(i), since the other can be proved in the same way.

Now from (7.4)(i), we observe that

$$\Gamma^q_i(p - r, q - s; \lambda)^p \equiv (-1)^{s+\beta+i-1}D_i(r, s; \lambda)$$

$$+ (-1)^{s+\beta+i-1} \sum_{(p-r,q-s) \leq \lambda} \sum_{(t,u) < (p,q)} \left( \begin{array}{c} r \\ p-t \end{array} \right) \left( \begin{array}{c} s \\ q-u \end{array} \right)$$

$$\cdot \sum_{\nu=1}^{\lambda-1} \Gamma^q_i(p - r, q - s; \nu)D_i(p - t, q - u; \lambda - \nu).$$

Since $D_j(t,u)^p = 0$ unless the $R_j$-row of the determinant $L$ is differentiated w.r.t. $x$ or $y$, we can prove by means of Lemma 5.5 that(7)

$$D_i(r, s; \lambda) = \sum_{(0,0) \leq \lambda} \sum_{(c,d) \leq (r,s)} (-1)^{i-1} \left( \begin{array}{c} r \\ c \end{array} \right) \left( \begin{array}{c} s \\ d \end{array} \right)$$

$$\cdot \sum_{j=1}^{\alpha} A_{x_j}^{-1}(c - 1, d) D_i(r - c, s - d; \lambda - 1)$$

$$+ (-c) \left\{ \sum_{j=1}^{\alpha} (-1)^{j+1} A_{x_j}^{-1}(c - 1, d) D_j(r - c, s - d; \lambda - 1)$$

$$+ \sum_{k=2}^{\beta} (-1)^{s+k} A_{y_k}^{-1}(c - 1, d) E_k(r - c, s - d; \lambda - 1) \right\}.$$
$E_k(a, b: \lambda - 1)$ $(k \geq 2)$ with coefficients $\delta_1(a, b: \lambda - v)$, $\delta_j(a, b: \lambda - 1)$ and $\varepsilon_k(a, b: \lambda - 1)$, respectively.

On the other hand, it follows from (6.1)(i) that

$$X^{\theta_{p^{r+s+1}}}(r, s: \lambda) \equiv (- r) \sum_{(0,0) \leq (c,d) \leq (r-1,s)} \binom{r-1}{c} \binom{s}{d} \cdot \left( \sum_{j=2}^{a} A^{l-1}_{x_{j}}(r-1-c, s-d) X^{\theta_{p^{+s}}}(c, d: \lambda - 1)^{p} \right.
+ \sum_{k=2}^{b} A^{l-1}_{y_{k}}(r-1-c, s-d) Y^{\theta_{p^{+s}}}(c, d: \lambda - 1)^{p}).$$

By the first inductive hypothesis, $X^{\theta_{p^{r+s}}}$ and $Y^{\theta_{p^{r+s}}}$ in the above formula may be replaced by $\Gamma_j^{p^{r-1-q}}$ and $\Lambda_k^{p^{r-1-q}}$ respectively. Further, since these $\Gamma$ and $\Lambda$ are linear combinations of $D_i$ and $E_k$ by Lemma 7.1, $X^{\theta_{p^{r+s+1}}}(r, s: \lambda)^{p}$ can be represented as a linear combination of $D_1(a, b: \lambda - v)$, $D_j(a, b: \lambda - 1)$ and $E_k(a, b: \lambda - 1)$ with coefficients $d_1(a, b: \lambda - v)$, $d_j(a, b: \lambda - 1)$ and $e_k(a, b: \lambda - 1)$ respectively. Thus the proof of (8.2)(i) is reduced to that of the following

$$\begin{align*}
\text{(i)} & \quad \delta_1(a, b: \lambda - v) \equiv d_1(a, b: \lambda - v), \\
\text{(ii)} & \quad \delta_j(a, b: \lambda - 1) \equiv d_j(a, b: \lambda - 1), \\
\text{(iii)} & \quad e_k(a, b: \lambda - 1) \equiv e_k(a, b: \lambda - 1).
\end{align*}$$

**Proof.** (i) The case $v = 1$.

$$\delta_1(a, b: \lambda - 1) = (-1)^{s+b+1} (-1)^{-1} \binom{r}{r-a} \binom{s}{s-b} \mathcal{A}^{i-1}(r-a, s-b)
+ (-1)^{s+b-1} \binom{r}{a} \binom{s}{b} \Gamma_i^{p-a\cdot q\cdot b}(p-r, q-s; 1) = 0.$$

On the other hand, $d_1(a, b: \lambda - 1) = 0$. The case $v \geq 2$.

$$\begin{align*}
d_1(a, b: \lambda - v) &= (- r) \sum_{(0,0) \leq (c,d) \leq (r-1,s)} \binom{r-1}{c} \binom{s}{d} \sum_{j=2}^{a} A^{l-1}_{x_{j}}(r-1-c, s-d)
\times (-1)^{s+b-1} \binom{c}{a} \binom{d}{b} \Gamma_j^{p-1-a\cdot q\cdot b}(p-1-c, q-d: v-1)
+ (- r) \sum_{(0,0) \leq (c,d) \leq (r-1,s)} \binom{r-1}{c} \binom{s}{d} \sum_{k=2}^{b} A^{l-1}_{y_{k}}(r-1-c, s-d)
\times (-1)^{s+b-1} \binom{c}{a} \binom{d}{b} \Lambda_k^{p-1-a\cdot q\cdot b}(p-1-c, q-d: v-1)
\end{align*}$$

(USING (8.2) AND THE SECOND INDUCTIVE HYPOTHESIS.)
\( \delta_1(a, b; \lambda - v). \)

(ii) \( \delta_j(a, b; \lambda - 1) \)

\[
= ( -1)^{r + \beta + 1} \begin{pmatrix} r & s \\ a & b \end{pmatrix} \Gamma_i^{p - a, q - b}(r - a, q - b; v)
\]

\[
= ( -1)^{r + \beta + 1} \begin{pmatrix} r & s \\ a & b \end{pmatrix} \Gamma_i^{a - p, b - q}(p - r, q - s; v)
\]

\[
= ( -1)^{r + \beta + 1} \begin{pmatrix} r & s \\ a & b \end{pmatrix} \Gamma_i^{a - p, b - q}(p - r, q - s; v)
\]

Thus the proof of (8.5), hence that of (8.2)(i) is completed.

The proof of Theorem 4.1 is thus completed.

9. Proof of Theorem 4.2. In this section, we assume that \( l \) is a link of multiplicity \( n \geq 2 \), and use the notation of §§2-3.

Let \( \mathcal{S} \) be the standard presentation of the group of \( l \) w.r.t. a Wirtinger presentation. Let \( M \) be the Jacobian matrix of \( \mathcal{S} \) at \( \psi \phi \) and let \( N \) be the matrix obtained from \( \tilde{M}(s_1, \ldots, x_{n, 1}) \) by adding the \( x_i, \ldots, x_{i, n, 1} \)-columns to \( x_i, \ldots, x_{i, n, 1} \)-column for \( 1 \leq i \leq n - 1 \). Interchange rows and columns of \( N \) to obtain the new matrix \( N' \) having the \( s_2, s_3, \ldots, s_n, s_{n, n} \)-rows and the \( x_1, \ldots, x_{n, 1} \)-columns in the top left corner. In other words, \( N' \) is of the form, \[ N_{ij} \], where

\[
N_{11} = \| \frac{\partial s_{i, j}}{\partial x_{k, 1}} \| (i \neq 1), \quad N_{12} = \| \frac{\partial s_{i, j}}{\partial x_{k, l}} \| (l \neq 1),
\]

\[
N_{21} = \| \frac{\partial s_{i, j}}{\partial x_{k, 1}} \| (j \neq \lambda), \quad N_{22} = \| \frac{\partial s_{i, j}}{\partial x_{k, l}} \| (l \neq 1).
\]

Then it is a straightforward matter to show the following

(i) Each of \((N_{11})^o, (N_{12})^o, (N_{21})^o\) is a zero matrix.

(ii) \( \det (N_{22})^o = ( -1)^{\lambda - n}, \) where \( \lambda = \Sigma_{i=1}^n \lambda_i. \)

Let \( L, L_{11} \) and \( L_{12} \) be matrices obtained from \( N', N_{11} \) and \( N_{12} \) by dividing the \( s_{n, n} \)-row by \( 1 - x_n \), respectively.

Consider a derivative \( D^p f(x_1, \ldots, x_n) \). It is obtained from \( \det L \) by differentiating each row w.r.t. \( x_1 \) in such a way that the total number of differentiations

\[(10) D^p f = \frac{d^p f}{dx_1^p}.\]
is exactly equal to \( p \). If the \( s_1, \ldots, s_i \)-row of \( \det L(2 \leq i \leq n - 1) \) is not differentiated, then \( [D^p \Delta(x_1, \ldots, x_n)]^\circ = 0 \). Therefore we obtain

\[
[D^p \Delta(x_1, \ldots, x_n)]^\circ = 0 \quad \text{for} \quad p < n - 2.
\]

Consider the case \( p \geq n - 2 \). It is easily verified that, for any \( q \), the nonzero elements of \( (D^q \det L_{11})^\circ \) occur only on the line just above the diagonal and in the last row, and that every element of the first column of \( (D^q \det N_{21})^\circ \) is zero. Thus we obtain that

\[
[D^p \Delta(x_1, \ldots, x_n)]^\circ = \sum_{q=0}^{p} \binom{p}{q} (D^q \det L_{11})^\circ (D^{p-q} \det N_{22})^\circ.
\]

Explicitly, \((D^q \det L_{11})^\circ\) is of the form

\[
(D^q \det L_{11})^\circ = (-1)^{q-2} \sum \frac{q!}{r_1! \cdots r_{n-1}!} (D^r \eta_2)^\circ \cdots (D^{n-r} \eta_n)^\circ \cdot \left( D^{r_{n-1}} \left( \sum_{i=1}^{r} \partial \eta_i / \partial x_i \right)^q \right)^\circ,
\]

where the summation runs over all sequences \( r_1, \ldots, r_{n-1} \) such that \( r_1 + \cdots + r_{n-1} = q \). Hence the proof of Theorem 4.2 will be completed if the following Lemma is proved.

**Lemma 9.1.**

(i) \( \frac{1}{p!} (D^p \det L_{11})^\circ \equiv \prod_{j=2}^{n-1} \mu([r_{j-1}]) \cdot \mu([r_{n-1} + 1]) \mod \bar{\Delta} \).

(ii) For \( 0 \leq r < q \),

\[
\frac{1}{r!} (D^r \det L_{11})^\circ \equiv 0 \mod \bar{\Delta},
\]

where \( \bar{\Delta} = \text{g.c.d.} \{\Delta^s([r_1]_2), \ldots, \Delta^s([r_{n-1} + 1])\} \).

**Proof.** (i) follows from (5.1), (5.2) and (5.3).

**Proof of (ii).** From (i), it follows that

\[
\frac{1}{r!} (D^r \det L_{11})^\circ \equiv \prod_{j=2}^{n-1} \mu([s_{j-1}]) \cdot \mu([s_{n-1} + 1]) \mod \bar{\Delta}.
\]

However, it vanishes \( \mod \bar{\Delta} \), because at least one of \( s_1, \ldots, s_{n-1}, s_r \), say, is less than \( r_1 \), as is seen from the fact that \( s_1 + \cdots + s_{n-1} < r_1 + \cdots + r_{n-1} \).

10. **Proof of Theorem 4.3.** Since the proof of Theorem 4.3 can be obtained in the same way as is used in §9, we omit the details. We have only to notice the following.
(10.1) The nonzero determinants in
\[ \left[ \frac{d^3}{dx_1 dx_2 dx_3} \Delta(x_1, x_2, x_3) \right] ^{\circ} \mod \Delta^8(123) \]
occurred only when the \( s_{2,1} \)-row is differentiated w.r.t. \( x_2 \), or \( x_1 \) and \( x_2 \), or \( x_2 \) and \( x_3 \).

**References**


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