LIMIT THEOREMS FOR MARKOV PROCESSES ON TOPOLOGICAL GROUPS(1)

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Summary. Limit theorems for $P^n(x, A)$, as $n \to \infty$, are established, where $P(x, A)$ is the transition probability of a Markov process on a topological group. The transition probability is assumed to satisfy certain commutativity relations with translations. Thus special cases of our investigation are spatially homogenous processes and processes induced by automorphisms of the group.

1. **Introduction.** Let G be a locally compact abelian group, Γ the dual group and m the Haar measure on G. Let $P(x,A)$ be the transition probability of a Markov process on G. Thus

1.1. *For every $x \in G$ the set function $P(x,\cdot)$, is a probability measure on the collection of Borel sets, Σ.*

1.2. *For every $A \in \Sigma$ the function $P(\cdot, A)$ is Σ measurable.*

The transition probability induces an operator that acts on bounded measurable functions and on measures by

$$(Pf)(x) = \int f(y)P(x,dy), \tag{1.3}$$

$$(\mu P)(A) = \int P(x,A)\mu(dx), \tag{1.4}$$

where the duality relation holds:

$$\int (Pf)(x)\mu(dx) = \int f(x)(\mu P)(dx). \tag{1.5}$$

If P is given by a transformation ϕ of G:

$$P(x,A) = I(\phi^{-1}(A))(x) \tag{1.6}$$

where $I(B)$ denotes the characteristic function of B. Then

$$(Pf)(x) = f(\phi(x)), \tag{1.7}$$

$$(\mu P)(A) = \mu(\phi^{-1}(A)). \tag{1.8}$$

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Let T_y denote the translation operator:

$$(T_y f)(x) = f(x+y). \tag{1.9}$$

Throughout the paper we shall assume:

CONDITION A. *For every $y \in G$ there exists a $z = z(y)$ and for every $z \in G$ there exists a $y = y(z)$ such that $PT_y = T_z P$.*

Let us introduce here two examples:

EXAMPLE 1. Let P be spatially homogeneus i.e. $P(x, A) = P(0, A - x)$. Then

$$(T_y Pf)(x) = \int P(x+y, dz) f(z) = \int P(x, dz - y) f(z) = \int P(x, dz) f(z+y)$$
$$= (PT_y f)(x).$$

Thus $z(y) = y$ and $y(z) = z$.

EXAMPLE 2. Let $\phi(x) = \psi(x) + a$ where ψ is an automorphism of G and a a fixed element. Then

$$(PT_y f)(x) = f(\psi(x) + y + a),$$
$$(T_z Pf)(x) = f(\psi(x) + \psi(z) + a);$$

thus $z = z(y)$ is the solution of $\psi(z) = y$.

From Condition A and the invariance of m under translations follows that $mPT_y = mP$. Therefore mP satisfies the main condition for a Haar measure. Now mP does not have to be regular or σ finite. However, we shall assume:

CONDITION B. $mP = m$.

Let us consider this condition for our two examples:

If $P(x, A) = \int_A p(x, \xi) m(d\xi)$ in Example 1 then $p(x, \xi) = p(0, \xi - x)$ and

$$(mP)(A) = \int \int_A p(x, \xi) m(d\xi) m(dx) = \int_A \int p(x, \xi) m(dx) m(d\xi)$$
$$= \int_A \left(\int p(0, \xi - x) m(dx) \right) m(d\xi) = \int_A \left(\int P(0, x) m(dx)\, m(d\xi) \right) = m(A).$$

If $G = R^n$ then ψ is an invertible matrix and Condition B is satisfied iff the determinant of ψ is 1.

Using Condition B we can apply the results of [3]:

The operator P defines by (1.3). *a contraction on $L_2 = L_2(G, \Sigma, m)$.*

With the notation of [3]:

$$K = \{f : f \in L_2 \text{ and } \|Pf\| = \|P^{*n} f\| = \|f\|, \quad n = 1, 2, \cdots\}. \tag{1.10}$$

$$H_0 = \{f : f \in L_2 \text{ and weak lim } Pf = 0\}, \tag{1.11}$$

$$H_1 = H_0^{\perp}. \tag{1.12}$$

Let us quote Theorem 1.1 of [3]:

THEOREM 1. *The sets K, H_0 and H_1 are subspaces invariant under P and P^*. The restriction of P to K is unitary and $H_1 \subset K$. The subspace K is of the form $L_2(G, \Sigma_1, m)$ where Σ_1 is a σ subfield of Σ. If $\sigma \in \Sigma_1$, then $PI(\sigma) = I(\tau)$ where $\tau \in \Sigma_1$, and P is an automorphism of Σ_1.*

2. **Convergence to zero.** Throughout this section we shall assume:

CONDITION C. *The group Γ does not contain an open nontrivial subgroup.* Let us note that

$$T_y^* = T_{-y}. \tag{2.1}$$

THEOREM 2. *Let $M = L_2(G, \Sigma', m)$ where Σ' is a σ subfield of Σ. If M is invariant under T_y for every y then either $M = 0$ or $M = L_2(G, \Sigma, m)$.*

Proof. Note that $f \in M$ iff $f \in L_2$ and f is Σ' measurable. Let $\tilde{f} \in M \cap L_1 \cap L_2$ and be bounded (e.g. a characteristic function). Put $\tilde{f}(x) = \tilde{f}(-x)$ then $f * \tilde{f}$ is continuous, vanishes at infinity and belongs to M: see [5, p. 4 and Theorem 7.1.2]. (The function $f * \tilde{f}$ is bounded and in L_1 hence in L_2.) Now the Fourier transform $(f * \tilde{f})^\wedge = |\hat{f}|^2 \neq 0$ hence $f * \tilde{f}$ is not identically zero. Let C be the collection of all continuous functions on G that vanish at infinity and belong to M. Thus $C \neq 0$ and C is an algebra under pointwise multiplication and is invariant under translations. Let us show that C separates points to conclude by means of the Stone-Weierstrass Theorem that the uniform closure of C contains every continuous function that vanishes at infinity:

If, for some $x \neq y$, $f(x) = f(y)$ for every $f \in C$ then $f(z) = f(z + (y - x))$ for every $f \in C$. Thus $\hat{f}(\gamma) = (y - x, \gamma)\hat{f}(\gamma)$ for every $\gamma \in \Gamma$. Therefore $(y - x, \gamma) = 1$ whenever $\hat{f}(\gamma) \neq$. The set $\{\gamma : (y - x, \gamma) = 1\}$ is a nontrivial subgroup of Γ that contains the open set $\{\gamma : \hat{f}(\gamma) \neq 0\}$ and this contradicts Condition C.

Let g be a continuous function with compact support and choose $f \in C$ such that $|f - g| < \varepsilon$. Put $A = \{x : |f(x)| \geqq \varepsilon\}$, then $I_A f \in M$ and $A \subset$ support g, and $|I_A f - g| < 2\varepsilon$. Hence

$$\int |I(A)f - g|^2 dm < (2\varepsilon)^2 m(\text{support } g)$$

and $g \in M$. Since continuous functions with compact support are dense in L_2 we have $M = L_2$.

REMARK. Theorem 2 can be rephrased to:

Σ does not contain nontrivial σ fields which are invariant under translations and are generated by sets of finite measure.

LEMMA 3. *The space K is invariant under translations.*

Proof. Let $f \in K$. Then

$$\| P^n T_y f \| = \| T_{z_n} P^n f \| = \| P^n f \| = \| f \|$$

by Condition A. Since $T_{-y}P^* = P^*T_{-z}$ the same argument applies to P^*.

THEOREM 4. *Either* $K = 0$ *or* $K = L_2$.

Proof. This follows immediately from Theorems 1 and 2 and Lemma 3.

COROLLARY I. *f for some set* A_0 *with* $0 < m(A_0) < \infty$ *and some int ger* m *the function* $P^m(x, A_0)$ *is not a characteristic function then, for every* $A \in \Sigma$ *with* $m(A) < \infty$, *the sequence* $P^n(x, A)$ *converges in measure to zero on every set of finite measure.*

Proof. $K = 0$ since $I(A_0) \notin K$. Thus $H_1 = 0$ and $H_0 = L_2$ by Theorem 1 and $P^n(x, A)$ converges weakly, in L_2 sense, to zero. The conclusion follows since $P^n(x, A) > 0$.

An application. Most of the ideas of this application were suggested to us by the eferee. Let μ be a probability measure which is absolutely continuous with respect to m and put $q = d\mu/dm$ (the Radon-Nikodym derivative). Let ψ be a continuous invertible automorphism of G that preserves m and put

$$P(x, A) = \mu(A - \psi(x)).$$

Condition A is clearly satisfied. Now if $m_1 = mP$ then

$$m_1(A) = \int P(x, A)m(dx) = \int P(0, A - \psi(x))m(dx) = \int P(0, A - x)m(dx)$$

because ψ preserves m, hence

$$\begin{aligned} m_1(A) &= \int \int_{A-x} q(\xi)m(d\xi)m(dx) = \int \int_A q(\xi - x)m(d\xi)m(dx) \\ &= \int_A \left(\int q(\xi - x)m(dx)\right) m(d\xi) = \int_A \left(\int q(x)m(dx)\right) m(d\xi) = m(A) \end{aligned}$$

where we used the invariance of m under translations. Let us now check the validity of the condition of the Corollary:

Let us assume, to the contrary, the existence of a sequence of compact sets with nonvoid interiors, A_n, such that $m(A_n) \to 0$ and $P(x_n, A_n) = 1$ for some x_n. Then $\mu(A_n - \psi(x_n)) = 1$ while $m(A_n - \psi(x_n)) = m(A_n) \to 0$ which contradicts the absolute continuity of μ with respect to m.

Therefore for every compact set A $P^n(x, A) \to 0$ a.e.

Let us assume in addition that for every compact set B the closure of $\bigcup \psi^n(B)$ is compact.

Let U be a compact neighborhood of zero and put $B = \mathrm{cl}(\bigcup \psi^n(U))$. Let A be any compact set and $x_0 \in U$ be such that $P^n(x_0, A + B) \to 0$. Then

$$P^n(0, A) \leqq P^n(0, A + B - \psi^n(x_0)) = P^n(x_0, A + B) \to 0$$

where we used $P^n(x, A) = P^n(0, A - \psi^n(x))$ which can be easily checked.

Therefore for every compact set A and every point x, $\lim P^n(x,A) = 0$.

The limit is uniform if x belongs to the compact U; this can be shown by a similar argument.

An easy induction shows that

$$P^n(x,A) = \int \cdots \int \mu(dy_{n-1}) \cdots \mu(dy_1)\mu(A - \psi(y_{n-1}) - \psi^2(y_{n-2}) - \cdots - \psi^n(x))$$

i.e. $P^n(x,\cdot)$ is the probability distribution of

$$y_n + \psi(y_{n-1}) + \cdots + \psi^{n-1}(y_1) + \psi^n(x)$$

where $y_1 \cdots y_n$ are independent identically distributed random variables each with distribution μ.

It should be noted that when $\psi = 0$ t e problem is one of n-fold convolutions of μ by itself and can be solved by standard Fourier transform techniques.

3. **Affine transformations on G.** Let us assume that P is given by $(Pf)(x) = f(\phi(x))$ where ϕ is a continuous invertible transformation of G. Now Condition A will imply $f(\phi(x+y)) = f(\phi(x) + z)$ for every function f. Let us assume then that $\phi(x+y) = \phi(x) + z$, $z = z(y)$ and thus $\phi(y) = \phi(0) + z$ hence

$$\phi(x+y) = \phi(x) + \phi(y) - \phi(0). \tag{3.1}$$

Thus

(3.2) $\phi(x) = \psi(x) + a$, *where ψ is a continuous invertible automorphism of G and a is a given element of G.*

Finally Condition B is equivalent to

$$m(\psi^{-1}(A)) = m(A). \tag{3.3}$$

Also throughout the rest of this paper we shall assume

CONDITION D. *The space G is connected.*

Thus by the Structure Theorem (Theorem 2.4.1 of [5]) G is the direct sum of the Euclidean space R^k and a compact group G^*. Every element of G will be written in the form(2)

$$x = (r,g)^{\vee} \; r \in R^k, \; g \in G^*.$$

If $a = (s,h)^{\vee}$ then

$$\phi(r,g)^{\vee} = \begin{pmatrix} \tau_1(r) + {}_2(g) + s \\ \tau_3(r) + \tau_4(g) + h \end{pmatrix} \tag{3.4}$$

where τ_i are continuous homomorphisms on their respective domains.

Now $\tau_2(G^*)$ is compact and contains $n\tau_2(g)$ for every $g \in G^*$. Thus:

(2) $(r,g)^{\vee}$ is the column vector of r and g.

$$\tau_2' = 0. \tag{3.5}$$

The homomorphism τ_1 is from R^k to R^k. Hence τ_1 can be represented by a $k \times k$ matrix.

LEMMA 5. *Let τ be a $k \times k$ matrix and s a given vector in R^k. Then either*
a. *For every bounded set $A \subset R^k$ the set*

$$\bigcup_{r \in A} \bigcup_{n=1}^{\infty} (\tau^n r + (\tau^{n-1} s + \cdots + s))$$

is bounded. Or
b. *$\tau^n r + (\tau^{n-1} s + \cdots + s) \to \infty$ as $n \to \infty$ for every r except in a linear manifold S (i.e. $S - S$ is a subspace of R^k different from R^k).*

Proof. Let $\tau = \Sigma(\lambda_i + N_i)E_i$ be the Jordan decomposition of τ over C^k, where $E_i^2 = E_i$, $E_iE_j = 0$, $i \neq j$, $N_iE_i = E_iN_i$, $N_i^k = 0$ (see [1, Theorem VII.1.8]). Let us study the various possibilities:

1. Let one of the eigenvalues, say λ_1, satisfy $|\lambda_1| > 1$. Then

$$E_1(\tau^n r + \tau^{n-1} s + \cdots + s) = \tau^n E_1 r + \tau^{n-1} E_1 s + \cdots + E_1 s = \tau^n E_1 r + (\tau^n - 1)(\tau - 1)^{-1} E_1 s$$
$$= \tau^n (E_1 r + (\tau - 1)^{-1} E_1 s) - (\tau - 1)^{-1} E_1 s,$$

where by $(\tau - 1)^{-1} E_1$ we mean the inverse of $(\tau - 1)E_1$ on $E_1 C^k$. Now on $E_1 C^k$ the matrix $\tau = \lambda + N$. Let $t = E_1 r + (\tau - 1)^{-1} E_1 s$. If $t \neq 0$ then

$$(\lambda + N)^n t = \lambda^n t + \binom{n}{1} \lambda^{n-1} N t + \cdots + \binom{n}{j} \lambda^{n-j} N^j t$$

where $j \leqq k$ and $N^{j+1} t = 0$ and $N^j t \neq 0$. Now the last term tends to infinity a $|\lambda^n| n^j$ and it dominates the other terms. Thus

$$S \subset \{r : E_1 r = -(\tau - 1)^{-1} E_1 s\}.$$

If S contains all of R^k then $E_1 r = \text{const}$ for every real vector but then $E_1 r = 0$ for every real vector and thus for every complex vector which is impossible. Thus $S \cap R^k$ is a linear manifold in R^k different from R^k and if $r \in R^k$ and $r \in {}'S \cap R^k$ then $\lim \tau^n r + \tau^{n-1} s + \cdots + s = \infty$.

2. Let one of the eigenvalues, say λ_1, satisfy $|\lambda_1| = 1$ and $N_1 \neq 0$. Then:

$$(\tau - 1)E_1(\tau^n r + \tau^{n-1} s + \cdots + s) = \tau^{n+1} E_1 r - \tau^n E_1 r + \tau^n E_1 s - E_1 s$$
$$= \tau^n (\tau E_1 r - E_1 r + E_1 s) - E_1 s.$$

Put $t = \tau E_1 r - E_1 r + E_1 s$ then

$$\tau^n t = \lambda^n t + \binom{n}{1} \lambda^{n-1} N t + \cdots + \binom{n}{j} \lambda^{n-j} N^j t$$

where again $j \geqq k$ and $N^{j+1}t = 0$ and $N^j t \neq 0$. The last term tends to infinity as n^j and it dominates the other terms. Thus

$$S \subset \{r: Nt = 0\} = \{r: N(\tau - 1)E_1 r = -NE_1 s\}.$$

Let us first consider the case where $\lambda_1 \neq 1$. If every real vector is in S then $N(\tau - 1)E_1 r =$ constant for every real vector and thus $N(\tau - 1)E_1 = 0$ but on $E_1 C^k$ the matrix $\tau - 1$ has an inverse and thus $NE_1 = 0$ which contradicts our assumption.

Let now $\lambda_1 = 1$ if $R^k \subset S$ then $N(\tau - 1)E_1 = N^2 E_1 = 0$ but then

$$\begin{aligned} E_1(\tau^n r + \tau^{n-1}s + \cdots + s) &= (1+N)^n E_1 r + (1+N)^{n-1} E_1 s + \cdots + E_1 s \\ &= E_1 r + nE_1 s + nNE_1 r + ((n-1) + \cdots + 1)NE_1 s \\ &= E_1 r + nE_1 s + nNE_1 r + \tfrac{1}{2}n(n-1)NE_1 s. \end{aligned}$$

If $NE_1 s \neq 0$ then clearly $S = 0$. If $NE_1 s = 0$ then

$$S \subset \{r: NE_1 r = -E_1 s\}$$

or $NE_1 r =$ const for every real vector and as before this leads to $NE_1 = 0$ which contradicts our assumptions.

3. Let $\lambda_1 = 1$ and $N_1 = 0$. Then

$$\tau^n E_1 r + \tau^{n-1} E_1 s + \cdots + E_1 s = E_1 r + nE_1 s.$$

Thus if $E_1 s \neq 0$, $S = 0$.

4. Let all eigenvalues λ_i satisfy $|\lambda_i| \leqq 1$ and for those eigenvalues on the circumference of the unit circle $N_i = 0$. Furthermore if $\lambda_j = 1$ for some j then $E_j s = 0$. Let $|\lambda_i| < 1$ then $\| E_i \tau^n r \| < \rho^n \| r$ for $\rho < 1$ and n large enough. Thus

$$\| \tau^n E_i r + \tau^{n-1} E_i s + \cdots + E_i s \| < \rho^n \| r \| + (\rho^{n-1} + \cdots + 1) \| s \|$$

and is uniformly bounded if r belongs to a bounded set.

Let $|\lambda_i| = 1$ and $\lambda_i \neq 1$. Then by assumption $N_i = 0$. Hence

$$\tau^n E_i r + \tau^{n-1} E_i s + \cdots + E_i s = \lambda_i^n E_i r + \frac{\lambda^n - 1}{\lambda_i - 1} E_i s$$

and is uniformly bounded.

Finally let $\lambda_i = 1$ and $N_i = 0$ and $E_i s = 0$. Then

$$\tau^n E_i r + \tau^{n-1} E_i s + \cdots + E_i s = E_i r.$$

THEOREM 6. *Either*

a. *For every compact set $A \subset G$ the closure of $\bigcup_{n=1}^{\infty} \phi^n(A)$ is compact or*

b. $\phi^n(x) \to \infty$ *a.e.*

Proof. Let Case a of Lemma 5 hold for $\tau_1 + s$. Then if $x = (r, g)^{\vee}$,

$$\phi^n(x) = \begin{bmatrix} \tau_1^n(r) + \tau_1^{n-1}(s) + \cdots + s, & 0 \\ \overset{(n)}{3}(r), & \tau_4^n g + \tau_4^{n-1} g + \cdots + g \end{bmatrix}$$

by 3.5(1). Now if $x \in A$ then r is in a bounded set and thus $(\tau_1 + s)^n r$ is in a compact set. Now $\tau_3^{(n)} r$ and $(\tau_4 + h)^n g$ both are in the compact set G^*.

If Case b holds then $(\tau_1 + s)^n r \to \infty$ except for $r \in S$. Thus $\phi^n(x) \to \infty$ except for $r \in S$. Let $r_0 \notin S - S$. Then if $a \neq b$ one gets $ar_0 + S \neq br_0 + S$. Hence put $G_1 = (S, G^*)^\vee$ then outside of $G_1, \phi^n x \to \infty$ and for $a \neq b$ one gets $ar_0 + G_1 \neq br_0 + G_1$.

Let A be any compact su set of G_1. Choose $\delta > 0$ so that $\bigcup\{ar_0 + A : 0 < a < \delta\}$ is contained in a compact set. Now $m(ar_0 + A) = m(A) = 0$ since

$$m \bigcup\{ar_0 + A : 0 < a < \delta\} < \infty.$$

Thus $m(G_1) = 0$ as m is a regular measure.

THEOREM 7. *If Case* b *holds then* $H_0 = L_2$.

Proof. Let A and B be compact sets. Then as $n \to \infty$, $I(A)(x)\, I(B)(\phi^n x) \to 0$ a.e. By the Lebesgue Dominated Convergence Theorem $\langle P^n I(B), I(A) \rangle \to 0$ where $\langle f, g \rangle$ is the inner product of f and g. Thus H_0 contains every characteristic function of a compact set and $H_0 = L_2$.

Throughout the rest of the paper we shall assume that Case a *of Theorem* 6 *holds.*

THEOREM 8. *For every* $x \in G$ *there exists a compact neighborhood of* x *which is invariant under* ϕ.

Proof. Let A be a compact neighborhood of x. Then $\bigcup_{n=0}^{\infty} \phi^n(A)$ has a compact closure and

$$\phi\left(\mathrm{cl}\left(\bigcup_{n=0}^{\infty} \phi^n(A)\right)\right) \subset \mathrm{cl}\left(\bigcup_{n=0}^{\infty} \phi^n(A)\right)$$

(where $\mathrm{cl}B =$ closure of B). Since ϕ is measure preserving $\mathrm{cl}(\bigcup_{n=0}^{\infty} \phi^n(A))$ is invariant.

Let $\gamma \in \Gamma$ then $\gamma(\phi(x)) = \gamma(\psi(x) + a) = \gamma(a)\gamma(\psi(x))$. Thus

$$\gamma(\phi^n(x)) = \gamma(\psi^n(x))\gamma(a)\gamma(\psi(a)) \cdots \gamma(\psi^{n-1}(a)).$$

Let us study the behaviour of $\gamma\psi^n$ as $n \to \infty$. Now by Theorem 2.2.2 of [5] $\Gamma = R^k \oplus \Gamma^*$ where Γ^* is the dual of G^* and is a discrete group. Every element of Γ has the form (r, γ^*) where $r \in R^k$ and $\gamma^* \in \Gamma^*$. Where (r, γ^*) applied to $(r_1, g)^\vee$ is $(r_1, r)(g, \gamma^*)$.

Now

$$\psi^n = \begin{bmatrix} \tau_1^n, & 0 \\ \tau_3^{(n)}, & \tau_4^n \end{bmatrix}$$

where $\tau_3^{(n)}$ is not the nth power of τ_3.

LEMMA 9. *For every $\gamma \in \Gamma$ either*
a. $\gamma\psi^n \to \infty$ *or,*
b. *the sequence $\gamma\psi^n$ has compact closure.*

Proof. Consider

$$\gamma\psi^n = (r, \gamma^*) \begin{bmatrix} \tau_1^n, & 0 \\ \tau_3^{(n)}, & \tau_4^n \end{bmatrix} = r\tau_1^n + \gamma^*\tau_3^{(n)} + \gamma^*\tau_4^n.$$

Since $\gamma^*\tau_4^n \in \Gamma^*$ this sequence either tends to infinity or is a finite set. Also $r\tau_1^n + \gamma^*\tau_3^{(n)} \in R^k$ hence if $\gamma^*\tau_4^n \to \infty$ so does $\gamma\psi^n$. Let us study the case where $\gamma^*\tau_4^n$ i a finite set. Let us assume that $\gamma^*\tau_4^m = \gamma^*\tau_4^{m+p} = (\gamma^*\tau_4^m)\tau_4^p$. In order o prove the lemma it is enough to show that $\gamma\psi^{m+np}$ either tends to infinity or has compact closure. If this is the case for one value of m, $0 \leqq m < p$, then it is so for every value of m, $0 \leqq m < p$. Thus let $\tau = \tau_1^p$ and replace γ by $\gamma\psi^m$ and we have:

$$\gamma = (r, \gamma^*), \qquad \gamma\psi^p = (r\tau + \gamma^*\tau_3^{(p)}, \gamma^*),$$

put $s = \gamma^*\tau_3^{(p)}$ we have to study the behaviour of $\gamma\psi^{pm} = (r\tau^n + s\tau^{n-1} + \cdots + s, \gamma^*)$. Thus it suffices to study $r\tau^n + s\tau^{n-1} + \cdots + s \in R^k$.

Now we have assumed that Case a of Theorem 6 (namely Case a of Lemma 5) holds. From the various steps in the proof of Lemma 5 follows that the Jordan decomposition of τ is

$$\sum_{|\lambda_i|<1} (\lambda_i + N_i)E_i + \sum_{|\lambda_i|=1} \lambda_i E_i.$$

Using the same argument as part 4 of Lemma 5 we see that the contribution of the first sum will be a bounded sequence. Also, for the second sum, if $\lambda_i \neq 1$ then

$$\lambda_i^n rE_i + \lambda_i^{n-1} sE_i + \cdots + sE_i = \lambda_i^n rE_i + \frac{\lambda_i^n - 1}{\lambda_i - 1} rE_i$$

is a bounded set. Finally if $\lambda_i = 1$ then $rE_i\tau^n + sE_i\tau^{n-1} + \cdots + sE_i = rE_i + nsE_i$. If $sE_i \neq 0$ then the sequence tends to infinity and if $sE_i = 0$ it is bounded.

REMARK. Since $\gamma\psi^n$ has compact closure only when $\gamma^*\tau_4^n$ assumes finitely many values, and in this case $\gamma\psi^n$ is contained in a finite sum of R^k spaces, the two notions of compactness and sequential compactness coincide.

Thus if $\gamma\psi^n$ does not tend to infinity, then every subsequence contains a subsequence of it, $\gamma\psi^{n_i}$, such that $(x, \gamma\psi^{n_i})$ converges uniformly for x in compact subsets of G.

In particular:

COROLLARY. *Let A be a compact subset of G. If $\gamma\psi^n$ does not tend to infinity then the sequence of functions*

$$(x,\gamma\phi^n)I(A)\,(x) = (x,\gamma\psi^n)(a,\gamma)(a,\gamma\psi)\cdots(a,\gamma\psi^{n-1})I(A)(x)$$

is precompact in L_2.

Let us use the following notation

(3.6) $\Gamma_0 = \{\gamma : \gamma \in \Gamma \text{ and } \gamma\psi \to \infty\}$,

(3.7) $\Gamma_1 = \Gamma - \Gamma_0 = \{\gamma : \gamma \in \Gamma \textit{ and } \gamma\psi \textit{ has compact closure}\}$,

(3.8) $\Xi = \{A : A \textit{ is compact and } \phi(A) = A\}$.

THEOREM 10.

1. *The space H_0 is generated by the collection of all functions of the form*

$$f(x) = (x,\gamma)\, I(A)(x) \textit{ where } \gamma \in \Gamma_0, A \in \Xi.$$

2. *The space H_1 is generated by the collection of all functions of the form*

$$f(x) = (x,\gamma)\, I(A)(x) \textit{ where } \gamma \in \Gamma_1, A \in \Xi.$$

3. *If $f \in H_1$ then the sequence $P^n f$ has compact closure in L_2 (or f belongs to the subspace generated by eigenfunctions of P).*

Proof. Let $\gamma \in \Gamma_0$ and $A \in \Xi$ and $f(x) = (x,\gamma)\, I(A)(x)$. Then

$$(P^n f)(x) = (a,\gamma)(a,\gamma\psi)\cdots(a,\gamma\psi^{n-1})\,(x,\gamma\psi^n)\, I(A)x$$

since $\phi^{-1}(A) = A$. Thus for every $g \in L_2 \cap L_1$

$$\langle P^n f, g\rangle = (a,\gamma)(a,\gamma\psi)\cdots(a,\gamma\psi^{n-1})\int (x,\gamma\psi^n)\,(I(A)(x)g(x))m(dx) \to 0$$

by Theorem 1.2.4 of [5]. Thus $f \in H_0$.

Let $\gamma \in \Gamma_1$ and $A \in \Xi$ and $f(x) = (x,\gamma)I(A)(x)$. Then, by the Corollary of Lemma 9, $P^n f$ has a compact closure in L_2. From 1.3 of [2] follows that $f \in H_1$. Finally if g is orthogonal to all functions of the form $(x,\gamma)I(A)(x)$ with $A \in \Xi$ then

$$\int g(x)(x,-\gamma)\, I(A)(x)m(dx) = 0;$$

hence by Plancherel Theorem $g(x) = 0$ on A a.e. Now if C is a compact set in G then $C \subset \bigcup_{i=1}^n A_i$ with $A_i \in \Xi$ by Theorem 8. Thus $g(x) = 0$ on C, or $g = 0$ a.e.

REMARK. *Theorem* 10 *and Theorem* 1.3 *of* [2] *imply that $f \in H_1$ if and only if* $\limsup |\langle P^n f, f\rangle| = \|f\|^2$.

We conjecture that this is correct for every P that is given by a measure preserving transformation.

LEMMA 11. *If $f \in H_1$ and α is a Borel set in the plane then $I(f(x) \in \alpha) \in H_1$.*

Proof. This follows from Theorem 2.1 of [4].

Let Σ_2 be the collection of Borel sets that are compact and their characteristic functions belong to H_1.

If $A \in \Xi$ and α is a closed set on the circumference of the unit circle then

$$\{x : (x, \gamma) \in \alpha\} \cap A = \{x : (x, \gamma) I(A)(x) \in \alpha\}$$

and the characteristic function of this set belongs to H_1 if $\gamma \in \Gamma_1$. Since these functions generate H_1 it follows that:

THEOREM 12. *The space H_1 is generated by the collection of all characteristic functions of sets in Σ_2.*

THEOREM 13. *If the set Γ_1 contains more than one element (the constant function) then it is infinite.*

Proof. Let $\gamma = (r, \gamma^*) \in \Gamma_1$ be nontrivial (we use here the notation employed in Lemma 9). Then for every $s \in R^k (r + s, \gamma^*) \in \Gamma_1$ as can be seen from the proof of Lemma 9. Thus Γ_1 is infinite unless $k = 0$ and G is compact. For compact groups the proof of Theorem 10 will yield:

The space H_1 is generated by the functions $f(x) = (x, \gamma)$, $\gamma \in \Gamma_1$. Now if Γ_1 is finite then H_1 is finite dimensional and Σ_2 contains finitely many disjoint compact sets $A_1 \cdots A_r$. Hence $m(G - \bigcup_{i=1}^r A_i) = 0$ but $G - \bigcup_{i=1}^r A_i$ is an open set thus $G = \bigcup_{i=1}^r A_i$ is not connected.

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