

## FUNCTIONS SATISFYING A WEIGHTED AVERAGE PROPERTY. II (¹)

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**Introduction.** In the previous paper [1] we were interested in characterizing the class of functions, defined in a given region (open connected set)  $R$  of the  $n$ -dimensional euclidean space  $E_n$ , which satisfy the following Weighted Average Property (W.A.P.):

$$(0.1) \quad u(P) = \frac{\int_{B(P,r)} uw d\rho}{\int_{B(P,r)} w d\rho}, \quad P \in R,$$

$$(0.2) \quad u(P) = \frac{\int_{S(P,r)} uw d\sigma}{\int_{S(P,r)} w d\sigma},$$

where  $B(P, r)$  and  $S(P, r)$  denote any ball and its surface with the point

$$P = P(x_1, x_2, \dots, x_n),$$

for its center and radius  $r$  which lies in  $R$ ;  $d\rho$  and  $d\sigma$  stand for the usual Lebesgue measure of  $B$  and  $S$  and  $w$  is a weight function (W.F.) defined in  $R$ .

For convenience we state here again the definitions of weight functions and functions satisfying a W.A.P.

**DEFINITION.**  $w$  is a W.F. in  $R$  means that

(a)  $w$  is a nonnegative, real-valued function defined in  $R$ , and

(b)  $w$  is locally summable in  $R$ , i.e., if  $P \in R$  and  $0 < r < d(P, T)$ ,  $T$  being the boundary of  $R$ , then the Lebesgue integral  $\int_{B(P,r)} w d\rho$  over  $B(P, r)$  exists and  $\int_{B(P,r)} w d\rho > 0$ .

**DEFINITION.** A real-valued function  $u$  is said to satisfy the W.A.P. with respect to a W.F.,  $w$  in  $R$  provided  $uw$  is locally summable in  $R$  and  $u$  satisfies the property (0.1) for each ball  $B(P, r)$  whose closure lies in  $R$ .

**DEFINITION.**

$$R^* = \{(P, r) : P \in R \text{ and } 0 < r < d(P, T)\}.$$

Received by the editors November 6, 1965.

(¹) This research was supported, in part, by the U. S. Army Research Office, Durham, N. C.

Suppose now that  $R$  be a region in  $E_n$  and  $w$  be a W.F. defined in  $R$ . Let  $S(w, R)$  denote the class of all functions satisfying W.A.P. with respect to  $w$  in  $R$ .

In the previous paper [1] we proved the following theorems and corollaries:

**THEOREM 2\*.**

- (i) If the W.F.  $w$  belongs to class  $C^m(R)$ , where  $m$  is a nonnegative integer and  $u \in S(w, R)$ , then  $u \in C^{m+1}(R)$ .
- (ii) If  $w$  be infinitely differentiable in  $R$  and  $u \in S(w, R)$ , then  $u$  is infinitely differentiable in  $R$ .
- (iii) If  $w$  be analytic in  $R$  and  $u \in S(w, R)$ , then  $u$  is analytic in  $R$ .

**REMARK 1\*.**

- (0.3) Constant functions belong to  $S(w, R)$ .
- (0.4)  $S(w, R)$  is a linear space over the reals and  $\dim S(w, R) \geq 1$ .
- (0.5) If  $w$  be a nonzero constant, then  $S(w, R)$  is the class of all harmonic functions defined on  $R$ .

**REMARK 2\*.** In proving Theorem 2\* the following was shown:

If  $w \in C^1(R)$  and  $u \in S(w, R)$ , then

$$(0.6) \quad u_{x_i}(P) = \frac{\int_{B(P,r)} u_{x_i} w d\rho}{\int_{B(P,r)} w d\rho} + \frac{\int_{B(P,r)} u w_{x_i} d\rho - u(P) \int_{B(P,r)} w_{x_i} d\rho}{\int_{B(P,r)} w d\rho},$$

$i = 1, 2, \dots, n$ , for all  $(P, r) \in R^*$ .

**THEOREM 4\*.** If the W.F.  $w \in C^1(R)$  and  $u \in S(w, R)$ , then  $u$  satisfies the second order elliptic differential equation

$$(0.7) \quad w \Delta u + 2 \sum_{i=1}^n w_{x_i} u_{x_i} = 0$$

in  $R$ , where  $\Delta u$  is the Laplacian of  $u$ .

**REMARK 3\*.** In proving Theorem 4\* incidentally the following was shown:

If the W.F.  $w \in C^1(R)$  and  $u \in S(w, R)$ , then

$$(0.8) \quad \Delta u(P) \cdot \int_{B(P,r)} w d\rho + 2 \sum_{i=1}^n u_{x_i}(P) \cdot \int_{B(P,r)} w_{x_i} d\rho = 0$$

for each  $(P, r) \in R^*$ .

**THEOREM 5\*.** Let  $w$  be a W.F. defined in  $R$  and  $\lambda$  be a real number such that  $w \in C^2(R)$  and is a solution of the differential equation

$$(0.9) \quad \Delta F + \lambda F = 0$$

in  $R$ . Then a necessary and sufficient condition that  $u \in S(w, R)$  is that

- (i)  $u \in C^2(R)$  and
- (ii)  $u$  satisfies the differential equation

$$(0.10) \quad w\Delta u + 2 \sum_{i=1}^n u_{x_i} w_{x_i} = 0$$

in  $R$ .

**REMARK 4\*.** In proving Theorem 5\* incidentally the following was shown: If the W.F.  $w \in C^2(R)$  and is a solution of (0.9), then  $w(P) > 0$  for all  $P$  in  $R$ .

**COROLLARY 1\*.** Let  $w$  be a W.F. defined in  $R$  and  $\lambda$  be a real number such that  $w \in C^2(R)$  and is a solution of the differential equation

$$(0.11) \quad \Delta F + \lambda F = 0.$$

in  $R$ . Then the following are true:

- (a) If  $f \in C^2(R)$  and is a solution of (0.11), then  $f/w \in S(w, R)$ .
- (b) If  $u \in S(w, R)$ , then  $uw \in C^2(R)$  and is a solution of (0.11).

**COROLLARY 2\*.** Let  $w$  be a W.F. defined in a region  $R$  of  $E_n$  ( $n > 1$ ) and  $\lambda$  a real number such that  $w \in C^2(R)$ , and is a solution of the equation

$$\Delta F + \lambda F = 0$$

in  $R$ . Then  $S(w, R)$  is infinite dimensional.

**REMARK 5\*.** In  $E_1$  the following can be proved easily:

If the W.F.  $w \in C^1(R)$ , then

- (i)  $1 \leq \dim S(w, R) \leq 2$
- (ii)  $\dim S(w, R) = 2$  if and only if  $w \in C^2(R)$  and is a solution of

$$(0.12) \quad \frac{d^2w}{dx^2} + \lambda \frac{dw}{dx} = 0$$

in  $R$ , for some real constant  $\lambda$ .

The importance of the weight functions which are solutions of the equation

$$(0.13) \quad \Delta w + \lambda w = 0$$

has been demonstrated in the previous paper [1].

In this paper we propose to deduce in terms of the W.A.P. a necessary and sufficient condition for a W.F. to be a solution of (0.13). From this result we will prove our main theorem of §I, namely, "S( $w, R$ ) is infinite dimensional if and only if  $w$  is a solution of (0.13)." (2)

In §II we will consider derivatives of functions belonging to  $S(w, R)$  and deduce necessary and sufficient conditions for these derivatives to belong to  $S(w, R)$ .

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(2) Added in proof. True only in  $E_2$ . See also footnote (3).

Here again we will emphasize the role of weight functions which are solutions of (0.13), specially the exponential weight functions  $\exp\{\sum_{i=1}^n a_i x_i\}$ .

### I. The differential equation $\Delta w + \lambda w = 0$ .

**THEOREM 1.** *Let  $w$  be a W.F. belonging to class  $C^1(R)$ . A necessary and sufficient condition that  $w \in C^2(R)$  and is a solution of*

$$(1.0) \quad \Delta w + \lambda w = 0$$

*in  $R$ , where  $\lambda$  is some real constant, is that*

- (i)  $w(P) > 0$  for all  $P(x_1, x_2, \dots, x_n)$  in  $R$  and
- (ii)  $w_{x_i}/w \in S(w, R)$  for  $i = 1, 2, \dots, n$ .

**Proof. Necessity.** Let  $w \in C^2(R)$  and be a solution of (1.0). Then  $w$  is analytic in  $R$  and each of the partial derivatives  $w_{x_i}$ ,  $i = 1, 2, \dots, n$ , is also a solution of (1.0). Again by Remark 4\*,  $w(P) > 0$  for all  $P = P(x_1, x_2, \dots, x_n)$  in  $R$ . Hence by Corollary 1\*,  $w_{x_i}/w \in S(w, R)$  for  $i = 1, 2, \dots, n$ .

**Sufficiency.** Suppose that  $w(P) > 0$  for all  $P = P(x_1, x_2, \dots, x_n)$  in  $R$  and  $w_{x_i}/w \in S(w, R)$  for  $i = 1, 2, \dots, n$ .  $w_{x_i}/w = F^i \in S(w, R)$  implies by Theorem 2\* that  $F^i \in C^2(R)$  and  $w \in C^3(R)$ . Also by Theorem 4\*,  $F^i$  is a solution of the differential equation

$$(1.1) \quad w \Delta F^i + 2 \sum_{j=1}^n F_{x_j}^i w_{x_j} = 0,$$

$i = 1, 2, \dots, n$ . Again  $F^i w = w_{x_i}$  implies that

$$(1.2) \quad \Delta w + \lambda w = 0,$$

where

$$\lambda = - \sum_{i=1}^n \{F_{x_i}^i + (F^i)^2\}.$$

Clearly  $\lambda$  belongs to class  $C^1(R)$  and

$$\frac{\partial \lambda}{\partial x_j} = \lambda_{x_j} = - \sum_{i=1}^n \{F_{x_i x_j}^i + 2F^i F_{x_j}^i\}, \quad j = 1, 2, \dots, n.$$

Using

$$F_{x_j}^i = F_{x_i}^j, F_{x_j x_i}^i = F_{x_i x_j}^j, \quad i, j = 1, 2, \dots, n;$$

we have

$$w \lambda_{x_j} = - \sum_{i=1}^n \{w F_{x_i x_j}^i + 2F_{x_i}^j w_{x_i}\} = - \left\{ w \Delta F^j + 2 \sum_{i=1}^n F_{x_i}^j w_{x_i} \right\} = 0.$$

Since

$$w \neq 0 \text{ in } R, \quad \lambda_{x_j} = 0 \text{ in } R \text{ for } j = 1, 2, \dots, n.$$

Therefore  $\lambda$  is a constant. This completes the proof.

*Dimension of  $S(w, R)$ .*

**THEOREM 2.** *Let  $R$  be a region in  $E_2$  and  $w$  be a positive W.F. belonging to class  $C^1(R)$ . Then  $S(w, R)$  is infinite dimensional if and only if  $w$  belongs to class  $C^2(R)$  and is a solution of*

$$(1.3) \quad \Delta w + \lambda w = 0$$

*in  $R$  for some real constant  $\lambda$ . Furthermore, if  $S(w, R)$  is finite dimensional then  $1 \leq \dim S(w, R) \leq 2$ .*

**Proof.** If  $w \in C^2(R)$  and is a solution of (1.3) then  $S(w, R)$  is infinite dimensional by Corollary 2\*. So now suppose that  $w$  be a positive W.F. belonging to class  $C^1(R)$  such that  $\dim S(w, R) > 2$ . Then there exists two nonconstant functions  $u_1$  and  $u_2$  both belonging to  $S(w, R)$  such that  $1, u_1, u_2$ , are linearly independent over  $R$ . By Theorem 4\* and Remark 3\* we have

$$(1.4) \quad w\Delta u_i + 2(w_x u_{ix} + w_y u_{iy}) = 0, \quad i = 1, 2, .$$

in  $R$  and

$$(1.5) \quad \Delta u_i(P) + 2u_{ix}(P) \left[ \frac{\int_{B(P,r)} w_x d\rho}{\int_{B(P,r)} w d\rho} \right] + 2u_{iy}(P) \left[ \frac{\int_{B(P,r)} w_y d\rho}{\int_{B(P,r)} w d\rho} \right] = 0$$

for each  $(P, r) \in R^*$ . Therefore we have

$$(1.6) \quad u_{ix}(P) \left[ \frac{w_x(P)}{w(P)} - \frac{\int_{B(P,r)} w_x d\rho}{\int_{B(P,r)} w d\rho} \right] + u_{iy}(P) \left[ \frac{w_y(P)}{w(P)} - \frac{\int_{B(P,r)} w_y d\rho}{\int_{B(P,r)} w d\rho} \right] = 0,$$

for each  $(P, r) \in R^*$ ,  $i = 1, 2$ .

Let  $R_1$  be the subset of  $R$  to which a point  $P$  belongs if and only if the Jacobian

$$J_{12} = \frac{D(u_1, u_2)}{D(x, y)} = \begin{vmatrix} u_{1x} & u_{1y} \\ u_{2x} & u_{2y} \end{vmatrix}$$

does not vanish at  $P$ .  $R_1$  is a dense subset of  $R$ . For, if not, then there exists a point  $P$  in  $R - R_1$  and a neighborhood  $N(P)$  of the point  $P$  lying in  $R$  such that the Jacobian  $J_{12}$  vanishes identically in  $N(P)$ . If at least one of the minors, say  $u_{1x}$ , is not identically zero in  $N(P)$ , then there is a point  $Q$  and a neighborhood  $N(Q)$  of the point  $Q$  lying in  $N(P)$  such that  $u_{1x} \neq 0$  in  $N(Q)$ . Hence there exists a functional relation  $u_2 = \phi(u_1)$  valid in  $N(Q)$ . Since each of  $u_1$  and  $u_2$  belong

to  $C^2(R)$  and  $u_{1x} \neq 0$  in  $N(Q)$ ,  $\phi$  is twice continuously differentiable and we have

$$w\Delta u_2 + 2u_{2x}w_x + 2u_{2y}w_y = w \cdot \frac{d^2\phi}{du_1^2} \{(u_{1x})^2 + (u_{1y})^2\} = 0$$

or

$$\frac{d^2\phi}{du_1^2} = 0.$$

Hence  $u_2 = c_1 u_1 + c_2$ , in  $N(Q)$ , where  $c_1$  and  $c_2$  are real constants. This means by the maximum principle of the elliptic equation

$$w\Delta u + 2w_xu_x + 2w_yu_y = 0$$

that  $u_2 = c_1 u_1 + c_2$  in  $R$ , contradicting the linear independence assumption. Therefore every minor of  $J_{12}$  is identically zero in  $N(P)$  implying that each of  $u_1$  and  $u_2$  is constant in  $N(P)$  and hence in  $R$  contradicting again our hypothesis. Therefore  $R_1$  is a dense subset of  $R$ . Now to prove our theorem consider the equations (1.6). Clearly

$$(1.8) \quad \int_{B(P,r)} w_x d\rho / \int_{B(P,r)} w d\rho = w_x(P)/w(P)$$

$$(1.9) \quad \int_{B(P,r)} w_y d\rho / \int_{B(P,r)} w d\rho = w_y(P)/w(P)$$

for each point  $P$  in  $R_1$  and each  $r$  satisfying  $0 < r < d(P, T)$ . Since  $R_1$  is dense, we conclude that the relations (1.8) and (1.9) are true for each  $(P, r) \in R^*$ . This means that each of the functions  $w_x/w$  and  $w_y/w$  belongs to  $S(w, R)$ . Hence by Theorem 1 there exists a real constant  $\lambda$  such that  $w \in C^2(R)$  and is a solution of (1.3). Therefore  $S(w, R)$  is infinite dimensional by Corollary 2\*.

It is clear also that if  $S(w, R)$  is finite dimensional then  $1 \leq \dim S(w, R) \leq 2$ .

**THEOREM 3**<sup>(3)</sup>. *Let  $R$  be a region in  $E_n$  ( $n > 2$ ) and  $w$  be a positive W.F. belonging to class  $C^1(R)$ . Then  $S(w, R)$  is infinite dimensional if and only if  $w$  belongs to class  $C^2(R)$  and is a solution of*

$$\Delta w + \lambda w = 0$$

*in  $R$ , for some real constant  $\lambda$ . If  $S(w, R)$  is finite dimensional then*

$$1 \leq \dim S(w, R) \leq 2n - 1.$$

**Proof.** For simplicity we will give the proof for  $n = 4$ . Proof for the general case is quite similar.

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<sup>(3)</sup> Added in proof. Theorem 3 is not possibly true. Mr. David Stanford of Denison University, Granville, Ohio, seems to have a counterexample. The author hopes to clarify this point in a future paper.

If  $w \in C^2(R)$  and is a solution of  $\Delta w + \lambda w = 0$ , for some real constant  $\lambda$ , then by Corollary 2\*,  $S(w, R)$  is infinite dimensional. So now suppose that  $w$  be a positive W.F. belonging to class  $C^1(R)$  such that  $\dim S(w, R) > 2 \cdot 4 - 1 = 7$ . Then there exists nonconstant functions  $u_1, u_2, \dots, u_7$  all belonging to  $S(w, R)$  such that  $1, u_1, u_2, \dots, u_7$  are linearly independent over  $R$ . As in Theorem 2, we have by Theorem 4\* and Remark 3\* the system of 7 equations:

$$(1.10) \quad \sum_{i=1}^n u_{jx_i}(P) \left[ w_{x_i}(P)/w(P) - \int_{B(P,r)} w_{x_i} d\rho / \int_{B(P,r)} w d\rho \right] = 0,$$

$j = 1, 2, \dots, 7$ ; for each  $(P, r) \in R^*$ .

Let  $(j_1, j_2, j_3, j_4)$  be a combination of four distinct integers taken from the seven positive integers  $1, 2, \dots, 7$ . The system of equations (1.10) gives rise to  ${}_7C_4 = 35$  Jacobians of the form

$$J_{j_1 j_2 j_3 j_4} = \frac{D(u_{j_1}, u_{j_2}, u_{j_3}, u_{j_4})}{D(x_1, x_2, x_3, x_4)}.$$

Let  $R_1$  be the subset of  $R$  to which a point  $P$  belongs if and only if at least one of the  ${}_7C_4 = 35$  Jacobians  $\{J_{j_1 j_2 j_3 j_4}\}$  does not vanish at  $P$ .  $R_1$  is a dense subset of  $R$ . For, if not, then there exists a point  $P$  and a neighborhood  $N(P)$  of the point  $P$  lying in  $R$  such that each of the 35 Jacobians  $\{J_{j_1 j_2 j_3 j_4}\}$  vanishes identically in  $N(P)$ . This again means that every first, second, and third minor of each of the 35 Jacobians  $\{J_{j_1 j_2 j_3 j_4}\}$  vanishes identically in  $N(P)$ . For, if possible, let one of the first minors of  $J_{1234}$ , say  $D(u_1, u_2, u_3)/D(x_1, x_2, x_3)$ , does not vanish identically in  $N(P)$ . Then there is a point  $Q$  and a neighborhood  $N(Q)$  of the point  $Q$  lying in  $N(P)$  such that  $D(u_1, u_2, u_3)/D(x_1, x_2, x_3) \neq 0$  in  $N(Q)$ . Hence considering also the Jacobians  $J_{1235}, J_{1236}, J_{1237}$ , there exists functional relations

$$(1.11) \quad \begin{aligned} u_4 &= \phi_4(u_1, u_2, u_3), \quad u_5 = \phi_5(u_1, u_2, u_3), \\ u_6 &= \phi_6(u_1, u_2, u_3), \quad u_7 = \phi_7(u_1, u_2, u_3) \end{aligned}$$

each valid in  $N(Q)$ . Also there is a functional relation

$$(1.12) \quad \phi(u_4, u_5, u_6, u_7) = 0,$$

valid in  $N(P)$ .

Therefore (1.11) and (1.12) imply that there is a functional relation

$$\psi(u_1, u_2, u_3) = 0,$$

valid in  $N(Q)$  contradicting the fact that  $D(u_1, u_2, u_3)/D(x_1, x_2, x_3) \neq 0$  in  $N(Q)$ . Hence every first minor of each of the 35 Jacobians vanishes identically in  $N(P)$ . If possible, suppose now that one of the second minors of  $J_{1234}$ , say,  $D(u_1, u_2)/D(x_1, x_2)$ , is not identically zero in  $N(P)$ . Then there exists a point  $Q$  and a neighborhood  $N(Q)$  of the point  $Q$  lying in  $N(P)$  such that  $D(u_1, u_2)/D(x_1, x_2)$

$\neq 0$  in  $N(Q)$ . Hence, considering also the Jacobian  $J_{1256}$ , there exists functional relations

$$(1.13) \quad u_3 = \phi_3(u_1, u_2), \quad u_4 = \phi_4(u_1, u_2), \quad u_5 = \phi_5(u_1, u_2), \quad u_6 = \phi_6(u_1, u_2)$$

valid in  $N(Q)$ . (1.12) and (1.13) lead to functional relation  $\psi(u_1, u_2) = 0$  valid in  $N(Q)$ , contradicting the nonvanishing of  $D(u_1, u_2)/D(x_1, x_2)$  in  $N(Q)$ . Hence every second minor of each of the 35 Jacobians  $\{J_{j_1 j_2 j_3 j_4}\}$  vanishes identically in  $N(P)$ . Finally suppose that one of the third minors of  $J_{1234}$ , say  $\partial u_1/\partial x_1$  is not identically zero in  $N(P)$ . Then there exists functional relations of the form

$$u_2 = \phi_2(u_1), \quad u_3 = \phi_3(u_1), \quad u_4 = \phi_4(u_1),$$

valid in some neighborhood  $N(Q)$  of a point  $Q$  lying in  $N(P)$ . Now applying similar arguments as in Theorem 2, we get a relationship  $u_2 = c_1 u_1 + c_2$  valid in  $R$ , where  $c_1$  and  $c_2$  are real constants which contradicts the linear independence assumption. Therefore every third minor of each of the 35 Jacobians  $\{J_{j_1 j_2 j_3 j_4}\}$  vanishes identically in  $N(P)$ , which means that each of the functions  $u_1, u_2, u_3, \dots, u_7$  is constant in  $N(Q)$  and hence, by the maximum principle of the elliptic equation (0.10), in  $R$ , implying again a contradiction. Therefore  $R_1$  is a dense subset of  $R$ .

Now arguing similarly as in Theorem 2, we conclude that each of the functions  $w_{x_i}/w$ ,  $i = 1, 2, 3, 4$ , belong to  $S(w, R)$ . Hence by Theorem 1, there exists a real constant  $\lambda$  such that  $w \in C^2(R)$  and is a solution of (1.0) in  $R$ . Hence  $S(w, R)$  is infinite dimensional by Corollary 2\*. It is also clear that if  $S(w, R)$  is finite dimensional then  $1 \leq \dim S(w, R) \leq 2 \cdot 4 - 1 = 7$ .

**II. Derivatives of functions satisfying W.A.P.** It is known that derivatives of harmonic functions, defined in a given region  $R$  of  $E_n$ , are also harmonic in that region. But this is not, in general, true for functions satisfying W.A.P. unless they satisfy a similar W.A.P. with respect to the derivatives of the W.F. For example, consider the W.F.,  $w(x, y) = x + y$ , defined in  $R$ , where  $R$  is the first quadrant of the plane  $E_2$ . The function  $u(x, y) = x^2 - 4xy + y^2$  is a solution of

$$(2.0) \quad w\Delta u + 2(w_x u_x + w_y u_y) = 0$$

in  $R$ . Also  $u \in C^2(R)$ . Since  $w \in C^2(R)$  and is a solution of  $\Delta w = 0$  in  $R$ , by Theorem 5\*,  $u \in S(w, R)$ . But  $u_x = 2x - 4y$  and  $u_y = -4x + 2y$ , do not satisfy (2.0) and hence cannot satisfy W.A.P. with respect to  $w$  in  $R$ .

On the other hand  $w(x, y) = \exp(x + y)$  is a W.F. in  $E_2$ ,  $w \in C^2(E_2)$  and is a solution of

$$\Delta w - 2w = 0$$

in  $E_2$ . The function  $u(x, y) = x^2 - 2xy + y^2 - x - y$  belongs to class  $C^2(E_2)$  and is a solution of

$$w\Delta u + 2w_x u_x + 2w_y u_y = 0$$

in  $E_2$ . Hence by Theorem 5\*,  $u \in S(w, E_2)$ . Also each of the derivatives  $u_x = 2x - 2y - 1$ ,  $u_y = -2x + 2y - 1$  belongs to class  $C^2(E_2)$  and is a solution of (2.0). Therefore each of the derivatives  $u_x$  and  $u_y$  belongs to  $S(w, E_2)$ .

**THEOREM 4.** *Let  $R$  be a region in  $E_n$  and  $w$  be a W.F. belonging to class  $C^1(R)$ . If  $u \in S(w, R)$ , then a necessary and sufficient condition that the partial derivative  $\partial u / \partial x_i = u_{x_i}$ ,  $1 \leq i \leq n$ , will belong to  $S(w, R)$  is that*

$$(2.1) \quad \int_{B(P,r)} uw_{x_i} d\rho = u(P) \int_{B(P,r)} w_{x_i} d\rho$$

for each  $(P, r) \in R^*$ .

**Proof.** By Remark 2\*, the partial derivative  $u_{x_i}$  satisfies the relation

$$(2.2) \quad u_{x_i}(P) \int_{B(P,r)} w d\rho = \int_{B(P,r)} u_{x_i} w d\rho + \left[ \int_{B(P,r)} uw_{x_i} d\rho - u(P) \int_{B(P,r)} w_{x_i} d\rho \right]$$

for each  $(P, r) \in R^*$ . It is clear from (2.2) that the theorem is true.

**REMARK 1.** If  $w$  be a positive constant, then  $S(w, R)$  is the class of all harmonic functions defined in  $R$  and  $w_{x_i} = 0$ , for  $i = 1, 2, \dots, n$ , implies that the relation (2.1) is true for constant W.F., which simply means that derivatives of harmonic functions are also harmonic as is well known.

**THEOREM 5.** *Let the W.F.  $w$  belong to class  $C^2(R)$  and be solution of*

$$(2.3) \quad \Delta w + \lambda w = 0$$

*in  $R$ , where  $\lambda$  is a real constant. If  $u \in S(w, R)$  then these are equivalent:*

- (i)  $u_{x_i}$ ,  $1 \leq i \leq n$ , belongs to  $S(w, R)$
- (ii)  $uw_{x_i}/w$  belongs to  $S(w, R)$
- (iii)  $uw_{x_i}$  is a solution of (2.3) in  $R$
- (iv)  $u_{x_i} w$  is a solution of (2.3) in  $R$
- (v)  $u$  is a solution of

$$(2.4) \quad w_{x_i} \Delta u + 2 \sum_{j=1}^n u_{x_j} w_{x_i x_j} = 0$$

in  $R$ .

**Proof.** By hypothesis and from Theorem 2\*, each of  $u$  and  $w$  is analytic in  $R$ . Also, by Theorem 1,  $w(P) > 0$  for each point  $P = P(x_1, x_2, \dots, x_n)$  in  $R$  and each of the functions  $w_{x_i}/w$ ,  $i = 1, 2, \dots, n$ , belongs to  $S(w, R)$ .

Now suppose that (i) is true. Using Remark 3\* we can write

$$u_{x_i}(P) \int_{B(P,r)} w d\rho = \int_{B(P,r)} \{u_{x_i} + (uw_{x_i})/w\} w d\rho - u(P) \int_{B(P,r)} \{w_{x_i}/w\} w d\rho,$$

or

$$\{u_{x_i}(P) + u(P)w_{x_i}(P)/w(P)\} \int_{B(P,r)} wd\rho = \int_{B(P,r)} \{u_{x_i} + (uw_{x_i})/w\} wd\rho$$

for each  $(P, r) \in R^*$  which implies that  $u_{x_i} + uw_{x_i}/w \in S(w, R)$ . Since  $S(w, R)$  is a linear space,  $uw_{x_i}/w \in S(w, R)$ . Hence (i) implies (ii). Next suppose that (ii) is true. Then by Corollary 1\*,  $uw_{x_i}$  is a solution of (2.3). Therefore (ii) implies (iii). Now suppose that (iii) is true. By hypothesis and from Corollary 1\*,  $uw$  is a solution of (2.3) and hence  $(uw)_{x_i} = u_{x_i}w + uw_{x_i}$  is also a solution of (2.3). We conclude from the linearity of (2.3) that  $u_{x_i}w$  is a solution of (2.3). Thus (iii) implies (iv).

Next suppose that (iv) is true. We have

$$(2.5) \quad \Delta(u_{x_i}w) + \lambda(u_{x_i}w) = 0.$$

Also by hypothesis  $\Delta w + \lambda w = 0$  and

$$(2.6) \quad w\Delta u + 2 \sum_{j=1}^n u_{x_j} w_{x_j} = 0.$$

Differentiation of (2.6) with respect to  $x_i$  yields

$$\left\{ w_{x_i} \Delta u + 2 \sum_{j=1}^n u_{x_j} w_{x_i x_j} \right\} + \{\Delta(u_{x_i}w) + \lambda(u_{x_i}w)\} = 0$$

or

$$w_{x_i} \Delta u + 2 \sum_{j=1}^n u_{x_j} w_{x_i x_j} = 0.$$

Therefore (iv) implies (v).

Finally suppose that (v) is true. Now differentiation of (2.6) with respect to  $x_i$  gives

$$\left\{ w_{x_i} \Delta u + 2 \sum_{j=1}^n u_{x_j} w_{x_i x_j} \right\} + \left\{ w\Delta u_{x_i} + 2 \sum_{j=1}^n u_{x_i x_j} w_{x_j} \right\} = 0.$$

Therefore we have  $w\Delta u_{x_i} + 2 \sum_{j=1}^n u_{x_i x_j} w_{x_j} = 0$ , which by Theorem 5\* means that  $u_i \in S(w, R)$ . This completes the cycle.

*Exponential Weight Functions, Definition.* Weight functions of the form

$$w(x_1, x_2, \dots, x_n) = k \exp(a_1 x_1 + a_2 x_2 + \dots + a_n x_n),$$

$(x_1, x_2, \dots, x_n) \in E_n$ , where  $a_i$ 's are real constants and  $k$  is a positive real constant, are called exponential weight functions.

*Properties of exponential weight functions.* If  $w$  be an exponential W.F., then (2.7)  $w$  is an analytic W.F. in any subregion  $R$  of  $E_n$ ,

(2.8)  $w$  is a solution of  $\Delta w + \lambda w = 0$  in  $E_n$ , where  $\lambda = -\sum_{i=1}^n a_i^2$ .

(2.9)  $w_{x_i}/w = a_i$ ,  $i = 1, 2, \dots, n$ .

The differential equation  $\Delta u + 2 \sum_{i=1}^n a_i u_{x_i} = 0$ .

**THEOREM 6.** For an exponential W.F.  $w(x_1, x_2, \dots, x_n) = k \exp(\sum_{i=1}^n a_i x_i)$ , a necessary and sufficient condition that  $u \in S(w, R)$  is that  $u \in C^2(R)$  and is a solution of

$$(2.10) \quad \Delta u + 2 \sum_{i=1}^n a_i u_{x_i} = 0,$$

in  $R$ .

**Proof.** The theorem follows at once from Theorem 5\*.

**THEOREM 7.** Let  $w(x_1, x_2, \dots, x_n) = k \exp(\sum_{i=1}^n a_i x_i)$  be an exponential W.F. If  $u \in S(w, R)$ , then

- (i)  $u$  is analytic in  $R$  and
- (ii) if  $m$  be a positive integer and  $D^m u$  be any  $m$ th order partial derivative of  $u$ , then  $D^m u \in S(w, R)$ .

**Proof.** Part (i) is an immediate consequence of Theorem 2\*. Again  $S(w, R)$  is a linear space and  $u \in S(w, R)$  implies that each of the functions  $uw_{x_i}/w = a_i u$ ,  $i = 1, 2, \dots, n$ , also belongs to  $S(w, R)$ . Hence by Theorem 5 each of the derivatives  $u_{x_i}$ ,  $i = 1, 2, \dots, n$ , belongs to  $S(w, R)$ . Applying mathematical induction, we see that the theorem is true.

**Characterisation of exponential W.F.** Property (2.9) and part (ii) of Theorem 7 characterises exponential weight function in the sense of the following theorem:

**THEOREM 8.** Let  $w$  be a W.F. belonging to class  $C^2(R)$  and is a solution of  $\Delta w + \lambda w = 0$  for some real constant. Then the following are equivalent:

- (i) If  $u \in S(w, R)$ , then each of the derivatives  $u_{x_i}$ ,  $i = 1, 2, \dots, n$ , also belongs to  $S(w, R)$ .
- (ii)  $w$  is an exponential W.F.

**Proof.** By Theorem 1,  $w(P) = w(x_1, x_2, \dots, x_n) > 0$  for all  $P(x_1, x_2, \dots, x_n)$  in  $R$  and each of the functions  $w_{x_i}/w \in S(w, R)$ ,  $i = 1, 2, \dots, n$ . Suppose that (i) is true. Then each of the partial derivatives  $(\partial/\partial x_i)(w_{x_i}/w)$ ,  $i = 1, 2, \dots, n$ , also belongs to  $S(w, R)$ . Therefore by Theorem 5 each of the functions  $(w_{x_i}/w)^2$  belongs to  $S(w, R)$ . Let  $P_0 \in R$ . Since  $S(w, R)$  is a linear space, each of the functions  $F_i = \{w_{x_i}/w - w_{x_i}(P_0)/w(p_0)\}^2$ ,  $i = 1, 2, \dots, n$ , also belongs to  $S(w, R)$ . Since  $F_i$  has a minimum (zero) at  $P_0$  in  $R$ , it follows from the minimum-principle of the differential equation

$$w\Delta u + 2 \sum_{i=1}^n w_{x_i} u_{x_i} = 0$$

that  $w_{x_i} = a_i w$ ,  $i = 1, 2, \dots, n$ , where  $a_i$ 's are real constants. Hence  $w(x_1, x_2, \dots, x_n)$

$= k \exp(\sum_{i=1}^n a_i x_i)$ , for each  $(x_1, x_2, \dots, x_n) \in R$ , where  $k$  is a positive real constant. Therefore (i) implies (ii). Now suppose that (ii) is true. Let  $w(x_1, x_2, \dots, x_n) = k \exp(\sum_{i=1}^n a_i x_i)$  and  $u \in S(w, R)$ . Then by Theorem 7 each of the derivatives  $u_{x_i}, i = 1, 2, \dots, n$ , belongs to  $S(w, R)$ . Hence (ii) implies (i). This completes the proof.

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