REGULARITY CRITERIA FOR INTEGRAL AND MEROMORPHIC FUNCTIONS

BY

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1. Introduction. In this paper we shall consider functions $f(z)$ which are meromorphic in the plane (hereinafter called meromorphic). In particular we consider integral functions. Throughout the paper we shall assume familiarity with the standard notation of the Nevanlinna theory,

$$T(r) = T(r, f), \quad N(r, a), \quad m(r, a), \quad \delta(a, f) \quad \cdots$$

and with the first fundamental theorem (see e.g. [7]). We define

$$M(r) = M(r, f) = \max |f(z)| \quad (|z| = r),$$

$$\mu(r) = \mu(r, f) = \min |f(z)| \quad (|z| = r),$$

using $\mu(r)$ instead of $m(r)$ for the minimum modulus to avoid confusion with the schmiegungsfunktion $m(r, f)$. We shall assume that $f(z)$ is transcendental i.e. that

$$\log r = o(T(r)) \quad (r \to \infty)$$

and also that $f(0) = 1$. It is easily seen in the sequel that this involves no loss of generality.

If $f(z)$ is an integral function then for $r$ sufficiently large [7, p. 18]

$$T(r, f) \leq \log M(r, f) \leq \frac{R + r}{R - r} T(R, f) \quad (0 < r < R).$$

From this it is easily deduced that the order or type of $f(z)$ is the same whether it is defined by $T(r, f)$ or $\log M(r, f)$. We note in particular that

$$\lim \inf_{r \to \infty} \frac{T(r, f)}{r^\rho} > 0 \iff \lim \inf_{r \to \infty} \frac{\log M(r, f)}{r^\rho} > 0,$$

$$\lim \inf_{r \to \infty} \frac{T(r, f)}{r^\rho} < \infty \iff \lim \inf_{r \to \infty} \frac{\log M(r, f)}{r^\rho} < \infty.$$

If $f(z)$ is an integral function of order $\rho < 1$ then [13], [14]

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Since \( T(r,f) \geq N(r,0) + O(1) \) we conclude

\[
\limsup_{r \to \infty} \frac{T(r)}{\log M(r)} \leq \frac{\sin \pi \rho}{\pi \rho}.
\]

Consider the integral function \( F(z) \) with real negative zeros for which \((0 < \rho < 1)\)

\[
n(r,0) \sim Ar^\rho \quad (r \to \infty).
\]

Then, as is well known (see e.g. [2])

\[
N(r,0) \sim \frac{A}{\rho} r^\rho \quad (r \to \infty),
\]

\[
\log M(r,F) = \log F(r) \sim \frac{A\pi}{\sin \pi \rho} r^\rho \quad (r \to \infty),
\]

\[
T(r,F) = m(r,F) \sim \frac{1}{2\pi} \frac{A\pi}{\sin \pi \rho} r^\rho \int_{-\pi}^{\pi} (\cos \rho \theta)^+ d\theta,
\]

\[
= \frac{A}{\rho} r^\rho \quad (r \to \infty) \quad \left(0 < \rho \leq \frac{1}{2}\right)
\]

\[
= \frac{A}{\rho \sin \pi \rho} r^\rho \quad (r \to \infty) \quad \left(\frac{1}{2} < \rho < 1\right).
\]

Thus (1.4) is best possible for \(0 < \rho < 1\), and (1.5) is best possible for \(0 < \rho \leq \frac{1}{2}\).

2. The classical Wiman-Heins theory [8] and its extensions by Kjellberg [9], [10] lead one to expect that the integral functions which only just attain the growth demanded by (1.4) and (1.5) would have regular growth. We have the following theorems.

**Theorem 1.** Let \( f(z) \) be an integral function with \( f(0) = 1 \) such that for some \( \rho \), \( 0 < \rho \leq \frac{1}{2} \),

\[
\pi \rho T(r) \leq \sin \pi \rho \log M(r)
\]

for all \( r > 0 \). Then

\[
\beta = \liminf_{r \to \infty} \frac{T(r)}{r^\rho} > 0.
\]

If, further, \( \beta < \infty \), then

\[
T(r) \sim \beta r^\rho \quad (r \to \infty).
\]

**Theorem 2.** Let \( f(z) \) be an integral function with \( f(0) = 1 \) and such that for
some $\rho$, $0 < \rho < 1$

(2.1) \[ \pi \rho N(r, 0) \leq \sin \pi \rho \log M(r) \]

for all $r > 0$. Then

\[ \beta' = \lim \inf_{r \to \infty} \frac{\log M(r)}{r^p} > 0. \]

If, further, $\beta' < \infty$, then

\[ \log M(r) \sim \beta' r^p \quad (r \to \infty), \]

\[ N(r, 0) \sim \frac{\beta' \sin \pi \rho}{\pi \rho} \quad (r \to \infty). \]

It is of interest to state the following corollary, which is implicit in some recent work of Edrei [3] (see also [6]).

**Corollary 1.** Let $f(z)$ be an integral function of lower order $\lambda$, $0 < \lambda < 1$, then

\[ \lim \sup_{r \to \infty} \frac{T(r)}{\log M(r)} \geq \lim \sup_{r \to \infty} \frac{N(r, 0)}{\log M(r)} \geq \frac{\sin \pi \lambda}{\pi \lambda}. \]

**Proof.** The first inequality is immediate. To prove the second let $\rho$ be any number greater than $\lambda$. Then

\[ \lim \inf_{r \to \infty} \frac{\log M(r)}{r^p} = 0. \]

Thus by Theorem 2 there exists a sequence $\{r_n\}$ say, of values of $r$ tending to infinity, such that

\[ \pi \rho N(r_n, 0) > \sin \pi \rho \log M(r_n), \]

i.e.

\[ \lim \sup_{r \to \infty} \frac{N(r, 0)}{\log M(r)} \geq \frac{\sin \pi \rho}{\pi \rho}. \]

The result follows on letting $\rho \to \lambda$.

If $\rho = 1$ the condition (2.1) implies that $f(z)$ has 0 as a Picard (and a fortiori as a Borel) exceptional value. For such functions

\[ \log M(r) \sim a r^n \quad (r \to \infty) \]

for some $a > 0$ and positive integer $n$. Thus

\[ \beta = \lim \inf_{r \to \infty} \frac{\log M(r)}{r} > 0 \]

and if $\beta < \infty$ then
Theorem 1 can thus be considered as an extension of this result to fractional orders.

Theorem 1 is easily deduced from Theorem 2 as follows: Suppose that for \( r \geq r_0 \)
\[ \pi \rho T(r) \leq \sin \pi \rho \log M(r). \]
Then, by the first fundamental theorem, since \( f(0) = 1 \)
\[ (2.2) \quad \pi \rho N(r, 0) \leq \pi \rho T(r) \leq \sin \pi \rho \log M(r). \]
Thus by Theorem 2
\[ \beta' = \lim_{r \to \infty} \inf \frac{\log M(r)}{r^{\rho}} > 0 \]
and so by (1.2)
\[ \beta = \lim_{r \to \infty} \inf \frac{T(r)}{r^{\rho}} > 0. \]
Also if \( \beta < \infty \) then by (1.3) \( \beta' < \infty \). Hence by Theorem 2
\[ \sin \pi \rho \log M(r) \sim \beta' \sin \pi \rho r^{\rho} \quad (r \to \infty), \]
\[ \pi \rho N(r, 0) \sim \beta' \sin \pi \rho r^{\rho} \quad (r \to \infty) \]
and thus by the inequality (2.2)
\[ T(r) \sim \frac{\beta' \sin \pi \rho}{\pi \rho} \quad (r \to \infty) \]
as required.

3. For the case \( \rho > \frac{1}{2} \) it is a conjecture of Paley [12] that for an integral function of order \( \rho \)
\[ \lim_{r \to \infty} \sup \frac{T(r)}{\log M(r)} \geq \frac{1}{\pi \rho}. \]
The example of §1 shows that the result would be sharp for \( \frac{1}{2} < \rho < 1 \). The Mittag-Leffler functions [7, p. 19] show that it would be sharp for \( \rho > 1 \). This conjecture is unproved, though Gol’dberg, [5], has shown that it is true with the additional assumption that there exists a \( \theta \) for which
\[ \log |f(re^{i\theta})| \sim \log M(r) \quad (r \to \infty). \]
It is clear from our proof of Theorem 1 that it remains true for \( \frac{1}{2} < \rho < 1 \).
If the above conjecture of Paley is correct, however, the theorem would be vacuously true. Unfortunately the present results shed no light on the conjecture.

4. Theorem 1 can be extended, at any rate in part, to meromorphic functions. We have

**Theorem 3.** Let \( f(z) \) be meromorphic with \( f(0) = 1 \) and such that for some \( \rho, \) \( 0 < \rho < \frac{1}{2}, \)

\[
\pi \rho T(r) \leq \sin \pi \rho \log M(r) + \pi \rho \cos \pi \rho N(r, \infty)
\]

for all \( r > 0. \) Then

\[
\beta = \lim \inf_{r \to \infty} \frac{T(r)}{r^\rho} > 0.
\]

If, further, \( \beta < \infty \) then

\[
\alpha = \lim \sup_{r \to \infty} \frac{T(r)}{r^\rho} < \infty.
\]

Theorem 3 is an immediate corollary of the following theorem.

**Theorem 4.** Let \( f(z) \) be meromorphic in the plane and such that for some \( \rho, \) \( 0 < \rho < 1, \) either

(4.1) \[
\pi \rho N(r, 0) \leq \sin \pi \rho \log M(r) + \pi \rho \cos \pi \rho N(r, \infty)
\]

or

(4.2) \[
\sin \pi \rho \log M(r) \leq \pi \rho \cos \pi \rho N(r, 0) - \pi \rho N(r, \infty)
\]

for all \( r > 0. \) Then

\[
\beta = \lim \inf_{r \to \infty} \frac{T(r)}{r^\rho} > 0.
\]

If, further, \( \beta < \infty \) then

\[
\alpha = \lim \sup_{r \to \infty} \frac{T(r)}{r^\rho} < \infty.
\]

**Remarks 1.** Condition (4.2) is just condition (4.1) applied to \( F(z) = (f(z))^{-1}, \) and so it suffices just to consider (4.1).

2. The inequality (4.2) and its conclusion have been used by Ostrovskii [11] to show that for a meromorphic function of lower order \( \lambda < \frac{1}{2}, \)

\[
\lim_{r \to \infty} \sup \frac{\log^+ \mu(r, f)}{T(r)} \geq \pi \lambda (\cosec \pi \lambda) (\cos \pi \lambda - 1 + \delta(\infty)).
\]

The result is sharp. An example to show this is easily constructed with the method of [7, p. 117].
It is an open question whether under the hypotheses of Theorems 3 and 4 we can conclude that $\alpha = \beta$, i.e., that $f(z)$ has perfectly regular growth in the sense of Valiron.

In the proof of Theorem 4 we can prove the following slightly more general theorem:

**Theorem 5.** Let $f(z)$ be meromorphic and such that, for some $\rho$, $0 < \rho < 1$, given $\varepsilon > 0$

$$
(4.3) \quad \int_{r_1}^{r_2} (\pi \rho N(r, 0) - \sin \pi \rho \log M(r) - \pi \rho \cos \pi \rho N(r, \infty)) \frac{dr}{r^{1+\rho}} < \varepsilon,
$$

for all $r_2 > r_1 > r(\varepsilon)$ or

$$
(4.4) \quad \pi \rho N(r, 0) \leq \sin \pi \rho \log M(r) + \pi \rho \cos \pi \rho N(r, \infty) + O(\log r) \quad (r \to \infty).
$$

If $\beta < \infty$ then $\alpha < \infty$ and if (4.4) holds $\beta > 0$.

5. The proof of Theorem 4 uses results similar to those in [6]. We also use the techniques developed by Kjellberg.

**Lemma 1.** Let

$$
F(z) = \frac{F_1(z)}{F_2(z)} = \prod_{n=1}^{\infty} \left( 1 + \frac{z}{a_n} \right) / \prod_{m=1}^{\infty} \left( 1 - \frac{z}{b_m} \right), \quad a_n > 0, \ b_m > 0,
$$

be meromorphic and of order less than one. Then there exist constants $K$, $k$, depending only on $F(z)$ satisfying $0 < k < K < \infty$, such that for any $r_2 > r_1 > 0$, $0 < \rho < 1$,

$$
\int_{r_1}^{r_2} (\pi \rho N(r, 0) - \sin \pi \rho \log |F(r)| - \pi \rho \cos \pi \rho N(r, \infty)) \frac{dr}{r^{1+\rho}}
$$

$$
> k \frac{T(r_1, F)}{r_1^\rho} - K \frac{T(2r_2, F)}{r_2^\rho}.
$$

**Proof.** Let $C$ be the contour consisting of the line segments $r_1 \leq t \leq r_2$, $-r_2 \leq t \leq -r_1$ and the semicircles $|z| = r_1$, $0 < \arg z < \pi$, and $|z| = r_2$, $0 < \arg z < \pi$, with indentations, of radius $\delta$ say, around the zeros and poles of $F(z)$. Consider

$$
\int_C \frac{\log F(z)}{z^{1+\rho}} \, dz.
$$

We consider that branch of $z^{1+\rho}$ which is real for $z > 0$ and that branch of $\log F(z)$ for which $\log F_1(z)$ is real for $z > 0$ and $\log F_2(z)$ real for $z < 0$. Since $F(z)$ is meromorphic $\log F(z)$ has only logarithmic singularities at the zeros and poles.
of \( F(z) \). The contribution to the integrand along any indentation is therefore 
\( O(\delta \log 1/\delta) \) as \( \delta \to 0 \). Since \( F(z) \) is analytic inside the contour we obtain on letting \( \delta \to 0 \),

\[
0 = \int_{r_1}^{r_2} \left( \log F(r) - e^{-ip\theta} \log F(-r) \right) \frac{dr}{r^{1+p}} 
+ \frac{i}{r_2^2} \int_0^{\pi} e^{-ip\theta} \log F(r_2 e^{i\theta}) \, d\theta 
- \frac{i}{r_1^2} \int_0^{\pi} e^{-ip\theta} \log F(r_1 e^{i\theta}) \, d\theta.
\]

On multiplying through by \( e^{ip\theta} \) and taking real parts we obtain,

\[
\int_{r_1}^{r_2} \left( \pi n(r, 0) - \sin \pi \rho \log |F(r)| - \pi \cos \pi \rho \pi(r, \infty) \right) \frac{dr}{r^{1+p}} 
= r_1^{-\rho} P(r_1) - r_2^{-\rho} P(r_2)
\]

where

\[
P(r) = - \int_0^{\pi} \left( \cos \rho (\pi - \theta) \log |F(r e^{i\theta})| - \sin \rho (\pi - \theta) \arg F(r e^{i\theta}) \right) \, d\theta.
\]

Now

\[
\int_{r_1}^{r_2} \frac{n(r, 0) \, dr}{r^{1+p}} = r_2^{-\rho} N(r_2, 0) - r_1^{-\rho} N(r_1, 0) + \rho \int_{r_1}^{r_2} \frac{N(r, 0) \, dr}{r^{1+p}}.
\]

Thus we obtain

\[
\int_{r_1}^{r_2} \left( \pi \rho N(r, 0) - \sin \pi \rho \log |F(r)| - \pi \rho \cos \pi \rho N(r, \infty) \right) \frac{dr}{r^{1+p}} 
= r_1^{-\rho} Q(r_1) - r_2^{-\rho} Q(r_2)
\]

(5.1)

where

\[
Q(r) = \pi N(r, 0) - \pi \cos \pi \rho N(r, \infty) + P(r).
\]

An application of Jensen’s theorem [7, formula (1.5)] yields

\[
Q(r) = \pi (1 - \cos \pi \rho) N(r, \infty)
+ \int_0^{\pi} (1 - \cos \rho (\pi - \theta)) \log |F(r e^{i\theta})| \, d\theta,
+ \sin \rho (\pi - \theta) \arg F(r e^{i\theta}) \, d\theta,
\]

since \( F(z) \) is symmetric with respect to the real axis. Now

\[
\pi (1 - \cos \pi \rho) N(r, \infty) < 2\pi T(r),
\]
\[
\int_0^\pi (1 - \cos \rho (\pi - \theta)) \log |F(re^{i\theta})| < 2\pi T(r),
\]
\[
\int_0^\pi \sin \rho (\pi - \theta) \arg F(re^{i\theta}) \, d\theta < \left[ \pi n(r,0) + \pi n(r, \infty) \right] \int_0^\pi \sin \rho (\pi - \theta) \, d\theta
\]
\[
< \pi^2 [n(r,0) + n(r, \infty)].
\]

Now

\[n(r,0) \log 2 = n(r,0) \int_r^{2r} \frac{dt}{t} < \int_r^{2r} \frac{n(t,0) \, dt}{t} < N(2r,0) < T(2r)\]

Similarly for \(n(r, \infty)\), and so we obtain

\[Q(r) < 4\pi T(r) + \frac{2\pi^2}{\log_2} T(2r) < KT(2r)\]

The left-hand inequality is not so immediate but it follows from the fact that \(\psi(\theta) = 1 - \cos \rho (\pi - \theta)\) is a decreasing function of \(\theta\) for \(0 < \theta < \pi\) and that

\[
m(r,F) = \frac{1}{\pi} \int_0^\gamma \log |F(re^{i\theta})| \, d\theta,
\]

\[
m\left(r, \frac{1}{F}\right) = \frac{1}{\pi} \int_0^\alpha \log |F(re^{i\theta})| \, d\theta
\]

for some \(\gamma = \gamma(r)\) satisfying \(0 \leq \gamma \leq \pi\).

We prove the inequality only in the case \(0 \leq \gamma \leq \pi/3\), the cases when \(\pi/3 < \gamma \leq 2\pi/3\) and \(2\pi/3 < \gamma \leq \pi\) being similar. Since

\[
\sin \rho (\pi - \theta) \arg F(re^{i\theta}) \geq 0
\]

for \(0 \leq \theta \leq \pi\) we have

\[
Q(r) \geq \pi (1 - \cos \pi \rho) N(r, \infty) + \int_0^\pi (1 - \cos \rho (\pi - \theta)) \log |F(re^{i\theta})| \, d\theta
\]

\[
\geq \pi (1 - \cos \pi \rho) N(r, \infty) + \pi (1 - \cos \rho (\pi - \gamma)) m(r, \infty)
\]

\[
+ \int_\gamma^\pi (1 - \cos \rho (\pi - \theta)) \log |F(re^{i\theta})| \, d\theta.
\]

Now \(\log |F(re^{i\theta})|\) is a decreasing function of \(\theta\) and is less than zero for \(\gamma < \theta \leq \pi\). Thus

\[\int_0^\gamma \log |F(re^{i\theta})| \, d\theta < \frac{1}{3} \int_\gamma^\pi \log |F(re^{i\theta})| \, d\theta = -\frac{\pi}{3} m(r,0).
\]

Hence,
\[
Q(r) \geq \pi(1 - \cos \rho(\pi - \gamma))T(r, F) \\
+ \left\{ \int_{\pi}^{2\pi/3} + \int_{2\pi/3}^{\pi} \right\} (1 - \cos \rho(\pi - \theta)) \log |F(re^{i\theta})| d\theta \\
\geq \pi(1 - \cos \rho(\pi - \gamma))T(r, F) + (1 - \cos \rho(\pi - \gamma)) \int_{\pi}^{2\pi/3} \log |F(re^{i\theta})| d\theta \\
+ \left( 1 - \cos \frac{\pi \rho}{3} \right) \int_{2\pi/3}^{\pi} \log |F(re^{i\theta})| d\theta \\
= \pi(1 - \cos \rho(\pi - \gamma))T(r, F) - \pi(1 - \cos \rho(\pi - \gamma))m(r, 0) \\
+ \left( \cos \rho(\pi - \gamma) - \cos \frac{\pi \rho}{3} \right) \int_{2\pi/3}^{\pi} \log |F(re^{i\theta})| d\theta \\
> \pi(1 - \cos \rho(\pi - \gamma))T(r, F) - \pi(1 - \cos \rho(\pi - \gamma))m(r, 0) \\
+ \frac{\pi}{3} \left( \cos \frac{\pi \rho}{3} - \cos \rho(\pi - \gamma) \right)m(r, 0)
\]

by (5.4). Therefore by the first fundamental theorem

\[
Q(r) \geq \frac{\pi}{3} \left[ \cos \frac{\pi \rho}{3} - \cos \rho(\pi - \gamma) \right] T(r, F) \\
\geq \frac{\pi}{3} \left( \cos \frac{\pi \rho}{3} - \cos \frac{2\pi \rho}{3} \right) T(r, F)
\]

since \(0 \leq \gamma < \pi/3\). This completes the proof of the lemma.

6. Proof of Theorem 4. If \(f(z)\) has only finitely many zeros and poles then, since we are assuming that \(f(z)\) is transcendental

\[
f(z) = \frac{P_1(z)}{P_2(z)} \exp \phi(z),
\]

where \(P_1, P_2\) are polynomials and \(\phi(z)\) is an integral function. From this we conclude that \(f(z)\) has lower order at least 1 and so

\[
\beta = \lim \inf_{r \to \infty} \frac{T(r)}{r^\rho} = \infty
\]

for any \(\rho, 0 < \rho < 1\), and so the theorem is proved.

Now, following Kjellberg we choose \(R\) sufficiently large so that \(f(z)\) has \(N\) zeros and \(M\) poles in \(|z| < R\) where \(\max(M, N) > 0, R\) being suitably chosen later. We denote zeros by \(a_n\) and poles by \(b_m\). Let

\[
f_1(z) = \prod_{n=1}^{N} \left( 1 - \frac{z}{a_n} \right) \prod_{m=1}^{M} \left( 1 - \frac{z}{b_m} \right),
\]
and define \( f_3(z) \) by

\[
f(z) = f_1(z) f_3(z).
\]

Then for \( 0 < r < \frac{1}{2} R \), [3, Lemma A]

\[
\log |f_3(re^{i\theta})| < \frac{14T(2R)}{R} r.
\]

Now

\[
T(R, f_2) \leq N(R, 0) + N(R, \infty) + \sum_{n=1}^{N} \log \left( 1 + \frac{R}{|a_n|} \right) + \sum_{m=1}^{M} \log \left( 1 + \frac{R}{|b_m|} \right)
\]

and

\[
\sum_{n=1}^{N} \log \left( 1 + \frac{R}{|a_n|} \right) = \int_{0}^{R} \log \left( 1 + \frac{r}{t} \right) dn(t, 0)
\]

\[
< n(R, 0) \log 2 + \int_{0}^{R} \frac{R}{R + t} \frac{n(t, 0) dt}{t}
\]

\[
< T(2R, f) + N(R, 0) \quad \text{by (5.2)}
\]

\[
< 2T(2R, f).
\]

Thus from (6.3) we obtain

\[
T(R, f_2) \leq 6T(2R, f).
\]

Also, by a result of Edrei [3, formula 8.4], we have for \( r \leq \frac{1}{2} R \)

\[
T(r, f) \leq T(r, f_2) + \frac{14r}{R} T(2R, f).
\]

We now apply Lemma 1 to \( f_2(z) \), which satisfies the hypotheses, to obtain, for any \( r_1, r_2, 0 < r_1 < r_2 < R \),

\[
\int_{r_1}^{r_2} (\pi \rho N(r, 0) - \sin \pi \rho \log |f_2(r)| - \pi \rho \cos \pi \rho N(r, \infty)) \frac{dr}{r^{1+\rho}}
\]

\[
> k \frac{T(r_1, f_2)}{r_1^2} - K \frac{T(2r_2, f_2)}{r_2^2},
\]

where \( k, K \) depend only on \( f_2 \), i.e. on \( f \). Thus by our hypothesis (4.1)

\[
0 \geq \sin \pi \rho \int_{r_1}^{r_2} (\log |f_2(r)| - \log M(r, f)) \frac{dr}{r^{1+\rho}}
\]

\[
+ k \frac{T(r_1, f_2)}{r_1^2} - K \frac{T(2r_2, f_2)}{r_2^2}.
\]
But from (6.1),

\[ \log M(r, f) \leq \log M(r, f_1) + \log M(r, f_2) \leq \log |f_2(r)| + \log M(r, f_3) \]

i.e.

\[ \log |f_2(r)| - \log M(r, f) \geq - \log M(r, f_3). \]

Thus for \( r_2 \leq \frac{1}{2} R \) we obtain by (6.2),

\[
\sin \pi \rho \int_{r_1}^{r_2} \left( \log |f_2(r)| - \log M(r, f) \right) \frac{dr}{r^{1+\rho}} > -14 \sin \pi \rho \frac{T(2R)}{R} \int_{r_1}^{r_2} \frac{dr}{r^\rho}
\]

\[
= -14 \sin \pi \rho \frac{T(2R)}{R} \left\{ r_2^{1-\rho} - r_1^{1-\rho} \right\}
\]

\[
-14 \sin \pi \rho \frac{T(2R)}{R} r_2^{1-\rho}.
\]

If we now choose \( r_2 = \frac{1}{2} R \) we obtain from (6.6)

\[
0 \geq k \frac{T(r_1, f_2)}{r_1^\rho} - 2^\rho K \frac{T(R, f_2)}{R^\rho} - 2^\rho - 1 \cdot 14 \sin \pi \rho \frac{T(2R, f)}{R^\rho}.
\]

We now use the estimates (6.4) and (6.5) to obtain

\[
0 \geq k \frac{T(r_1, f)}{r_1^\rho} - 14 \left( \frac{r_1}{R} \right)^{1-\rho} \frac{T(2R, f)}{R^\rho} - \left( 6.2^\rho K + \frac{14.2^\rho - 1}{1-\rho} \right) \frac{T(2R, f)}{R^\rho}.
\]

Finally since \( r_1 < \frac{1}{2} R \) we have, for some suitable constant \( K_1 > 0 \).

\[
(6.7) \quad 0 \geq k \frac{T(r_1, f)}{r_1^\rho} - K_1 \frac{T(2R, f)}{(2R)}.
\]

This holds for all \( R \) and all \( r_1 < \frac{1}{2} R \).

7. Theorem 4 now follows from (6.7). Suppose that \( r_1 \) is fixed, then by the definition of \( \beta \) there exist arbitrarily large values of \( R \) such that for any \( \varepsilon > 0 \)

\[ T(2R, f) < (\beta + \varepsilon)(2R)^\rho. \]

Thus for any \( \varepsilon > 0 \)

\[
(7.1) \quad k \frac{T(r_1, f)}{r_1^\rho} \leq K_1(\beta + \varepsilon).
\]

The left-hand side of this inequality is a fixed positive number and so if \( \beta = 0 \) this gives a contradiction. Thus

\[ \beta = \lim \inf_{r \to \infty} \frac{T(r)}{r^\rho} > 0. \]

Also (7.1) holds for any \( r_1 \). Thus if \( \beta < \infty \) we obtain
\[
\alpha = \limsup_{r \to \infty} \frac{T(r)}{r^\rho} \leq \frac{K_1 \beta}{k} < \infty.
\]

This completes the proof of Theorem 4.

We note that for a given \( f(z) \) explicit values of \( K_1 \) and \( k \) could be calculated. This, however, sheds no light on the more interesting question of whether or not \( \alpha = \beta \) under the hypotheses of Theorem 4.

If instead of (4.1) we use the assertions (4.3) and (4.4) we obtain instead of (6.7) the assertions

\[
\nu \geq k \frac{T(r_1, f)}{r_1^\rho} - K_1 \frac{T(2R, f)}{(2R)^\rho}, \quad r_1 > r(\delta)
\]

and

\[
\frac{K_2 \log r_1}{r_1^\rho} \geq k \frac{T(r_1, f)}{r_1^\rho} - K_1 \frac{T(2R, f)}{(2R)^\rho}
\]

respectively for some \( K_2 < \infty \). The conclusions then follow as before since we are assuming that \( f(z) \) is transcendental. Thus Theorem 5 is proved.

8. It remains to prove Theorem 2. By Theorem 4 and (1.1) we have \( \beta' > 0 \), and if \( \beta' < \infty \), then

\[
(8.1) \quad \alpha' = \limsup_{r \to \infty} \frac{\log M(r)}{r^\rho} < \infty
\]

and we have to show that \( \alpha = \beta \). Now \( f(z) \) has genus zero by (8.1). Let

\[
\begin{align*}
    f(z) &= \prod_{n=1}^{\infty} \left( 1 - \frac{z}{a_n} \right), \\
    F(z) &= \prod_{n=1}^{\infty} \left( 1 + \frac{z}{|a_n|} \right).
\end{align*}
\]

Then, [8], p. 204, we have

\[
(8.2) \quad \log M(r, f) \leq \log F(r),
\]

Hence by (2.1)

\[
(8.3) \quad N(r, 0) \leq \frac{\sin \pi \rho}{\pi \rho} \log F(r).
\]

Also from (8.1)

\[
\alpha_1 = \limsup_{r \to \infty} \frac{\log F(r)}{r^\rho} < \infty.
\]

Now \( F(z) \) satisfies the hypotheses of Lemma 1, and so, with the notation of that lemma
\( (8.4) \quad \int_{r_1}^{r_2} (\pi \rho N(r,0) - \sin \pi \rho \log F(r)) \frac{dr}{r^{1+\rho}} = \frac{Q(r_1)}{r_1^\rho} - \frac{Q(r_2)}{r_2^\rho}. \)

Now by (5.2),

\[ Q(r) < KT(2r) < K \log F(2r) \]

and so from (8.4)

\[ \alpha'' = \limsup_{r \to \infty} \frac{Q(r)}{r^\rho} < \infty. \]

By (8.3) we obtain

\[ \frac{Q(r_1)}{r_1^\rho} \leq \frac{Q(r_2)}{r_2^\rho} \]

for all \( r_2 > r_1 > 0 \). Now let \( r_2 \to \infty \) through a suitable sequence of values so that \( r_2^{-\rho} Q(r_2) \) tends to its lower limit \( \beta'' \) say. We obtain \( \alpha'' \leq \beta'' \) and so \( Q(r) \sim \alpha'' r^{\rho}(r \to \infty) \).

Applying this to (8.5) we see that given any \( \varepsilon > 0 \) there exists an \( r(\varepsilon) \) such that for \( r_2 > r_1 > r(\varepsilon) \),

\[ \varepsilon > \int_{r_1}^{r_2} (\pi \rho N(r,0) - \sin \pi \rho \log F(r)) \frac{dr}{r^{1+\rho}} > -\varepsilon. \]

By our hypothesis (2.1) and (8.2)

\[ -\varepsilon < \int_{r_1}^{r_2} (\pi \rho N(r,0) - \sin \pi \rho \log M(r,f)) \frac{dr}{r^{1+\rho}} \leq 0 \]

and so we obtain by subtraction that

\[ 0 \geq \sin \pi \rho \int_{r_1}^{r_2} (\log F(r) - \log M(r,f)) \frac{dr}{r^{1+\rho}} \geq -2\varepsilon. \]

Thus the integral

\[ \int_0^\infty (\log F(r) - \log M(r,f)) \frac{dr}{r^{1+\rho}} \]

exists and is finite. Now,

\[ \log F(r) = r \int_0^\infty \frac{n(t,0)dt}{t(t+r)} = r \int_0^\infty \frac{N(t,0)dt}{(t+r)^2} \]

and so we may write (8.3) as

\[ r^{-\rho} N(r,0) \leq \frac{\sin \pi \rho}{\pi \rho} r^{1-\rho} \int_0^\infty \frac{N(t,0)}{t^\rho} \frac{t^\rho dt}{(t+r)^2} \]

\[ = \frac{\sin \pi \rho}{\pi \rho} \int_0^\infty \frac{N(t,0)}{t^\rho} \frac{(t/r)^{1+\rho}}{((t/r) + 1)^2} \frac{dt}{t}. \]
We let $r = e^s$, $t = e^u$ and let $\psi(s) = r^{-p}N(r, 0)$. Thus (8.3) becomes

$$\psi(s) \leq \frac{\sin \pi \rho}{\pi \rho} \int_{-\infty}^{\infty} \psi(u) \frac{(e^{u-p})^{1+p}}{(e^u + 1)^2}$$

i.e.

$$(8.5) \quad \psi(s) \leq \int_{-\infty}^{\infty} \psi(u)K(u - s) \, du,$$  

where

$$K(x) = \frac{\sin \pi \rho}{\pi \rho} e^{x(1+\rho)}(e^x + 1)^{-2}.$$  

Convolution inequalities like (8.5) have been studied by Essén [4]. He has the following lemma.

**Lemma 2.** Let $\psi(s)$ be bounded and slowly decreasing, i.e.

$$\lim \inf_{x \to \infty} \lim \inf_{y \to x; y > x} |\psi(y) - \psi(x)| = 0.$$  

If $K(x) \in L(-\infty, \infty)$ and satisfies

$$\int_{-\infty}^{\infty} K(x) \, dx = 1,$$  

$$\int_{-\infty}^{\infty} |x| K(x) \, dx < \infty,$$  

$$\int_{-\infty}^{\infty} x K(x) \, dx = m \neq 0,$$

then the inequality (8.5) implies that $\lim_{x \to \infty} \psi(x)$ exists.

If we apply the lemma to $\psi(s) = r^{-p}N(r, 0)$ as above, we obtain

$$N(r, 0) \sim lr^p \quad (r \to \infty)$$  

for some $l$. We must show that $\psi$ and $K$ satisfy the hypothesis of the lemma.

It is easy to verify that

$$\int_{-\infty}^{\infty} \frac{e^{x(1+\rho)}}{(e^x + 1)^2} \, dx = \pi \rho (\csc \pi \rho).$$  

Differentiating with respect to $\rho$ we obtain ($0 < \rho < 1$)

$$\int_{-\infty}^{\infty} \frac{xe^{x(1+\rho)}}{(e^x + 1)^2} \, dx = \pi (\sin \pi \rho - \pi \rho \cos \pi \rho) \csc^2 \pi \rho \neq 0.$$
The other condition on $K$ is also clearly satisfied. It follows from (8.1) that $\psi(s)$ is bounded. Also

$$\psi(y) - \psi(x) = \frac{N(r_2)}{r_2^p} - \frac{N(r_1)}{r_1^p} > \frac{N(r_1)}{r_1^p} \left[ \left( \frac{r_1}{r_2} \right)^p - 1 \right].$$

As $r_2/r_1 \to 1$ and $r \to \infty$ the above expression tends to zero, and so $\psi$ is slowly decreasing. Thus

$$N(r, 0) \sim lr^p \quad (r \to \infty)$$

and elementary Tauberian and Abelian arguments enable us to conclude that

$$n(r, 0) \sim lp^p \quad (r \to \infty),$$
$$\log F(r) \sim \pi lp (\csc \pi p)r^p \quad (r \to \infty).$$

But from (8.4) we conclude by a well-known argument [1, §4] that

$$\lim_{r \to \infty} (\log F(r) - \log M(r, f)) = 0$$

as $r \to \infty$ outside an open set, $E$ say, of finite logarithmic length. Thus for $r \in E$

$$\log M(r, f) \sim \pi lp (\csc \pi p)r^p \quad (r \to \infty).$$

But if $r \in E$ there exists $r_1, r_2 \notin E$ with $r_1 < r < r_2$ and such that $\log(r_2/r_1) \to 0$ ($r_1 \to \infty$). Now $\log M(r, f)$ is a monotonic increasing function of $r$. Thus given $\varepsilon > 0$ we have, for $r$ sufficiently large,

$$\log M(r, f) > \log M(r_1, f) \sim \pi lp (\csc \pi p)r_1^p > \pi lp (\csc \pi p)(1 - \varepsilon)r^p$$

and

$$\log M(r, f) < \log M(r_2, f) \sim \pi lp (\csc \pi p)r_2^p < \pi lp (\csc \pi p)(1 + \varepsilon)r^p.$$ 

Hence

$$\log M(r, f) \sim \pi lp (\csc \pi p)r^p \quad (r \to \infty).$$

Thus, by the definition of $\beta'$,

$$\log M(r, f) \sim \beta'r^p \quad (r \to \infty),$$
$$N(r, 0) \sim \frac{\beta' \sin \pi p}{\pi p}r^p \quad (r \to \infty),$$

which is the required result.

REFERENCES


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