Let $G$ be a locally compact group with a left Haar measure $\mu$. Let $p$ and $r$ be two real numbers $> 1$. By $L^{r,p}$ conjecture we mean the following assertion: whenever $f \in L^r(G)$ and $g \in L^p(G)$ we have that the convolution product $f * g$ of $f$ and $g$ is defined and belongs to $L^q(G)$ again if and only if $G$ is compact. By $L^p$ conjecture we mean the assertion above with $r = p$. Both the conjectures were widely believed to be true though there was no written statement about these conjectures until recently. But in 1960, Kunze and Stein [3] showed that the $L^{r,p}$ conjecture is false for the unimodular group of $2 \times 2$ real matrices. This naturally raises the question whether the $L^p$-conjecture is true in general. The first published result on the $L^p$-conjecture is by Zelazko [9] and Urbanik [7] in 1961. They proved the conjecture to be true for the abelian case. Then in [5] the author established the truth of the $L^p$-conjecture for discrete groups when $p \geq 2$. The author announced in that paper that the conjecture is true for all groups when $p > 2$ and presented this result to Amer. Math. Soc. in August of 1963 at the Boulder meeting. At the same time Zelazko [10] established the conjecture for all $p > 2$ for all unimodular groups. He claims to have established the conjecture for $p > 2$ for all groups in that paper but his crucial Lemma 1 of that paper contains a gap in the proof. In a private communication, Zelazko agreed to this gap. The problem is still open in general when $p > 1$.

In this paper we prove the following:

The $L^p$-conjecture is true for all locally compact groups when $p > 2$.

The $L^p$-conjecture is true for totally disconnected groups when $p = 2$.

The methods used in this paper yield the truth of the conjecture for all nilpotent groups, and all unimodular $C$-groups of Iwasawa when $p > 1$. But this result will appear elsewhere.

This is a portion of the author's doctoral dissertation submitted to Yale University in 1963 under the guidance of Professor C. E. Rickart. The author had a grant AFOSR 62–20 from the Air Force when this work was done. The author thanks Professor C. Ionescu-Tulcea for having brought the paper [9] of Zelazko to his attention.

Notations and conventions. All purely topological notions are taken from [2]. All topological spaces occurring in this paper are taken to be Hausdorff. All notions in topological groups and integration on locally compact groups are
taken in general from [8]. By a normal algebra we mean a Banach space which is also a ring where multiplication is bicontinuous. A Banach algebra is a normed algebra where we have further the inequality \( \|xy\| \leq \|x\| \|y\| \) for all elements ‘\( x \)’ and ‘\( y \)’ of the algebra. The symbols \( \mu, \nu, \theta \) are used for measures. When only one left Haar measure \( \mu \) is used on a locally compact group \( G \) we write sometimes \( \int_G f(x) \, dx \) or \( \int f(x) \, dx \) instead of \( \int_G f(x) \, d\mu(x) \). If \( H \subseteq G \) is a subset of a group \( G \), then \( \chi_H(x) \) will denote the characteristic function of \( H \). \( L(G) \) will denote the class of complex valued continuous functions on \( G \) with compact support.

If \( G \) is a locally compact group with a left Haar measure ‘\( \mu \)’ and \( 1 \leq p < \infty \) then \( L^p(G) \) will denote the equivalence classes of Borel measurable functions \( f \) on \( G \) with complex values such that \( \int |f(x)|^p \, dx < \infty \). If \( f \in L^p(G) \) then \( \|f\|_p \) will denote \( (\int_G |f|^p \, dx)^{1/p} \) when \( 1 \leq p \leq \infty \).

1. \( L^p \)-conjecture for the case \( p > 2 \), and some general results.

**Definition 1.1.** Let \( G \) be a locally compact group with a left Haar measure ‘\( \mu \)’. Let \( f \) and \( g \) be two Borel measurable complex valued functions on \( G \). Then the convolution \( f \ast g \) of ‘\( f \)’ and ‘\( g \)’ is said to exist if the integral \( \int_G |f(y)g(y^{-1}x)| \, dy \) exists for almost all \( x \in G \). In this case \( f \ast g(x) \) is defined to be \( \int_G f(y)g(y^{-1}x) \, dy \). If \( 1 \leq p < \infty \), we say that \( L^p(G) \) is closed under convolution if whenever \( f \) and \( g \) belong to \( L^p(G) \) we have that \( f \ast g \) exists and again belongs to \( L^p(G) \).

\( L^p \)-conjecture 1.2. This is the following statement: Let \( G \) be a locally compact group with a left Haar measure \( \mu \). Then \( L^p(G) \) is closed under convolution for some \( p \) such that \( 1 < p < \infty \) if and only if \( G \) is compact.

**Remark.** The “if” part of the \( L^p \)-conjecture is trivial to establish. So we consider the “only if” part in this paper.

**Theorem 1.3.** Let \( G \) be a locally compact group with a left Haar measure ‘\( \mu \)’. Let \( L^p(G) \) be closed under convolution for some \( p > 1 \) and \( < \infty \). Then \( L^p(G) \) is a normed algebra with convolution as multiplication. Moreover, in this case we can choose a suitable left Haar measure ‘\( \mu_1 \)’ such that \( L^p(\mu_1) \) is a Banach algebra.

**Proof.** Let \( 1/p + 1/q = 1 \) and \( \Delta(x) \) the modular function of \( G \). Let \( f, g \in L^p(G) \) and \( h \in L^q(G) \). Let \( (f, h) = \int f(x)h(x) \, dx \) and let \( T_f \) be the operator \( g \to f \ast g \) in \( L^p(G) \). Let \( f(x) = f(x^{-1})\Delta(x^{-1}) \). Then by a routine calculation it follows that \( T_f(g), h) = (f \ast g, h) = (g, f \ast h) \) for all \( f, g \in L^p(G) \) and \( h \in L^q(G) \). So by an easy application of the closed graph theorem we get that \( T_f \) is continuous in \( L^p(G) \). Similarly we get that the right multiplication is continuous in \( L^p(G) \). So by an application of the principle of uniform boundedness we get \( L^p(G) \) is a normed algebra. So there is a constant \( K \) such that \( \|f \ast g\|_p \leq K \|f\|_p \|g\|_p \) for all \( f, g \in L^p(G) \). Now choose a left Haar measure ‘\( \mu_1 \)’ on \( G \) by the relation \( d\mu_1(x) = K^p \, d\mu(x) \). Then \( L^p(\mu_1) \) will be a Banach algebra under convolution.
Lemma 1.4. Let $G$ be a locally compact group with a left Haar measure $\mu$. Let $H \subset G$ be an open subgroup of $G$. Let $L^p(G)$ be closed under convolution for some $p > 1$. Then $L^p(H)$ is also closed under convolution. If $G$ is the direct product $G_1 \times G_2$ of two closed subgroups $G_1$ and $G_2$ with left Haar measures $\mu_1$ and $\mu_2$ respectively and if $L^p(G)$ is closed under convolution then $L^p(G_1)$ and $L^p(G_2)$ are also closed under convolution.

Proof. Obvious.

Lemma 1.5. Let $G$ be a locally compact group with a left Haar measure $\mu$. Let $\lambda$ be a number such that $1 < \lambda < \infty$. Let $H \subset G$ be a compact normal subgroup of $G$. Let $L^\lambda(G)$ be closed under convolution. Then $L^\lambda(G/H)$ is also closed under convolution.

Proof. Let $\nu$ be the normalized Haar measure of $H$ and $\phi: G \to G/H$ be the canonical map from $G$ onto $G/H$. Let $\theta$ be a left Haar measure on $G/H$ such that the relation $\int_G f(x) \, d\mu(x) = \int_{G/H} \left( \int_H f(tx) \, d\theta(t) \right) \, d\phi(x)$ holds for all $f \in L(G)$ where $\phi(x) = \bar{x}$. Then the following relations are easily deduced: If $T(f) = \int_H f(tx) \, d\nu(t)$ then

1. $Tf \in L(G/H)$ whenever $f \in L(G)$.
2. $T$ is linear from $L(G)$ onto $L(G/H)$.
3. $T(f * g) = T(f) * T(g)$ for all $f, g \in L(G)$.
4. $\|Tf\|_p = \|f\|_p$ for all $f \in L(G)$.

Now $L^\lambda(G)$ is closed under convolution. So there is a constant $K$ such that $\|f \ast g\|_p \leq K \|f\|_p \|g\|_p$ from Theorem 1.3. So we have that $\|f \ast g\|_p \leq K \|f\|_p \|g\|_p$ for all $f$ and $g \in L(G/H)$ from 1, 2, 3, and 4 above. Since $p < \infty$, we have that $L(G/H)$ is dense in $L^\lambda(G/H)$. Then we get by repeated use of Fatou's lemma, Fubini's theorem and monotone convergence theorem that if $\tilde{f}$ and $\tilde{g}$ belong to $L^\lambda(G/H)$ then $\tilde{f} \ast \tilde{g}$ is defined and again belongs to $L^\lambda(G/H)$.

Lemma 1.6. Suppose that the $L^p$-conjecture is true for a number $p$ ($1 < p < \infty$) for all totally disconnected locally compact groups and all connected Lie groups. Then the conjecture is true for that $p$ for all locally compact groups.

Proof. By a theorem of Yamabe [4] every locally compact group $G$ contains an open subgroup $H$ and a compact normal subgroup $N \subset H$ such that $H/N$ is a connected Lie group ($N$ is normal with respect to $H$). So if $L^p(G)$ is closed under convolution then $L^p(H)$ is closed under convolution by Lemma 1.4. So $L^p(H/N)$ is closed under convolution by Lemma 1.5. So $H/N$ is compact by hypothesis of the lemma. So the connected component $G_0$ of the identity 'e' of $G$ is compact. So $L^p(G/G_0)$ is closed under convolution by Lemma 1.5. So $G/G_0$ is compact by hypothesis of the lemma. So $G$ is compact.
Lemma 1.7. Let $G$ be a locally compact group with a left Haar measure $\mu$. Let $V$ be a compact symmetric neighborhood of the identity $e$ of $G$ such that the group generated by $V$ is not compact. Then the following are true:

(i) If the set \( \{ \mu(V^{n+1})/\mu(V^n) : n=1, 2, \ldots \} \) is bounded then $L^p(G)$ is not closed under convolution for any $p>2$.

(ii) If the set \( \{ \mu(V^{2n})/\mu(V^n) : n=1, 2, 3, \ldots \} \) is bounded then $L^p(G)$ is not closed under convolution for any $p>1$.

Proof. Let there be a constant $k>0$ such that $\frac{\mu(V^{n+1})}{\mu(V^n)} \leq k$ for all $n = 1, 2, 3, \ldots$. Let $\chi_{V^n}(x)$ be the characteristic function of $V^n$ for $n=1, 2, 3, \ldots$. Then

\[
\chi_{V^n} \ast \chi_{V^{n+1}}(x) = \int_G \chi_{V^n}(y) \chi_{V^{n+1}}(y^{-1}x) \, d\mu(y)
\]

\[
\geq \mu(V^n) \chi_V(x) \quad \text{for all } x \in G \text{ and } n = 1, 2, \ldots
\]

So $\|\chi_{V^n} \ast \chi_{V^{n+1}}\|_p \geq \mu(V^n) (\mu(V))^{1/p}$. So

\[
\frac{\|\chi_{V^n} \ast \chi_{V^{n+1}}\|_p}{\|\chi_{V^n}\|_p \|\chi_{V^{n+1}}\|_p} \geq \frac{\mu(V^n) (\mu(V))^{1/p}}{(\mu(V^n))^{1/p} (\mu(V^{n+1}))^{1/p}}
\]

\[
= \left( \frac{\mu(V^n)}{\mu(V^{n+1})} \right)^{1/p} (\mu(V))^{1/p} (\mu(V^n))^{1-(2/p)}
\]

\[
\geq \left( \frac{\mu(V)}{k} \right)^{1/p} (\mu(V^n))^{1-(2/p)}.
\]

Thus $\lim_{n \to \infty} \left( \|\chi_{V^n} \ast \chi_{V^{n+1}}\|_p / \|\chi_{V^n}\|_p \|\chi_{V^{n+1}}\|_p \right) = \infty$ if $p>2$. Thus (i) follows from Theorem 1.3. The statement (ii) can be proved likewise.

Lemma 1.8. Let $G$ be a totally disconnected locally compact group with a left Haar measure $\mu$. Let $L^p(G)$ be closed under convolution for a real number $p \ (1 < p < \infty)$. Then there is a maximal compact open subgroup $H$ of $G$. (That is $H$ is a compact, open subgroup of $G$ and any open compact subgroup of $G$ containing $H$ is $H$ itself.)

Proof. Since $G$ is totally disconnected, there are compact open subgroups in $G$ (see p. 54 of [4]). Suppose there is an ascending sequence $H_1 \leq H_2 \leq \cdots \leq H_n \leq \cdots$ of compact open subgroups $H_1, H_2, \ldots, H_n, \ldots$ of $G$. Let $\chi_{H_n}(x) = 1$ if $x \in H_n$ and 0 if $x \in G - H_n$. Put $\varphi_n(x) = (\chi_{H_n}(x)/\mu(H_n))$ for $n = 1, 2, 3, \ldots$. Then $\varphi_n \ast \varphi_n = \varphi_n$ for all $n = 1, 2, 3, \ldots$. So

\[
\frac{\|\varphi_n \ast \varphi_n\|_p}{\|\varphi_n\|_p^2} = \frac{\|\varphi_n\|_p}{\|\varphi_n\|_p^2} = \frac{1}{\|\varphi_n\|_p} = \frac{\mu(H_n)}{(\mu(H_n))^{1/p}} = (\mu(H_n))^{1-(1/p)}.
\]
By Theorem 1.3 the set \( \{\mu(H_n) \mid n = 1, 2, 3, \ldots\} \) is bounded. Hence there is an \( n_0 \) such that \( H_{n_0} = H_{n_0 + 1} = \cdots \). So every ascending sequence of compact, open subgroups of \( G \) is finite. Hence the lemma.

**Lemma 1.9.** Let \( G \) be a totally disconnected locally compact group with a left Haar measure \( \mu \). Let \( p \) be a real number \( 2 < p < \infty \). Let \( L^p(G) \) be closed under convolution. Then \( G \) contains an open compact normal subgroup.

**Proof.** By Lemma 1.8 there is a maximal, compact, open subgroup \( H \). Now take any element \( \alpha \in G - H \) and consider the group generated by \( H \cup \alpha^{-1}H \). This group should be compact. If not put \( V = H \cup \alpha^{-1}H \). Then \( V \) is a compact symmetric open neighborhood of the identity \( e \in G \). Since \( V^2 \) is compact there is a finite number of elements \( a_1, a_2, \ldots, a_k \) of \( G \) such that \( V^{n+1} \subset (a_1V) \cup (a_2V) \cup \cdots \cup a_kV \). So \( \mu(V^{n+1}) \leq \mu(V^n) \) for all \( n = 1, 2, 3, \ldots \). So \( \mu(V^{n+1})/\mu(V^n) \leq k \) for all \( n = 1, 2, 3, \ldots \). Then \( L^p(G) \) cannot be closed under convolution by Lemma 1.7 and Lemma 1.4 which contradicts our hypothesis on \( G \). Since \( H \) is a maximal open compact subgroup of \( G \) we get that \( H \cup \alpha^{-1}H \subset H \). So \( \alpha^{-1}H \subset H \) for all \( \alpha \in G \). So \( H \) is a compact, open, normal subgroup of \( G \).

**Theorem 1.10.** Let \( G \) be a locally compact group with a left Haar measure \( \mu \). Let \( L^p(G) \) be closed under convolution for a real number \( p \) \( 2 < p < \infty \). Then \( G \) must be compact.

**Proof.** Let us assume first that \( G \) is connected. Let \( V \) be a compact symmetric neighborhood of the identity \( e \). Then adopting the proof of Lemma 1.9 we get that the set \( \{\mu(V^{n+1})/\mu(V^n) \mid n = 1, 2, 3, \ldots\} \) is bounded. Then by the connectedness of \( G \) and by Lemma 1.7 we get that \( G \) is compact. Now let us assume that \( G \) is totally disconnected. Then by Lemma 1.9 there exists a compact, open, normal subgroup \( H \) of \( G \). Then \( G/H \) is a discrete group and \( L^p(G/H) \) is closed under convolution by Lemma 1.5. Then \( G/H \) is finite by Theorem 3 of [5]. Then \( G \) must be compact. So if \( G \) is either connected or totally disconnected the theorem is true. Now the result follows from Lemma 1.6.

2. **The case \( p = 2 \) of the \( L^p \)-conjecture.**

**Definition 2.1.** An involution \( * \) in an algebra \( A \) over complex numbers is a one-to-one map from \( A \) onto \( A \) such that the following hold:

(i) \( (x^*)^* = x \) for all \( x \in A \).

(ii) \( (\lambda x + \mu y)^* = \lambda^* x^* + \mu^* y^* \) for all complex numbers \( \lambda \) and \( \mu \) and \( x, y \in A \).

(iii) \( (xy)^* = y^* x^* \) for all \( x, y \in A \).

An \( A^* \)-algebra is a Banach algebra \( B \) with an involution \( * \) and an auxiliary norm \( \| \cdot \| \) such that \( \| xy \| \leq \| x \| \| y \| \) and \( \| xx^* \| = \| x \|^2 \) for all \( x, y \in B \).
Let $B$ be a Banach algebra over the complex numbers. An element $x$ is said to be in the radical of $B$ if there is an ideal $I \subseteq B$ such that $x \in I$ and $\lim_{n \to \infty} (\|y^n\|)^{1/n} \to 0$ as $n \to \infty$ for all $y \in I$. The algebra $B$ is said to be semisimple if 0 is the only element in the radical of $B$.

**Lemma 2.2.** Let $G$ be a unimodular locally compact group with a left Haar measure $\mu$. Let $L^p(G)$ be closed under convolution for some $p$ ($1 < p < \infty$). Then $L^p(G)$ is a semisimple Banach algebra assuming that $\mu$ was properly chosen to make $L^p(G)$ a Banach algebra.

**Proof.** Let $1/p + 1/q = 1$. Let $f^*(x) = \overline{f(x^{-1})}$ for all $f \in L^p(G)$. Then, from the fact that $G$ is unimodular, one can check that $f \mapsto f^*$ is an involution in $L^p(G)$. Moreover, by using standard theorems on integration one can show that $(f \ast g, h) = (g, f^* \ast h) = (f, h \ast g^*)$ for all $f, g \in L^p(G)$ and $h \in L^q(G)$ where $(f, h) = \int G f(x)h(x) \, dx$.

From this it follows easily that if $f \in L^p(G)$ and $g \in L^q(G)$ then $f \ast g \in L^q(G)$ and $\|f \ast g\|_q \leq \|f\|_p \|g\|_q$. From this and the fact that $L^p(G)$ is a Banach algebra and the Riesz convexity theorem it follows that $f \ast g \in L^q(G)$, and $\|f \ast g\|_2 \leq \|f\|_p \|g\|_2$ for all $f \in L^p(G)$ and $g \in L^q(G)$. Now put $\|f\|_2 = \sup \{\|f \ast g\|_2 \mid g \in L^q(G) \text{ and } \|g\|_2 = 1\}$. Then it easily follows that $\|\cdot\|_2$ is a norm in $L^p(G)$ and $\|f \ast g\|_2 \leq \|f\|_2 \|g\|_2$ for all $f, g \in L^p(G)$. So $L^p(G)$ is an $A^*$-algebra. So it is semisimple by a theorem of Rickart (Theorem 4.1.15 of [6]).

**Theorem 2.3.** Let $G$ be a totally disconnected locally compact group with a left Haar measure $\mu$. Let $L^2(G)$ be closed under convolution. Then $G$ is compact.

**Proof.** Assume for the moment that $G$ is unimodular. We may as well assume that $\mu$ was properly chosen so as to make $L^2(G)$ a Banach algebra. Let $f^*(x) = f(x^{-1})$ for all $f \in L^2(G)$. Then, as was shown in the proof of Lemma 2.2, $*$ is an involution in $L^2(G)$ and $(f \ast g, h) = (g, f^* \ast h) = (f, h \ast g^*)$ for all $f, g, h \in L^2(G)$ where $(f, g)$ is the inner product in $L^2(G)$. By Lemma 2.2 we have that $L^2(G)$ is semisimple and hence it is a semisimple $H^*$-algebra of Ambrose (see [1]). Now let $K$ be a compact, open subgroup of $G$ and let $\varphi(x) = \chi_K(x)/\mu(K)$ where $\chi_K(x)$ is the characteristic function of $K$.

Then $\varphi = \varphi^*$ and $\varphi \ast \varphi = \varphi$ and $\varphi \in L^2(G)$. So $\varphi \ast L^2(G) \ast \varphi$ is a semisimple $H^*$-algebra with an identity element and hence is finite dimensional (see [1]). But $\varphi \ast L^2(G) \ast \varphi$ consists exactly of those functions in $L^2(G)$ which are constant on double cosets modulo $K$. So the number of such cosets has to be finite and hence $G$ must be compact.

In the general case let $\Delta(x)$ be the modular function of $G$ and let

$$H = \{x \mid \Delta(x) = 1; x \in G\}.$$
Then $H$ contains all compact, open subgroups of $G$. So $H$ is an open subgroup of $G$. So $L^2(H)$ is closed under convolution by Lemma 1.4. Clearly $H$ is unimodular. So $H$ is compact by what was shown above. So $L^2(G/H)$ is closed under convolution, by Lemma 1.5. But $G/H$ is a discrete subgroup of the reals. So it is finite by Theorem 2 of [5]. So $G$ is compact again.

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