THE FREDHOLM METHOD IN POTENTIAL THEORY(1)

BY
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Introduction. Let $G$ be an open set with a compact boundary $B$ in $\mathbb{R}^m$, the Euclidean $m$-space. If $h$ is a harmonic function in $G$ such that

\[(1) \int_P |\text{grad } h(x)| \, dx < \infty\]

for every bounded open set $P \subset G$, one may form the distribution $Nh$ over the space $D$ of all infinitely differentiable functions $\psi$ with compact support in $\mathbb{R}^m$ defining

\[\langle \psi, Nh \rangle = \int_G \text{grad } \psi(x) \cdot \text{grad } h(x) \, dx.\]

This distribution will be termed the generalized normal derivative of $h$ (compare [CC], [M], [Y]). It is easily seen that $Nh$ has support in $B$. In general, $Nh$ need not be a measure in the sense usual in distribution theory [S]. §1 of the present paper deals with generalized normal derivatives of Newtonian potentials. We denote by $C^*(B)$ the Banach space of all finite signed Borel measures with support in $B$; total variation is taken as a norm in $C^*(B)$. With every $\mu \in C^*(B)$ we associate the corresponding Newtonian potential

\[U_\mu(x) = \int_{\mathbb{R}^m} p(x-y) \, d\mu(y),\]

where $p(z) = |z|^{2-m}/m-2$ or $p(z) = \log (1/|z|)$ according as $m > 2$ or $m = 2$, and we ask what necessary and sufficient condition is to be imposed on $G$ in order that $NU_\mu$ be a measure for every $\mu \in C^*(B)$. For this purpose it is useful to introduce the concept of a hit of a half-line $\{y+t\theta : t > 0\}$ on $G$ (cf. Definition 1.5). If $n(\theta, y)$ denotes the number of such hits, then $n(\theta, y)$ is a Baire function of the variable $\theta$ on $\Gamma = \mathbb{R}^m \cap \{\theta : |	heta| = 1\}$ and the above mentioned condition reads as follows:

\[(2) \sup_{y \in B} \int_\Gamma n(\theta, y) \, dH_{m-1}(\theta) < \infty,\]

where $H_{m-1}$ stands for the $(m-1)$-dimensional Hausdorff measure.

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If $G$ fulfills (2), then the operator
$$NU: \mu \rightarrow NU\mu$$
is bounded on $C^*(B)$ and has the form $\frac{1}{2}AI + \overline{W}^*$, where $A = H_{m-1}(\Gamma)$, $I$ is the identity operator and $\overline{W}^*$ is adjoint to an operator $\overline{W}$ acting on the space $C(B)$ of all continuous functions on $B$. Some properties of $\overline{W}$, which is connected with the classical double-layer potential, are investigated in §§2–3. In particular, in §3 we show that, in case $B$ has no isolated points, the Fredholm radius of $\overline{W}$ is the reciprocal of the quantity
\[ V_0 = \limsup_{r \to 0} \left\{ A[d(y) - \frac{1}{2}] + \int_\Gamma n_\gamma(\theta, y) dH_{m-1}(\theta) \right\}, \]
where $d(y)$ denotes the $m$-density of $G$ at $y$ and $n_\gamma(\theta, y)$ is the number of hits of $\{y + \theta t : 0 < t < r\}$ on $G$. Relations between $V_0$ and the geometric structure of $B$ are also investigated in §3. In case $V_0$ is sufficiently small, these results apply to the Neumann problem where the boundary condition is given by an arbitrary measure $\nu \in C^*(B)$, as treated in §4. By duality based on the Fredholm theory one obtains, as a by-product, representation of solutions of the Dirichlet problem by means of double-layer potentials.

Methods and concepts employed here are those of geometric measure theory; they have their origin in investigations connected with the Gauss-Green theorem, sets with finite perimeter and functions whose partial derivatives are measures [DG], [F], [FL], [FY], [KR], [MA], [P].

1. Normal derivatives of potentials.

1.1. Terminology and notation. The symbols $R^m$, $C^*(B)$, $p$, $U_\mu$, $D$ will have the meaning described in the introduction. For $M \subset R^m$ we shall denote by $\text{cl} M$, $\text{int} M$, $\text{fr} M$ and $\text{diam} M$ the closure, interior, boundary and diameter of $M$, respectively. $H_k$ will stand for the $k$-dimensional Hausdorff measure; $H_m$ coincides with the Lebesgue measure in $R^m$. We put $\Omega_r(y) = R^m \cap \{z : |z - y| < r\}$, $\Omega = \Omega_2(0)$, $\Gamma_r(y) = \text{fr} \Omega_r(y)$, $\Gamma = \Gamma_1(0)$, $A = H_{m-1}(\Gamma)$. Throughout this paragraph $G \subset R^m(m \geq 2)$ will be a fixed set with a compact boundary $B$. We shall tacitly assume that $G$ is open. On several places, however, it will be useful to allow $G$ to be a Borel set; this will be always pointed out explicitly.

The generalized normal derivative of a harmonic function $h$ (satisfying (1) for every bounded open $P \subset G$) is defined as in the introduction; we shall write $N^G h$ instead of $Nh$ if it is necessary to specify $G$. The reason for the terminology is obvious: if $G$ has a smooth boundary with exterior normal $n$ and $h$ is smooth up to $B$, then
\[
\langle \psi, Nh \rangle = \int_B \psi(\partial h/\partial n) dH_{m-1}.
\]
If spt $\psi$ (the support of $\psi$) does not meet $B$, then there is an open set $Q$ with a smooth boundary such that spt $\psi \cap G \subset Q$, cl $Q \subset G$, so that

$$\langle \psi, N^Q \rangle = \langle \psi, N^Q h \rangle = 0.$$ 

In particular, if $N^Q h$ is a (Borel) measure $\nu$, which means that

$$\langle \psi, N^Q h \rangle = \int_{\mathbb{R}^m} \psi \ d\nu$$

for every $\psi \in D$, then $\nu \in C^*(B)$.

Variation of a (signed) measure $\mu$ on a Borel set $M$ will be denoted by $|\mu|(M)$; for $\mu \in C^*(B)$, $|\mu|(B) = \|\mu\|$ is the norm of $\mu$.

Simple calculation shows that, for $\mu \in C^*(B)$ and $x \in G$,

$$|\text{grad } U\mu(x)| \leq \int_B |x-y|^{1-m} \ d|\mu|(y),$$

whence we obtain for any bounded Borel $P \subset G$

$$(1.1) \quad \int_P |\text{grad } U\mu(x)| \ dx \leq A \text{ diam } (B \cup P) \|\mu\|.$$ 

We see that $NU\mu$ is meaningful for every $\mu \in C^*(B)$. Our main objective in §1 is to answer the following question:

1.2. What necessary and sufficient restrictions are to be imposed on $G$ in order that $NU\mu$ be a measure for every $\mu \in C^*(B)$?

1.3. REMARK. Let us agree to denote by $\delta_y$ the Dirac measure concentrated at $y \in \mathbb{R}^m$. We have for any $\psi \in D$ and any $y \in B$

$$\langle \psi, NU \delta_y \rangle = \int g \text{ grad } \psi(x) \cdot \frac{y-x}{|y-x|^m} \ dx.$$ 

Direct calculation shows that, in case $Q = \mathbb{R}^m - \{y\}$, $N^Q U \delta_y = A \delta_y$.

Let us also observe that, for $\psi \in D$ and $\mu \in C^*(B)$,

$$(1.2) \quad \langle \psi, NU \mu \rangle = \int_B \langle \psi, NU \delta_y \rangle \ d\mu(y).$$ 

Indeed, if $P = G \cap \text{spt } \psi$ and $K = \sup |\text{grad } \psi|$, then

$$\iint_{G \times B} |\text{grad } \psi(x) \cdot \frac{y-x}{|y-x|^m}| \ dx \ d|\mu|(y) \leq KA \text{ diam } (P \cup B) \|\mu\|,$$

so that Fubini's theorem applies to

$$(1.3) \quad \int_{G \times B} \text{grad } \psi(x) \cdot (y-x) |y-x|^{-m} \ dx \ d\mu(y);$$

it remains to notice that the two repeated integrals derived from (1.3) occur in (1.2).
Before investigating the problem 1.2 we shall answer the following simpler question:

1.4. Fix \( y \in B \). What must be the shape of \( G \) in order that \( N\delta_y \) be a measure?

Let us first introduce a concept which will be useful later.

1.5. Definition. If \( M \subset \mathbb{R}^k \) is a Borel set and \( S \subset \mathbb{R}^k \) is an open segment or half-line then \( z \in S \) will be termed a hit of \( S \) on \( M \) provided both \( S \cap M \cap \Omega_r(z) \) and \( (S-M) \cap \Omega_r(z) \) have a positive linear measure for every \( r>0 \).

An answer to 1.4 is included in the following proposition, which will be needed later.

1.6. Proposition. Suppose that \( G \) is a Borel set. Fix \( y \in \mathbb{R}^m, r>0 \) and put

\[
E_r(y) = D \cap \{ \psi : \text{spt } \psi \subset \Omega_r(y), |\psi| \leq 1 \},
\]

\[
D_r(y) = E_r(y) \cap \{ \psi : y \notin \text{spt } \psi \}.
\]

If \( n_r(\theta, y) \) denotes the number (possibly 0 or \( \infty \)) of all hits of \( \{ y + p\theta : 0 < p < r \} \) on \( G \), then \( n_r(\theta, y) \) is a Baire function of the variable \( \theta \) on \( \Gamma \), the integral

\[
v_r(y) = \int_{\Gamma} n_r(\theta, y) \, dH_{m-1}(\theta)
\]

is equal to

\[
\sup \left\{ \int_G \frac{\nabla \psi(x) \cdot \frac{y-x}{|y-x|^m}}{|y-x|^m} \, dx : \psi \in D_r(y) \right\}
\]

and

\[
\sup \left\{ \int_G \frac{\nabla \psi(x) \cdot \frac{y-x}{|y-x|^m}}{|y-x|^m} \, dx : \psi \in E_r(y) \right\} \leq A + v_r(y).
\]

If \( y \in B \) and \( G \) is open, then \( N\delta_y \) is a measure if and only if \( v_\infty(y) < \infty \).

1.7. Remark. If it is necessary to specify the set \( G \), we write \( n_r(\theta, y) \) and \( v_r(y) \) instead of \( n_r(\theta, y) \) and \( v_r(y) \).

We postpone the proof of Proposition 1.6 to 1.11. First we establish two lemmas.

1.8. Notation. If \( f \) is a function in \( \mathbb{R}^1 \) we denote by \( \text{var}_r [f; (a, b)] \) its variation on \( (a, b) = \mathbb{R}^1 \cap \{ t : a < t < b \} \). If \( f \) is known to be summable over every compact subset in \( (a, b) \), we shall use \( \text{var} \, [f; (a, b)] \) to denote \( \sup \int_0^b \phi(t) f(t) \, dt, \phi \) ranging over all infinitely differentiable functions with compact \( \text{spt } \phi \subset (a, b) \) such that \( |\phi| \leq 1 \).

Remark. It follows easily from the Riesz representation theorem and elementary distribution theory that \( \text{var ess} \, [f; (a, b)] < \infty \) implies the existence of a function \( g \) in \( (a, b) \) such that \( g = f \) a.e. in \( (a, b) \) and \( \text{var} \, [g; (a, b)] = \text{var ess} \, [f; (a, b)] \).

Clearly, \( \text{var} \, [f; (a, b)] = \text{var ess} \, [f; (a, b)] \) whenever \( f \) is continuous in \( (a, b) \).

1.9. Lemma. If \( c_M \) is the characteristic function of a Borel set \( M \subset \mathbb{R}^1 \), then \( \text{var ess} \, [c_M; (a, b)] \) equals the number of hits of \( (a, b) \) on \( M \).
Proof. Let \( q \) stand for the number of all hits of \((a, b)\) on \( M\). If \( q < \infty \) and \( a_1 < \cdots < a_q \) are all the hits, then no \((a_i, a_i+1)\) can meet both \( M \) and \((a, b) - M\) in a set of positive linear measure. It follows that either \( M \) or \((a, b) - M\) is equivalent with \( \bigcup_k (a_{2k-1}, a_{2k}) \), where \( 1 \leq k, 2k \leq q \). Consequently, \( \text{var ess} [c_M; (a, b)] = q \). Conversely, if \( \text{var ess} [c_M; (a, b)] < \infty \), then there is a \( g \) with \( \text{var} [g; (a, b)] < \infty \) such that \( g = c_M \) a.e. in \((a, b)\).

Clearly, this implies \( q < \infty \).

1.10. Lemma. Let \( f \) be a bounded Baire function in \( R^m \), \( y \in R^m \), \( 0 \leq a < b \leq \infty \). For \( \theta \in \Gamma \) put

\[
(1.4) \quad f_\theta(t) = f(y + t\theta), \quad t \in R^1.
\]

Then \( \text{var ess} [f_\theta; (a, b)] \) is a Baire function of the variable \( \theta \) on \( \Gamma \) and the integral

\[
\int_\Gamma \text{var ess} [f_\theta; (a, b)] \, dH_{m-1}(\theta)
\]

equals

\[
v(a, b, f) = \sup _\psi \int_{R^m} f(x) \text{grad } \psi(x) \cdot \frac{x - y}{|y - x|^m} \, dx,
\]

\( \psi \) ranging over all functions in \( D \) with

\[
(1.5) \quad \text{spt } \psi \subset R^m \cap \{x: a < |x - y| < b\}, \quad |\psi| \leq 1.
\]

Proof. We may assume \( y = 0 \), \( b < \infty \). Using the notation from (1.4) we obtain for any \( \psi \in D \) satisfying (1.5)

\[
\int_{R^m} f(x) \text{grad } \psi(x) \cdot \frac{x}{|x|^m} \, dx = \int_\Gamma \left( \int_a^b f_\theta(t)\psi_\theta(t) \, dt \right) \, dH_{m-1}(\theta),
\]

\[
\int_a^b f_\theta(t)\psi_\theta(t) \, dt \leq \text{var ess} [f_\theta; (a, b)].
\]

Assuming that we know already that \( \text{var ess} [f_\theta; (a, b)] \) is measurable \( (H_{m-1}) \) on \( \Gamma \) we get

\[
v(a, b, f) \leq \int_\Gamma \text{var ess} [f_\theta; (a, b)] \, dH_{m-1}(\theta).
\]

It remains to prove that \( \text{var ess} [f_\theta; (a, b)] \) is a Baire function of \( \theta \) and

\[
(1.6) \quad \int_\Gamma \text{var ess} [f_\theta; (a, b)] \, dH_{m-1}(\theta) \leq v(a, b, f).
\]

To show this we first assume, in addition, that

(I). \( f_\theta \) has a continuous derivative on \((a, b)\) for every \( \theta \in \Gamma \) and

\[
\sup \{|f_\theta'(t)| : \theta \in \Gamma, c < t < d\} = K(c, d) < \infty
\]

whenever \( a < c < d < b \).
For every positive integer $N$ we subdivide $(a, b)$ by means of points

$$a_k = a_k^N = a + k2^{-N}(b-a), \quad 1 \leq k < 2^N.$$

Consider $k < 2^N - 2$. Since $\text{sign} [f_\theta(a_{k+1}) - f_\theta(a_k)]$ is a Baire function of $\theta$, there are functions $\phi_{ka} \in D$ such that $|\phi_{ka}| \leq 1$ and

$$\lim_{s \to \infty} \phi_{ka}(\theta) = \text{sign} [f_\theta(a_{k+1}) - f_\theta(a_k)] \quad \text{a.e. } (H_{m-1})$$

on $\Gamma$. Further express the characteristic function of $(a_k, a_{k+1})$ as $\lim_{s \to \infty} \rho_{ks}$, where $\rho_{ks}$ are infinitely differentiable functions in $R^1$ with

$$\text{spt } \rho_{ks} \subset (a_k, a_{k+1}), \ |\rho_{ks}| \leq 1,$$

and define

$$\psi_s(t\theta) = - \sum_{k=1}^{2^N-2} \phi_{ks}(\theta)\rho_{ks}(t), \quad t \geq 0, \ \theta \in \Gamma.$$

Then

$$\psi_s \in D, \ |\psi_s| \leq 1, \ \text{spt } \psi_s \subset R^n \cap \{x : a < |x| < b\}.$$  

Consequently,

$$v(a, b, f) \geq \int_{\Gamma} \left[ \int_a^b f_\theta(t)\psi_s(t) \, dt \right] dH_{m-1}(\theta).$$

The sequence of integrals

$$\int_a^b f_\theta(t)\psi_s(t) \, dt = \sum_{k=1}^{2^N-2} \phi_{ks}(\theta) \int_{a_k}^{a_{k+1}} \rho_{ks}(t)f_\theta(t) \, dt$$

is dominated by $(b-a)K(a_1, a_{2^N-1})$ and converges, as $s \to \infty$, to

$$\sigma_N(\theta) = \sum_{k=1}^{2^N-2} |f_\theta(a_{k+1}) - f_\theta(a_k)|$$

a.e. $(H_{m-1})$ on $\Gamma$. Hence we conclude

$$v(a, b, f) \geq \int_{\Gamma} \sigma_N(\theta) \, dH_{m-1}(\theta).$$

Noting that $\sigma_N(\theta) \uparrow \text{var } [f_\theta; (a, b)]$ as $N \to \infty$ we see that $\text{var } [f_\theta; (a, b)]$ is a Baire function of $\theta$ and (1.6) holds in this special case.

Let us now drop the additional assumptions (I) on $f$. For every positive integer $N$ we fix a symmetric infinitely differentiable function $\omega_N$ in $R^1$ with

$$\text{spt } \omega_N \subset (-1/N, 1/N), \int_{R^1} \omega_N(t) \, dt = 1.$$
and define $f_N$ so that $f_{N\theta}=f_\theta \ast \omega_N$ (the convolution of $f_\theta$ and $\omega_N$) on the positive real axis, $f_N(0)=0$. Let $a^N=a+1/N$, $b^N=b-1/N$, $2/N<b-a$. It follows from the first part of the proof that

$$\int \var [f_{N\theta}; (a^N, b^N)] dH_{m-1}(\theta) = v(a^N, b^N, f_N).$$

If $\psi_N$ is obtained from $\psi$ in the same way as $f_N$ from $f$, then

$$\psi \in D, \ |\psi| \leq 1, \ \text{spt } \psi \subset R^m \cap \{x: a^N < |x| < b^N\}$$

imply

$$\psi_N \in D, \ |\psi_N| \leq 1, \ \text{spt } \psi_N \subset R^m \cap \{x: a < |x| < b\}$$

and

$$\int_{a^N}^{b^N} \psi_N(t) f_{N\theta}(t) dt = \int_{a}^{b} \psi(t) f_\theta(t) dt.$$ 

Consequently,

$$v(a^N, b^N, f_N) \leq v(a, b, f).$$

The same argument shows that

$$\var \text{ ess } [f_{N\theta}; (a^N, b^N)] \leq \var \text{ ess } [f_\theta; (a, b)].$$

It is easy to see that

$$\lim \inf \var \text{ ess } [f_{N\theta}; (a^N, b^N)] \geq \var \text{ ess } [f_\theta; (a, b)],$$

which together with (1.9) yields

$$\lim_{N \to \infty} \var \text{ ess } [f_{N\theta}; (a^N, b^N)] = \var \text{ ess } [f_\theta; (a, b)].$$

In particular, $\var \text{ ess } [f_\theta; (a, b)]$ is a Baire function of $\theta$. (1.7), (1.8), and (1.10) imply (1.6).

**REMARK.** The above lemma could also be derived from general theorems on functions, whose partial derivatives are measures; cf. [FL], [KR], [P] on the subject.

Now it is easy to present the following.

**1.11. Proof of Proposition 1.6.** Let $f$ be the characteristic function of $G$. By 1.9 and 1.10

$$\var \text{ ess } [f_\theta; (0, r)] = n_r(\theta, y),$$

$$v_r(y) = v(0, r, f).$$

If $n_r(\theta, y)<\infty$, then $G \cap \{y+t\theta: 0<t<r\}$ is equivalent $(H_1)$ with a finite union
of disjoint segments, whose end points are hits of \( \{y + t \theta : 0 < t < r\} \) on \( G \) and, possibly, \( y \) and \( y + r \theta \). Hence we conclude for \( \psi \in E(y) \)

\[
\left| \int_0^\infty f_\psi(t) \psi_\theta(t) \, dt \right| \leq 1 + n_r(\theta, y), \quad \theta \in \Gamma,
\]

\[
\int_G \text{grad } \psi(x) \cdot (y - x)|y - x|^{-n} \, dx \leq \int_\Gamma [1 + n_r(\theta, y)] \, dH_{m-1}(\theta) = A + v_r(y).
\]

It remains to note that, in case \( y \in B \) and \( G \) is open, \( NU_\psi \) is a measure if and only if

\[
\sup \{ \langle \psi, NU_\psi \rangle : \psi \in D, |\psi| \leq 1 \} < \infty.
\]

1.12. Remark. Let us observe that, in case \( y \in B \) and \( NU_\psi \in C^*(B) \),

\[
(1.11) \quad v_\infty(y) \leq \|NU_\psi\| \leq A + v_\infty(y).
\]

Now we are in position to answer the question raised in 1.2.

1.13. Theorem. \( NU_\mu \) is a measure for every \( \mu \in C^*(B) \) if and only if

\[
V = \sup_{y \in B} v_\infty(y) < \infty.
\]

If this is the case, then

\[
NU : \mu \rightarrow NU_\mu
\]

is a bounded linear operator on \( C^*(B) \),

\[
\|NU\| \leq A + V
\]

and (1.2) holds for every bounded Baire function \( \psi \) on \( B \). In particular,

\[
NU_\mu(M) = \int_B NU_\psi(M) \, d\mu(y)
\]

for \( \mu \in C^*(B) \) and every Borel set \( M \subset B \).

Proof. With every \( \psi \in D \) we associate a linear functional \( L_\psi \) over \( C^*(B) \) defined by

\[
\langle \mu, L_\psi \rangle = \langle \psi, NU_\mu \rangle, \quad \mu \in C^*(B).
\]

Denoting

\[
P_\psi = G \cap \text{spt } \psi, \quad s_\psi = \sup |\text{grad } \psi|,
\]

we obtain from (1.1)

\[
|\langle \mu, L_\psi \rangle| \leq s_\psi A \text{ diam } (B \cup P_\psi) \|\mu\|
\]

which shows that every \( L_\psi \) is bounded on \( C^*(B) \). Let \( E = D \cap \{\psi : |\psi| \leq 1\} \). Then \( NU_\mu \) is a measure if and only if

\[
\sup_{\psi \in E} \langle \psi, NU_\mu \rangle < \infty.
\]
In particular, if \( NU_\mu \) is a measure for every \( \mu \in C^*(B) \), then the class of functionals \( \{L_\varphi\}_{\varphi \in B} \) must be pointwise bounded on \( C^*(B) \) and, by the uniform boundedness principle,

\[
\sup_{\varphi \in B} \|L_\varphi\| = K < \infty.
\]

Employing (1.11) we get for every \( y \in B \)

\[
v_\varphi(y) \leq \sup_{\varphi \in B} \langle \psi, NU\delta_\varphi \rangle \leq K.
\]

Conversely, if (1.12) holds, then (1.2) together with (1.11) imply

\[
\sup_{\varphi \in B} |\langle \psi, NU\mu \rangle| \leq (A + V)\|\mu\|
\]

for every \( \mu \in C^*(B) \). It is also easily seen that in this case (1.2) extends to any bounded Baire function \( \psi \).

2. Double layer potentials.

2.1. Notation. Throughout this paragraph \( C \subseteq \mathbb{R}^m \) will denote a Borel set with a compact boundary \( B \). Given \( z \in \mathbb{R}^m \) we put

\[
D(z) = D \cap \{\psi : z \notin \text{spt } \psi\}
\]

and define

(2.1) \[
W_\psi(z) = \int_C \text{grad } \psi(x) \cdot \frac{x - z}{|x - z|^m} \, dx, \quad \psi \in D(z).
\]

If it is necessary to specify \( C \) we write \( W_\psi^C \) instead of \( W_\psi \). In case \( C \) has a smooth boundary with exterior normal \( n \) the integral (2.1) reduces to

\[
\int_B \psi(y) \frac{(y - z) \cdot n(y)}{|y - z|^m} \, dH_{m-1}(y),
\]

which is the classical double-layer potential. If \( \psi \) vanishes in some neighborhood of \( B \) then there is a \( Q \subseteq \mathbb{R}^m \) with a smooth boundary such that

\[
\text{spt } \psi \cap C \subseteq \text{int } Q, \quad \text{cl } Q \subseteq \text{int } C,
\]

whence

\[
W_\psi(z) = W_\tilde{\psi}(z) = 0.
\]

If \( z \notin B \), we use this observation to extend \( W_\psi(z) \) from \( D(z) \) to \( D \) defining

\[
W_\psi(z) = W_\tilde{\psi}(z),
\]

where \( \tilde{\psi} \) is an arbitrary function in \( D(z) \) coinciding with given \( \psi \in D \) in some neighborhood of \( B \). \( W_\psi(z) \) may thus be considered as a distribution over \( D \) with support in \( B \) (compare [D, Chapter III, p. 157]).
For fixed \( \psi \in D \), \( W_\psi(z) \) is a harmonic function of \( z \) in \( R^m - B \). Indeed, if \( O \) is an open set with \( B \cap \text{cl } O = \emptyset \), then there is a \( \tilde{\psi} \in D \) coinciding with \( \psi \) in some neighborhood of \( B \) and vanishing on \( O \); clearly,

\[
W_\psi(z) = W_{\tilde{\psi}}(z) = \int_{\mathcal{C} - O} \text{grad } \tilde{\psi}(x) \cdot \frac{x - z}{|x - z|^m} \, dx
\]

is a harmonic function of \( z \) in \( O \).

Our main objective in this paragraph is to find necessary and sufficient geometric conditions on \( C \) securing natural extendability of \( W_\psi \) from \( D \) to broader class of continuous functions and also "nice behaviour" (e.g., boundedness) of \( W_\psi \) near \( B \) for each continuous \( \psi \).

2.2. **Lemma.** Fix \( z \in R^m \). Then

\[
(2.2) \quad \nu^C_\infty(z) < \infty
\]

is a necessary and sufficient condition to secure

\[
\lim_{k \to \infty} W_{\psi_k}(z) = W_\psi(z)
\]

for every sequence of \( \psi_k \in D(z) \) converging uniformly (as \( k \to \infty \)) to \( \psi \in D(z) \). If (2.2) holds then there is a \( \nu_z \in C^*(B) \) such that

\[
(2.3) \quad W_\psi(z) = \int_B \psi(y) \, dv_z(y), \quad \psi \in D(z),
\]

\[
(2.4) \quad \nu_z(\{z\}) = 0,
\]

\[
(2.5) \quad \|\nu_z\| = \nu^C_\infty(z).
\]

(2.3) together with any of the two conditions (2.4), (2.5) determine \( \nu_z \) uniquely.

**Proof.** This follows at once from the equality

\[
(2.6) \quad \nu^C_\infty(z) = \sup \{W_\psi(z) : \psi \in D(z), |\psi| \leq 1\}
\]

established in 1.6.

2.3. **Remark.** If (2.2) holds we extend \( W \cdots (z) \) defining

\[
Wf(z) = \int_B f(y) \, dv_z(y)
\]

for any bounded Baire function \( f \) on \( B \).

In order to present another integral representation for \( Wf(z) \) we introduce the following.

2.4. **Notation.** Fix \( z \in R^m \) and \( \theta \in \Gamma \). We put for \( t > 0 \)

\[
s(t; z, \theta) = \sigma (\pm 1)
\]
if there is a \( \delta > 0 \) such that

\[
\text{if there is a } \delta > 0 \text{ such that } z + (t + \sigma t) \theta \in \mathbb{R}^m - C, \quad z + (t - \sigma t) \theta \in C,
\]

for a.e. \( \tau \in (0, \delta) \); otherwise we set \( s(t; z, \theta) = 0 \).

Clearly, \( s(t; z, \theta) \neq 0 \) only if \( z + t\theta \) is a hit of \( \{z + \tau \theta : \tau > 0\} \) on \( C \).

2.5. Lemma. If \( \nu_\omega^\circ(z) < \infty \) then

\[
Wf(z) = \int_{\Gamma} \left\{ \sum_{t > 0} f(z + t\theta)s(t; z, \theta) \right\} dH_{m-1}(\theta)
\]

for any bounded Baire function \( f \) on \( B \).

Proof. Let \( \nu_\omega^\circ(z) < \infty \). If \( f \in D(z) \) then

\[
Wf(z) = \int_{C} \text{grad} f(x) \cdot \frac{x - z}{|x - z|^m} \, dx
\]

\[
= \int_{\Gamma} \left\{ \int_{C_\theta} \partial_\theta f(z + t\theta) \, dt \right\} dH_{m-1}(\theta),
\]

where

\[
C_\theta = \{t : t > 0, z + t\theta \in C\}, \quad \partial_\theta f = \theta \cdot \text{grad} f.
\]

Noting that \( \nu_\omega^\circ(\theta, z) < \infty \) implies

\[
\int_{C_\theta} \partial_\theta f(z + t\theta) \, dt = \sum_{t > 0} f(z + t\theta)s(t; z, \theta)
\]

we obtain (2.7).

If \( \{f_k\} \) is a pointwise convergent sequence of functions on \( B \) such that, for all \( k \), \( |f_k| \leq K \) and (2.7) holds with \( f \) replaced by \( f_k \), then

\[
\left| \sum_{t > 0} f_k(z + t\theta)s(t; z, \theta) \right| \leq K\nu_\omega^\circ(\theta, z)
\]

a.e. \( \{H_{m-1}\} \) on \( \Gamma \) and, by the Lebesgue convergence theorem, (2.7) holds for \( f = \lim_k f_k \) as well.

We conclude that (2.7) is valid for every bounded Baire function \( f \) vanishing at \( z \); in view of (2.4), vanishing at \( z \) is irrelevant.

2.6. Proposition. Let \( \nu_\omega^\circ(z) < \infty \). Denote by \( K_z \) and \( L_z \) the set of all \( \theta \in \Gamma \) for which there is an \( \varepsilon = \varepsilon(\theta) > 0 \) such that

\[
H_1(\{z + t\theta : 0 < t < \varepsilon\} \cap C) = 0
\]

and

\[
H_1(\{z + t\theta : 0 < t < \varepsilon\} - C) = 0,
\]
respectively. Then $K_z, L_z$ are measurable $(H_{m-1})$,

$$(2.9) \quad H_{m-1}(\Gamma - (K_z \cup L_z)) = 0$$

and $\nu_z(B) = H_{m-1}(L_z)$ or $\nu_z(B) = -H_{m-1}(K_z)$ according as $C$ is bounded or not. If $\psi \in D$, then

$$(2.10) \quad \int_C \text{grad} \psi(x) \cdot \frac{x - z}{|x - z|^m} \, dx = W_\psi(z) - H_{m-1}(L_z)\psi(z).$$

If $Q$ is a convex Borel set, then

$$(2.11) \quad |\nu_z(B \cap Q)| \leq A.$$  

**Proof.** It is easily seen that

$$\Gamma \cap \{\theta : n_\infty(\theta, z) < \infty\} \subset L_z \cup K_z$$

whence (2.9) follows at once.

Fix now a $\theta \in \Gamma$ with $n_\infty(\theta, z) < \infty$. Let

$$(2.12) \quad t_1 < \cdots < t_q$$

be all the points $t \in (0, \infty)$ with $s(t; z, \theta) \neq 0$ (cf. 2.4). Clearly,

$$(2.13) \quad s(t_{j+1}; \cdots) = -s(t_j; \cdots), \quad 1 \leq j < q$$

and $s(t_1; \cdots) = 1$ or $s(t_1; \cdots) = -1$ according as $\theta \in L_z$ or $\theta \in K_z$. If $C$ is bounded, then $s(t_q; \cdots) = 1$, while $s(t_q; \cdots) = -1$ in the opposite case. We conclude that

$$\sum_{\theta} s(t; z, \theta)$$

almost $(H_{m-1})$ equals the characteristic function of $L_z$ if $C$ is bounded, while

$$-\sum_{\theta} s(t; z, \theta)$$

almost equals the characteristic function of $K_z$ in the opposite case. Employing (2.7) with $f \equiv 1$ we get the first part of our proposition.

Let now $f$ be the characteristic function of a convex Borel set $Q$. Consider again a fixed $\theta \in \Gamma$, $n_\infty(\theta, z) < \infty$, and the corresponding sequence (2.12). If $t_i$ and $t_k$ are the first and the last members of (2.12) with $z + t_i \theta \in Q$, respectively, then (2.13) implies

$$\left| \sum_{j=1}^q f(z + t_j \theta)s(t_j; z, \theta) \right| = \left| \sum_{f=1}^k s(t_f; z, \theta) \right| \leq 1,$$

whence (2.11) follows by 2.5. If $\psi \in D$ then we have with the notation from (2.8)

$$\int_{C_z} \phi(z + t \theta) \, dt = \sum_{t > 0} \psi(z + t \theta)s(t; z, \theta)$$

for $\theta \in K_z \cap \{\theta : n_\infty(\theta, z) < \infty\}$, while

$$\int_{C_z} \phi(z + t \theta) \, dt = \sum_{t > 0} \psi(z + t \theta)s(t; z, \theta) - \psi(z)$$
for $\theta \in L_z \cap \{\theta : n^\Phi_\omega(\theta, z) < \infty\}$. Hence

$$
\int \nabla \psi(x) \cdot \frac{x-z}{|x-z|^m} \, dx = \int \left( \int_{c_\theta} \partial_n \psi(z + t\theta) \, dt \right) \, dH_{m-1}(\theta)
$$

$$
= W\psi(z) - H_{m-1}(L_z)\psi(z).
$$

2.7. **Lemma.** Let $v^\Phi_\omega(z) < \infty$ and define $L_z$ as in 2.6. If $M \subset \Gamma$ is measurable ($H_{m-1}$), $H_{m-1}(M) > 0$ and

$$
\Lambda_M = \{z + t\theta : \theta \in M, t > 0\},
$$

then

$$(2.14) \quad \lim_{r \to 0^+} \frac{H_m(\Omega_r(z) \cap C \cap \Lambda_M)}{H_m(\Omega_r(z) \cap \Lambda_M)} = \frac{H_{m-1}(L_z \cap M)}{H_{m-1}(M)}.
$$

In particular, $C$ has an $m$-dimensional density

$$
d_c(z) = H_{m-1}(L_z)/A
$$

at $z$.

**Proof.** Let $\varepsilon(\theta)$ have the meaning described in the definition of $K_z, L_z$ in 2.6 and put

$$
K^r = M \cap \{\theta : \theta \in K_z, \varepsilon(\theta) > r\},
$$

$$
L^r = M \cap \{\theta : \theta \in L_z, \varepsilon(\theta) > r\}.
$$

We have

$$
H_m(\Omega_r(z) \cap C \cap \Lambda_M) \geq m^{-1}r^m \text{inn} \, H_{m-1}(L^r),
$$

$$
H_m((\Omega_r(z) - C) \cap \Lambda_M) \geq m^{-1}r^m \text{inn} \, H_{m-1}(K^r),
$$

where inn $H_{m-1}$ stands for the inner $(m-1)$-dimensional Hausdorff measure. Denoting

$$
d_r = \frac{H_m(\Omega_r(z) \cap C \cap \Lambda_M)}{H_m(\Omega_r(z) \cap \Lambda_M)}
$$

and noting that

$$
K^r \uparrow (K_z \cap M), \quad L^r \uparrow (L_z \cap M)
$$

as $r \downarrow 0$, we obtain

$$
\lim_{r \to 0^+} \inf d_r \geq H_{m-1}(L_z \cap M)/H_{m-1}(M),
$$

$$
\lim_{r \to 0^+} \inf (1-d_r) \geq H_{m-1}(K_z \cap M)/H_{m-1}(M),
$$

whence (2.14) follows by (2.9).
2.8. Notation. \( P(C) \) will denote the perimeter of \( C \) defined by
\[
P(C) = \sup_w \int_C \text{div } w(x) \, dx,
\]
where \( w = [w_1, \ldots, w_m] \) ranges over all vector-valued functions with \( m \) components \( w_j \in D \) satisfying
\[
\left( \sum_{j=1}^m w_j^2 \right)^{1/2} = |w| \leq 1.
\]
(Further information on sets with finite perimeter may be found in [DG], [F3], [FL], [MA].)

For \( M \subset \mathbb{R}^m \) and \( z \in \mathbb{R}^n \) we let
\[
\text{dist} (z, M) = \inf \{|z - y| : y \in M\}.
\]

2.9. Lemma. \( v_\infty^0(z) \) is a lower semicontinuous function of \( z \) on \( \mathbb{R}^m \) satisfying the inequality
\[
v_\infty^0(z) \leq P(C)(\text{dist} (z, B))^{1-m}, \quad z \notin B.
\]

**Proof.** If \( K < v_\infty^0(z) \), then there is a \( \psi \in D(z) \) such that \( |\psi| \leq 1 \) and \( W_\psi(z) > K \) (see (2.6)). Hence
\[
\lim_{y \to z} \inf_{y \to z} v_\infty^0(y) \geq \lim_{y \to z} W_\psi(y) = W_\psi(z) > K.
\]

Suppose now that \( z \notin B \), fix an arbitrary \( \psi \in D(z) \) with \( |\psi| \leq 1 \) and a positive \( \rho < \text{dist} (z, B) \). Then there is a \( \tilde{\psi} \in D, |\tilde{\psi}| \leq 1 \), which coincides with \( \psi \) in some neighborhood of \( B \) and vanishes on \( \Omega_\rho(z) \). Let us define \( w(z) = O (\in \mathbb{R}^m) \),
\[
w(x) = \tilde{\psi}(x) \frac{x - z}{|x - z|^m}, \quad x \neq z,
\]
and observe that \( |w| \leq \rho^{1-m} \),
\[
\text{grad } \tilde{\psi}(x) \cdot \frac{x - z}{|x - z|^m} = \text{div } w(x).
\]

Consequently,
\[
W_\psi(z) = W_\tilde{\psi}(z) = \int_C \text{div } w(x) \, dx \leq \rho^{1-m} P(C).
\]

**Remark.** We see that \( v_\infty^0(z) \) is finite on \( \mathbb{R}^m - B \) provided \( P(C) < \infty \). The converse is also true as it follows from the following

2.10. Proposition. If
\[
\sum_{j=1}^{m+1} v_\infty^0(z_j) < \infty
\]
for an \((m+1)\)-tuple of points \(z_1, \ldots, z_{m+1}\) in general position (i.e., not situated on a single hyperplane), then

\[(2.15) \quad P(C) < \infty.\]

**Proof.** To prove (2.15) it is sufficient to show that

\[
\sup \left\{ \int_C \partial \psi(x) \, dx : \psi \in D, |\psi| \leq 1 \right\} < \infty
\]

for every \(\theta \in \Gamma\). Fix \(\theta \in \Gamma\). Let \(\Pi_j\) denote the hyperplane determined by \(\{z_k : k \neq j\}\). Since

\[
\bigcup_{j=1}^{m+1} (\mathbb{R}^m - \Pi_j) = \mathbb{R}^m,
\]

there are \(\alpha_j \in D\) such that

\[
\Pi_j \cap \text{spt } \alpha_j = \emptyset
\]

and

\[
\alpha = \sum_{j=1}^{m+1} \alpha_j = 1
\]

in some neighborhood of \(B\).

Noting that

\[
\int_C \alpha(x) \partial \psi(x) \, dx = \int_C \partial \psi(x) \, dx
\]

we see that it is sufficient to prove that

\[
\sup \left\{ \int_C \alpha_j(x) \partial \psi(x) \, dx : \psi \in D, |\psi| \leq 1 \right\} < \infty
\]

for \(j=1, \ldots, m+1\). Consider, for instance, \(j=1\). If \(x \in \text{spt } \alpha_1\), then \(x - z_{2}, \ldots, x - z_{m+1}\) are linearly independent. Consequently,

\[
\theta = \sum_{k=2}^{m+1} a_k(x) \frac{x - z_k}{|x - z_k|^m}
\]

where \(a_k\) are infinitely differentiable in some neighborhood of \(\text{spt } \alpha_1\). Extending \(a_k\) arbitrarily to \(\mathbb{R}^m\) we get

\[
\int_C \alpha_1(x) \partial \psi(x) \, dx = \sum_{k=2}^{m+1} \int_C \alpha_1(x) a_k(x) \, \text{grad } \psi(x) \cdot \frac{x - z_k}{|x - z_k|^m} \, dx.
\]

Fix \(k \in \langle 2, m+1 \rangle\) and define \(F(x) = \alpha_1(x) a_k(x)\). Then \(F \in D(z_k)\) and denoting \(K = \max |F|\) we obtain for any \(\psi \in D\) with \(|\psi| \leq 1\)

\[
\int_C F(x) \, \text{grad } \psi(x) \cdot \frac{x - z_k}{|x - z_k|^m} \, dx = I_1 + I_2,
\]
where
\[
I_1 = \int_C \text{grad } (F(x)\psi(x)) \cdot \frac{x-z_k}{|x-z_k|^m} \, dx \leq K\nu_\infty(z_k),
\]
\[
I_2 = -\int_C \psi(x) \text{grad } F(x) \cdot \frac{x-z_k}{|x-z_k|^m} \, dx \leq \int_C |\text{grad } F(x)| \cdot |x-z_k|^{1-m} \, dx < \infty. \tag{2}
\]

2.11. Remark. It follows from 2.2, 2.9, and 2.10 that (2.12) is a necessary and sufficient condition to secure continuous dependence (with respect to uniform convergence) of $W\psi(z)$ on $\psi$ for every $z \notin B$. For this reason we agree to impose (2.15) on $C$ throughout the rest of the present paragraph.

Let us recall that $\theta \in \Gamma$ is called the exterior normal of $C$ at $y$ in the sense of Federer provided the symmetric difference of $C$ and the half-space
\[
R^m \cap \{x : (x-y) \cdot \theta < 0\}
\]
has $m$-dimensional density 0 at $y$ (cf. [F1]).

In what follows the term exterior normal is always to be interpreted in this sense. We put $n^e(y) = n(y) = \theta$ if $\theta$ is the exterior normal of $C$ at $y$; otherwise $n(y)$ denotes the zero vector. The set of all $y$ with $n(y) \neq 0$ is called the reduced boundary of $C$ and will be denoted by $\hat{B}$. It is known from [DG2] and [F3] that
\[
H_{m-1}(\hat{B}) < \infty
\]
and
\[
\int_C \text{div } w(x) \, dx = \int_B w(y) \cdot n(y) \, dH_{m-1}(y)
\]
for every vector-valued function $w = [w_1, \ldots, w_m]$ with components $w_j \in D$.

2.12. Lemma. For every $z \in R^m$
\[
(2.16) \quad \nu_\infty(z) = \int_B \frac{|n(y) \cdot (y-z)|}{|y-z|^m} \, dH_{m-1}(y).
\]

If $\nu_\infty(z) < \infty$ and $M \subset B$ is a Borel set, then
\[
\nu_\infty(M) = \int_M \frac{|n(y) \cdot (y-z)|}{|y-z|^m} \, dH_{m-1}(y).
\]

Proof. Fix $z \in R^m$. Let $\psi \in D(z)$ and put $w(z) = O (\in R^m)$,
\[
w(x) = \psi(x) \frac{x-z}{|x-z|^m}, \quad x \neq z.
\]

(*) The author is indebted to Herbert Federer for simplification of this proof.
Then
\[ W_{\psi}(z) = \int_{\mathbb{R}^n} \text{div} w(x) \, dx = \int_{\mathbb{R}^n} \psi(y) \frac{n(y) \cdot (y-z)}{|y-z|^m} \, dH_{m-1}(y) \]
and (2.16) follows from (2.6). Let now \( v_{\omega}(z) < \infty \). As we have just seen,
\[ \int_{\mathbb{R}^n} f \, dv_{\omega} = \int_{\mathbb{R}^n} f(y) \frac{n(y) \cdot (y-z)}{|y-z|^m} \, dH_{m-1}(y) \]
promvided \( f \in D(z) \); it is easily seen that this formula extends to any bounded Baire function \( f \).

The following result will be useful below:

2.13. Theorem. Let
\[ V^C = \sup \{ v_{\omega}(y) : y \in \mathbb{R}^n \}. \]
Then \( v_{\omega}(z) \leq A + V^C \) for every \( z \in \mathbb{R}^n \).

Proof. We may assume \( V^C < \infty \). Fix \( z \in \mathbb{R}^n - B \) and let \( d \) be an arbitrary number less than \( v_{\omega}(z) \). Then there exist mutually disjoint closed parallelepipeds \( K_1, \ldots, K_q \) such that
\[ \sum_{j=1}^{q} |v_x(B \cap K_j)| > d. \]
Put \( \sigma_j = \text{sign } v_x(B \cap K_j) \) and consider the function
\[ h(x) = \sum_{j=1}^{q} \sigma_j \rho_x(B \cap K_j), \]
which is harmonic on \( \mathbb{R}^n - \bigcup_{j=1}^{q} B \cap K_j \supset \mathbb{R}^n - B. \)
Fix an arbitrary \( y \in B \). If \( y \notin \bigcup_{j=1}^{q} K_j \), then
\[ \lim_{x \to y} h(x) = h(y) \leq \|v_x\| \leq V^C. \]
In the opposite case we may assume that \( y \in K_1 \), so that
\[ \lim_{x \to y} \sum_{j=2}^{q} \sigma_j \rho_x(B \cap K_j) = \sum_{j=2}^{q} \sigma_j \rho_x(B \cap K_j) \leq \|v_x\| \leq V^C \]
and, by Proposition 2.6,
\[ \sup_{x} |v_x(B \cap K_1)| \leq A. \]
We see that
\[ \lim_{x \to y: x \in B} \sup h(x) \leq A + V^C. \]
Noting that \( h(x) \rightarrow 0 \) as \( |x| \rightarrow \infty \) we conclude that \( h \leq A + V^c \) on \( \mathbb{R}^m - B \). In particular, \( d < h(z) \leq A + V^c \).

2.14. COROLLARY. If \( r > 0 \) and \( z \in \mathbb{R}^m \), then

\[
H_{m-1}(\Omega_r(z) \cap \hat{B}) \leq m(m+1)^n(A + V^c)r^{m-1}.
\]

**Proof.** To prove (2.17) we may clearly assume that \( z = 0 \). Noting that \( V^c \) is invariant with respect to dilations of \( C \) we observe that it is sufficient to establish (2.17) for \( r = 1 \) only. Let \( e^i \) denote the point in \( \mathbb{R}^m \) all of whose coordinates vanish with the exception of the \( i \)th one which is equal to \( m+1 \). We have then for \( \theta \in \Gamma \) and \( y \in \Omega = \Omega_1(0) \)

\[
\sum_{i=1}^m |\theta \cdot (y - e^i)| \geq 1,
\]

so that

\[
H_{m-1}(\hat{B} \cap \Omega) \leq \sum_{i=1}^m \int_{B} |n(y) \cdot (y - e^i)| \, dH_{m-1}(y)
\]

\[
\leq (m+1)^n \sum_{i=1}^m \int_{B} \frac{|n(y) \cdot (y - e^i)|}{|y - e^i|^m} \, dH_{m-1}(y)
\]

\[
= (m+1)^n \sum_{i=1}^m v^c_{\theta}(e^i) \leq m(m+1)^n(A + V^c).
\]

2.15. THEOREM. Let \( C(B) \) denote the Banach space of all continuous functions \( f \) on \( B \) with the norm \( \|f\| = \sup |f| \). If \( Wf \) is bounded on \( \mathbb{R}^m - B \) for every \( f \in C(B) \) then

\[
(2.18) \quad V^c < \infty.
\]

If

\[
C_i = \mathbb{R}^m \cap \{z : d_C(z) = i\} \quad (i = 0, 1)
\]

and (2.18) holds, then \( Wf \) is bounded and uniformly continuous on each of the sets \( C_0, C_1 \) and

\[
(2.19) \quad \lim_{z \rightarrow y; \ z \in C_1} Wf(z) = Wf(y) + A(1 - d_C(y))f(y) \quad \text{for} \ y \in B \cap \text{cl} \ C_1,
\]

\[
(2.20) \quad \lim_{z \rightarrow y; \ z \in C_0} Wf(z) = Wf(y) - Ad_C(y)f(y) \quad \text{for} \ y \in B \cap \text{cl} \ C_0
\]

whenever \( f \in C(B) \).

**Proof.** If \( Wf(z) = \langle f, \nu_z \rangle \) is a bounded function of \( z \) on \( Q \subset \mathbb{R}^m \) for every \( f \in C(B) \) then, by the uniform boundedness principle, \( \|\nu_z\| = v^c_0(z) \) is bounded on \( Q \). In view of 2.9, \( v^c_0(z) \) must be bounded on \( \text{cl} \ Q \) as well. For \( Q = \mathbb{R}^m - B \) we get the first part of our theorem. Assume (2.18) and fix \( y \in B \). If \( f \equiv 1 \) on \( B \) then (2.19),
(2.20) follow from 2.6, 2.7. It is therefore sufficient to prove (2.19), (2.20) assuming \( f \in C(B) \), \( f(y)=0 \). For every \( k \) we have the decomposition \( f=f_k+g_k \), where \( f_k \in C(B) \) vanishes in some neighborhood of \( y \) in \( B \) and \( \|g_k\| \leq 1/k \). Then \( Wf_k \) is continuous at \( y \) and \( |Wg_k| \leq (A+V^c)/k \). We see that \( Wf=\lim_{k \to \infty} Wf_k \) is continuous at \( y \). The rest is obvious.

3. The Fredholm radius of an operator.

3.1. Notation. As in the introduction, \( G \) will stand for a fixed open set with a compact boundary \( B \) in \( \mathbb{R}^n \). We put \( C=R^m-G \) and write \( v_t(y)=v_t^p(y) \) (\( =v_t^c(y) \)), \( V=V^c \) (cf. 1.6, 1.7, (1.12), 2.13). We always assume

\[
V < \infty.
\]

In view of 1.13,

\[
NU : \mu \mapsto NU\mu
\]

is a bounded linear operator on \( C^*(B) \). By 2.7, \( G \) has an \( m \)-dimensional density \( d_\sigma(y) \) at any \( y \in \mathbb{R}^m \).

3.2. Lemma. If \( f \) is a bounded Baire function on \( B \) then

\[
<f, NU\delta_y> = Ad_\sigma(y)f(y) + W^cf(y), \quad y \in B.
\]

Proof. It is sufficient to prove (3.3) for \( f \in D \) only. Employing 1.3, 2.6, and 2.7 we obtain

\[
<f, NU\delta_y> = \int_G \text{grad} f(x) \cdot \frac{y-x}{|y-x|^m} \, dx
\]

\[
= Af(y) + \int_G \text{grad} f(x) \cdot \frac{x-y}{|x-y|^m} \, dx
\]

\[
= Ad_\sigma(y)f(y) + W^cf(y).
\]

3.3. Definition. If \( f \in C(B) \) we define

\[
WF(y) = <f, NU\delta_y> - \frac{1}{2}Af(y), \quad y \in B.
\]

3.4. Lemma. \( WF \in C(B) \) whenever \( f \in C(B) \). The operator

\[
WF : f \mapsto WF
\]

is bounded on \( C(B) \) and the operator (3.2) is adjoint to \( \frac{1}{2}AI + \overline{\theta} \), where \( I \) is the identity operator on \( C(B) \). If \( f \in C(B) \) and \( C_1 \) has the meaning described in 2.15, then

\[
WF(y) = \lim_{z \to y; z \in C_1} W^cf(z) - \frac{1}{2}Af(y), \quad y \in B \cap \text{cl} C_1,
\]

\[
WF(y) = W^cf(y) + A(d_\sigma(y) - \frac{1}{2})f(y)
\]

\[
= \lim_{z \to y; z \in G} W^cf(z) + \frac{1}{2}Af(y), \quad y \in B.
\]
Proof. (3.7), (3.6) follow from (3.4), (3.3), and (2.19), (2.20). By (3.7), \( Wf \in C(B) \) for \( f \in C(B) \). If \( \nu_y \) has the meaning described in 2.2 and
\[
\nu_y = A(d_G(y) - \frac{1}{2}) \delta_y + \nu_y,
\]
then
\[
Wf(y) = \langle f, \nu_y \rangle, \quad f \in C(B), \quad y \in B,
\]
whence
\[
\|W\| = \sup_{y \in B} \|\nu_y\| = \sup_{y \in B} (A|d_G(y) - \frac{1}{2}| + \nu_\nu(y)).
\]
By 1.13, the formula (1.2) holds for any \( \psi \in C(B) \). This together with (3.4) implies
\[
NU = (\frac{1}{2}AI + W)^*,
\]
where \((\cdots)^*\) denotes the operator adjoint to \((\cdots)\).

3.5. Remark. In §4 we shall be engaged with the Neumann problem in the following formulation: Given \( v \in C^*(B) \) find a \( \mu \in C^*(B) \) with \( NU\mu = v \). By (3.11), this problem reduces to solving the equation
\[
(\frac{1}{2}AI + W)^*\mu = v.
\]
In connection with this equation it is useful to know the Fredholm radius of \( W \), i.e., the reciprocal of
\[
\omega W = \inf_T \|W - T\|,
\]
where \( T \) ranges over all compact operators on \( C(B) \) (cf. [RS]). Our main objective in §3 is to express \( \omega W \) in terms of geometric quantities connected with \( G \) and investigate relations between \( \omega W \) and regularity of \( B \).

3.6. Theorem. Let \( I_B \) denote the set of all isolated points of \( B \) and put \( E = B - I_B \) if \( I_B \) is finite, \( E = B \) in the opposite case. Let \( V_r = 0 \) or
\[
V_r = \sup_{y \in E} [A|\frac{1}{2} - d_G(y)| + \nu_r(y)]
\]
according as \( E = \emptyset \) or not and define
\[
V_0 = \lim_{r \to 0^+} V_r.
\]
Then \( \omega W = V_0 \).

Proof will be divided into two steps.

Step 1. We first prove that
\[
\omega W \leq V_r
\]
for every \( r > 0 \) satisfying
\[
H_{m-1}(B \cap \{z : |z - y| = r\}) = 0 \quad \text{for all } y,
\]
where \( B \) is the reduced boundary defined in 2.11. If \( R \) is the set of all \( r > 0 \) enjoying (3.13) then \( (0, \infty) - R \) is at most countable, because spherical shells with different radii meet each other in a set of \( H_{m-1} \)-measure zero and \( H_{m-1}(B) < \infty \). Hence \( V_0 = \inf \{ V_r : r \in R \} \) and

\[
(3.14) \quad \omega W \leq V_0
\]

will follow from (3.12). So let us fix \( r \in R \). If \( I_B \) is finite we assume, as we may, \( r < \operatorname{dist}(I_B, E) = \inf \{ \operatorname{dist}(z, E) : z \in I_B \} \). Let \( c_y \) denote the characteristic function of \( B - (\Omega_r(y) \cap E) \) and put

\[
W_r f(y) = \int_B c_y f d\nu_y, \quad f \in C(B),
\]

where \( \nu_y \) is defined by (3.8). Absolute values of all the functions in

\[
(3.15) \quad \{ W_r f : f \in C(B), \| f \| \leq 1 \}
\]

are bounded by \( \sup_{y \in B} \| \nu_y \| \leq \frac{1}{2} \Lambda + V \). If \( f \in C(B) \) and \( x, y \) are arbitrary points in \( E \) with \( |x - y| = d \leq \frac{1}{2} r \), then we obtain from 2.12

\[
W_r f(x) - W_r f(y) = J_1(f) + J_2(f),
\]

where

\[
J_1(f) = \int_B f(z) [c_x(z) - c_y(z)] \frac{n(z) \cdot (z - x)}{|z - x|^m} dH_{m-1}(z),
\]

\[
J_2(f) = \int_B f(z) c_y(z) \left[ \frac{z - x}{|z - x|^m} - \frac{z - y}{|z - y|^m} \right] \cdot n(z) dH_{m-1}(z).
\]

Denoting

\[
\alpha(d) = \sup_{z} H_{m-1}[(\overline{\Omega_{r} + d(z)} - \Omega_{r - d}(z)) \cap \bar{B}]
\]

we get for \( \| f \| \leq 1 \)

\[
|J_1(f)| \leq (\frac{1}{2} r)^{1-m} \alpha(d),
\]

\[
|J_2(f)| \leq (m + 1) d (\frac{1}{2} r)^{-m}.
\]

Since \( r \in R \), an easy compactness argument yields

\[
\lim_{d \to 0^+} \alpha(d) = 0.
\]

We see that all the functions in (3.15) are equicontinuous on \( E \); noting that \( B - E \) is finite we conclude that they are equicontinuous on \( B \) as well and the operator

\[
W_r : f \mapsto W_r f
\]

is compact. Hence

\[
\omega W \leq \| W - W_r \|.
\]
If \( f \in C(B) \) then
\[
W_r f(y) = \overline{W} f(y) \quad \text{for } y \in B - E
\]
while (3.9) shows that, for \( y \in E \),
\[
(W - W_r) f(y) = \int f \, d\bar{v}_y
\]
with the integral extended over \( B \cap \Omega_r(y) \). Consequently,
\[
\| W - W_r \| = \sup_{y \in E} |\bar{v}_y| (\Omega_r(y) \cap B) = V_r
\]
and (3.12) is established.

**Step 2.** Now we are going to prove the inequality
\[
(3.16)
\]
which is trivial if \( E = \emptyset \). Therefore we assume \( E \neq \emptyset \), so that \( E \) is infinite. A point \( y \in B \) will be termed a discontinuity for a \( \mu \in C^*(B) \) if \( \mu(\{y\}) \neq 0 \). By the Radon theorem, every compact operator on \( C(B) \) can be arbitrarily closely approximated by operators of finite rank. If \( Q \) is such an operator, sending \( f \in C(B) \) into
\[
(3.17)
Qf = \sum_{k=1}^{q} g_k \langle f, m_k \rangle
\]
where \( g_k \in C(B) \) and \( m_k \in C^*(B) \), then every \( m_k \) can be arbitrarily closely (in the norm of \( C^*(B) \)) approximated by \( \bar{m}_k \in C^*(B) \) having only a finite number of discontinuities. Defining
\[
\bar{Q} \cdots = \sum_{k=1}^{q} g_k \langle \cdots, \bar{m}_k \rangle
\]
we see that the deviation \( \| Q - \bar{Q} \| \) can be made as small as we want. It follows from these observations that, in order to prove (3.16), it is sufficient to show that
\[
(3.18)
\]
for every \( Q \) of the type (3.17), where \( m_k \in C^*(B) \) have only a finite number of discontinuities each. Let us fix such a \( Q \) and denote by \( K \) the (finite) set of all \( y \in B \) which represent a discontinuity for some of the measures \( m_k \). Every \( m_k \) splits into \( m_k^\sharp \) having no discontinuities and a finite combination of Dirac measures, to be denoted by \( m_k^\triangle \). Since \( y \) is the only possible discontinuity for \( \bar{v}_y \), for \( y \in B - K \)
\[
\left\| \bar{v}_y - \sum_{k} g_k(y) m_k^\sharp \right\| = \left\| \bar{v}_y - \sum_{k} g_k(y) m_k^\sharp \right\| + \left\| \sum_{k} g_k(y) m_k^\triangle \right\|
\]
whence
\[
\| W - Q \| \geq \sup \left\{ \left\| \bar{v}_y - \sum_{k} g_k(y) m_k^\sharp \right\| : y \in E - K \right\}.
\]
Since the operator

\[ f \rightarrow \left\langle f, \bar{v}_y - \sum_k g_k(y)m_k \right\rangle \]

sends each \( f \in C(B) \) into a continuous function of \( y \) we conclude that

\[ a_r(y) = \left| \bar{v}_y - \sum_k g_k(y)m_k \right| (\Omega_r(y) \cap B) \]

is a lower semicontinuous function of \( y \) for every \( r > 0 \). Consequently,

\[ \| W - Q \| \geq \sup \{ a_r(y) : y \in E - K \} \]

\[ = \sup \{ a_r(z) : z \in E - (I_B \cap K) \}. \]

(3.19)

Consider now an arbitrary \( y \in E \cap I_B \cap K \) and note that \( E \cap I_B \neq \emptyset \) implies

\[ \emptyset \neq E \cap (I_B - K) \subset E - (I_B \cap K). \]

If

\[ r < \text{dist} \ (I_B \cap K, E - (I_B \cap K)), \]

then

\[ \sum_k g_k(y)m_k \mid (\Omega_r(y) \cap B) = \left| \sum_k g_k(y)m_k \right| (\Omega_r(y) \cap I_B) = 0, \]

\[ a_r(y) = |\bar{v}_y| (\Omega_r(y) \cap I_B) = \frac{1}{2}A. \]

(3.20)

On the other hand, we have for any \( z \in I_B \)

\[ \frac{1}{2}A \leq A(\frac{1}{2} - d_0(\delta))((\Omega_r(z) \cap B) + \mid v_z - \sum_k g_k(z)m_k \mid (\Omega_r(z) \cap B) = a_r(z), \]

because \( v_z - \sum_k g_k(z)m_k \) has no discontinuities. Combining this with (3.20) we get

\[ a_r(y) \leq \sup \{ a_r(z) : z \in E - (I_B - K) \} \leq \sup \{ a_r(z) : z \in E - (I_B \cap K) \}. \]

We have thus for small \( r > 0 \)

\[ \sup \{ a_r(z) : z \in E - (I_B \cap K) \} = \sup \{ a_r(y) : y \in E \}. \]

(3.21)

Note that

\[ V_r = \sup_{y \in \mathbb{K}} |\bar{v}_y|((\Omega_r(y) \cap B)). \]

If \( M = \max \{|g_k(x)| : x \in B, 1 \leq k \leq q\} \), then

\[ \sup \{ a_r(y) : y \in E \} \geq V_r - M \sum_k \sup_{y \in \mathbb{K}} |m_k|((\Omega_r(y) \cap B)). \]

(3.22)
Since $m_k^2 (k = 1, \ldots, q)$ have no discontinuities,
\[
\limsup_{r \to 0^+} \sup_{y \in B} |m_k^2| (\Omega_r(y) \cap B) = 0.
\]
Making $r \to 0^+$ in (3.22) and using (3.21), (3.19) we arrive at (3.18).

**Remark.** The basic idea of the above proof goes back to J. Radon (cf. [RS]).

3.7. Lemma. Let us define $\hat{B}$ as in 2.11 and put
\[
B^* = B \cap \{ y : |dG(y) - \frac{1}{r}| < \frac{1}{r} \}.
\]
Then $\hat{B}$ is dense in $B^*$ (moreover, every ball of center in $B^*$ meets $\hat{B}$ in a set of positive $H_{m-1}$-measure) and
\[
H_{m-1}(B^* - \hat{B}) = 0.
\]
**Proof.** If $y \in B^*$ then there is an $\varepsilon > 0$ such that
\[
H_m(\Omega_r(y) \cap G) > \varepsilon H_m(\Omega_r(y)),
\]
\[
H_m(\Omega_r(y) \cap C) > \varepsilon H_m(\Omega_r(y))
\]
for $0 < r < \varepsilon$. By the relative isoperimetric inequality for sets with finite perimeter (cf. Theorem (4.3) in [MI]; general isoperimetric inequalities for currents may be found in [FF, §6]) we conclude that
\[
H_{m-1}(\Omega_r(y) \cap \hat{B}) \geq \alpha r^{m-1}, \quad 0 < r < \varepsilon,
\]
where $\alpha > 0$ does not depend on $r$. Hence it follows by [F2, §3] that
\[
H_{m-1}(B^* - \hat{B}) = 0.
\]

3.8. Notation. For $z \in \mathbb{R}^m$, $r > 0$ and $\theta \in \Gamma$ we put
\[
\Omega_r(z, \theta) = \Omega_r(z) \cap \{ x : (x - z) \cdot \theta > 0 \}.
\]
We denote by $a(\theta, \eta) = \arccos(\theta \cdot \eta)$ the nonoriented angle enclosed by $\theta, \eta \in \Gamma$.

It is easily seen that
\[
a(\theta, \eta) = \frac{H_m(\Omega_r(z, \theta) \cap \Omega_r(z, -\eta))}{H_m(\Omega_r(z))}
\]
(3.23)
The symbol $n$ will always have the meaning described in 2.11. The symmetric difference of $P, Q \subset \mathbb{R}^m$ will be denoted by $P \Delta Q$.

3.9. Lemma. Let $z \in \hat{B}, \theta = n(z)$. Then
\[
H_m(\Omega_r(z, \theta) \cap \text{int } C) + H_m(\Omega_r(z, -\theta) \cap G) \leq H_m(\Omega_r(z)) \frac{v_r(z)}{A}.
\]
**Proof.** Let
\[
\gamma_1 < \frac{H_m(\Omega_r(z, \theta) \cap \text{int } C)}{H_m(\Omega_r(z))}, \quad \gamma_2 < \frac{H_m(\Omega_r(z, -\theta) \cap G)}{H_m(\Omega_r(z))}.
\]
Put \( \Gamma_+ = \Gamma \cap \{ \eta : \eta \cdot \theta > 0 \} \), \( \Gamma_- = \Gamma \cap \{ \eta : \eta \cdot \theta < 0 \} \), \( S(\rho) = \{ x : |x-z| = \rho \} \) and define \( K_\rho, L_\rho \) as in 2.6. There are \( \rho_1, \rho_2 \in (0, r) \) such that

\[
(3.24) \quad H_{m-1}(S(\rho_1) \cap \Omega_\rho(z, \theta) \cap \text{int } C) > \gamma_1 H_{m-1}(S(\rho_1)),
\]

\[
(3.25) \quad H_{m-1}(S(\rho_2) \cap \Omega_\rho(z, -\theta) \cap G) > \gamma_2 H_{m-1}(S(\rho_2)).
\]

By virtue of 2.7

\[
H_{m-1}(L_\rho \cap \Gamma_+) = \frac{1}{A} \lim_{\rho \to 0^+} \frac{H_m(\Omega_\rho(z, \theta) \cap C)}{H_m(\Omega_\rho(z, \theta))} = 0,
\]

\[
H_{m-1}(L_\rho \cap \Gamma_-) = \frac{1}{A} \lim_{\rho \to 0^+} \frac{H_m(\Omega_\rho(z, -\theta) \cap C)}{H_m(\Omega_\rho(z, -\theta))} = \frac{1}{A}.
\]

We see that \( L_\rho \) is equivalent \((H_{m-1})\) with \( \Gamma_- \) and \( K_\rho \) is equivalent \((H_{m-1})\) with \( \Gamma_+ \). If \( \eta \in L_\rho \) and \( \{ z + \rho \eta : 0 < \rho < r \} \cap G \neq \emptyset \) then, with the notation from 1.6, \( n_\rho(\eta, z) \geq 1 \). Employing (3.25) we obtain

\[
\int_{L_\rho} n_\rho(\eta, z) \, dH_{m-1}(\eta) > \gamma_2 A.
\]

Similarly, (3.24) implies

\[
\int_{K_\rho} n_\rho(\eta, z) \, dH_{m-1}(\eta) > \gamma_1 A,
\]

so that

\[
v_\rho(z) = \int_{\Gamma} n_\rho(\eta, z) \, dH_{m-1}(\eta) > (\gamma_1 + \gamma_2)A.
\]

3.10. Lemma. Let \( N \in \Gamma \), \( y \in R^n \), \( r > 0 \) and suppose that

\[
(3.26) \quad \sup_{z \in B} v_\rho(z) \leq u_0 A,
\]

\[
(3.27) \quad H_m(\Omega_\rho(y, N) \cap C) \leq u_1 H_m(\Omega_\rho(y)),
\]

\[
(3.28) \quad H_m(\Omega_\rho(y, -N) \cap \text{cl } G) \leq u_2 H_m(\Omega_\rho(y)).
\]

If \( s = u_0 + u_1 + u_2 < \frac{1}{2} \), then for every \( \gamma > s \) there is a \( \delta > 0 \) (depending on \( (\gamma - s)r \) only) such that

\[
a(n(\rho, z), N) \leq \pi \gamma \quad \text{for } z \in \bar{B} \cap \Omega_\rho(y). \]

Proof. Let \( \gamma = s + 2\epsilon \), \( \epsilon > 0 \), and consider a \( \theta \in \Gamma \) with \( a(N, \theta) > \gamma \pi \). We have by (3.23)

\[
H_m(\Omega_\rho(y, -N) \cap \Omega_\rho(y, \theta)) > \frac{1}{2} \gamma H_m(\Omega_\rho(y)),
\]

\[
H_m(\Omega_\rho(y, N) \cap \Omega_\rho(y, -\theta)) > \frac{1}{2} \gamma H_m(\Omega_\rho(y)).
\]
Let us fix $\delta > 0$ small enough to secure

$$H_m(\Omega_r(z, \eta) - \Omega_r(y, \eta)) < \epsilon H_m(\Omega_r(y))$$

for $|z - y| < \delta$ and any $\eta \in \Gamma$. We have then for $z \in \Omega_\delta(y)$

$$H_m(\Omega_r(y, -N) \cap \Omega_r(z, \theta)) > \frac{1}{2}s H_m(\Omega_r(z)),$$
$$H_m(\Omega_r(y, N) \cap \Omega_r(z, -\theta)) > \frac{1}{2}s H_m(\Omega_r(z)),$$

whence we obtain on account of (3.27), (3.28)

\begin{align}
H_m(\Omega_r(z, \theta) \cap \text{int } C) & \geq H_m(\Omega_r(z, \theta) \cap \Omega_r(y, -N)) - H_m(\Omega_r(y, -N) \cap \text{cl } G) \\
& > (\frac{1}{2}s - u_0) H_m(\Omega_r(z)),
\end{align}

(3.29)

\begin{align}
H_m(\Omega_r(z, -\theta) \cap G) & \geq H_m(\Omega_r(z, -\theta) \cap \Omega_r(y, N)) \\
& > (\frac{1}{2}s - u_1) H_m(\Omega_r(z)).
\end{align}

(3.30)

Suppose now that $z \in \mathcal{B}$ and $\theta = n(z)$. Employing (3.29), (3.30) and Lemma 3.9 we arrive at $v_1(z) > u_0 A$, which violates (3.26).

3.11. Notation. Let $P_N$ stand for the orthogonal projection of $R^m$ onto $R^m \cap \{x : x \cdot N = 0\}$. With every $\alpha \in (0, \frac{1}{2})$ we associate $B(\alpha) \subset B$ as follows. We let $y \in B(\alpha)$ if for every $\gamma \in (\alpha, \frac{1}{2})$ there is a neighborhood $Q$ of $y$ in $B$ and an $N \in \Gamma$ such that $|P_N(x) - P_N(z)| \geq |x - z| \cos \pi \gamma$ whenever $x, z \in Q$. By Theorem 5.1 in [MI] we get the following corollary of Lemma 3.10:

3.12. Corollary. If (3.26), (3.27), (3.28) hold and $s = u_0 + u_1 + u_2 < \frac{1}{2}$, then $y \in B(s)$; moreover, for every $\gamma \in (s, \frac{1}{2})$ there is a $\delta > 0$ (depending on $r(y - s)$ only) such that $B \cap \Omega_\delta(y) \subset B(\gamma)$.

3.13. Theorem. If $V_0 < \frac{1}{4} A$ then $I_B$ is finite and

$$H_{m-1}(B - B(V_0/A)) = 0.$$

If $V_0 < \frac{1}{4} A$ then $B = B(2V_0/A)$.

Proof. Let $V_0 < \frac{1}{4} A$. Then $I_B$ must be finite, $B - I_B = E \subset B^*$ and, by 3.7, $H_{m-1}(B - \mathcal{B}) = 0$. To prove (3.31) it is therefore sufficient to show that

$$\mathcal{B} \subset B\left(3e + \frac{V_0}{A}\right)$$

for every small $\epsilon > 0$. Fix $y \in \mathcal{B}$, $N = n(y)$ and $\epsilon > 0$, $3e + V_0/A < \frac{1}{4}$. We have then for sufficiently small $r > 0$

$$H_m(\Omega_r(y, N) \cap C) \leq \epsilon H_m(\Omega_r(y)),$$
$$H_m(\Omega_r(y, -N) \cap \text{cl } G) = H_m(\Omega_r(y, -N) \cap G) \leq \epsilon H_m(\Omega_r(y)),$$

(3.32)

$$V_r < V_0 + \epsilon A.$$
Employing 3.12 we get $y \in B(3e + V_0/A)$. Suppose now that $\alpha = V_0/A < \frac{1}{4}$ and fix an $r > 0$ with (3.32). By 3.9 we have for all $y \in \hat{B}$

$$H_m(\Omega_r(y), n(y)) \cap C) + H_m(\Omega_r(y), -n(y)) \cap cl G) < (\alpha + \varepsilon)H_m(\Omega_r(y)).$$

By 3.12 there is a $\delta > 0$ independent of $y$ such that $\Omega_r(y) \cap B \subset B(3e + 2\delta)$ for every $y \in \hat{B}$. It remains to note that $\hat{B}$ is dense in $E$ by 3.7.

3.14. Corollary. If $V_0 < \frac{1}{4} A$ then

$$\lim_{r \to 0^+} \sup \{p^{1-m}H_{m-1}(\Omega_\rho(y) \cap B) : y \in B, 0 < \rho < r \} \leq b_{m-1} \sec (2V_0\pi/A),$$

where $b_{m-1}$ denotes the volume of the unit ball in $R^{m-1}$.

3.15. Theorem. Let $V_0 = 0$ (which means that $W$ is compact). Then $I_B$ is finite and $E = B - I_B$ is a surface of class $C^1$.

Proof. For every $\varepsilon > 0$, $\varepsilon < \frac{1}{4}$ there is an $r > 0$ such that $V_r < \varepsilon A$. By 3.9 (note also that $H_m(B) = 0$)

$$H_m(\Omega_r(y), n(y)) \cap C) + H_m(\Omega_r(y), -n(y)) \cap cl G) < \varepsilon H_m(\Omega_r(y))$$

for all $y \in \hat{B}$. Employing 3.10 we get a $\delta > 0$ depending on $r\varepsilon$ only such that, for every couple of points $y, z \in \hat{B}$, $a(n(z), n(y)) \leq 3\pi\varepsilon$ whenever $|y - z| < \delta$. We see that $n$ is uniformly continuous on $\hat{B}$. By 3.7, $n$ extends to a continuous function $N$ on $E = \text{cl } B$ and, for every $y \in E$,

$$N(y) = \lim_{r \to 0^+} \frac{\int N(z) \, dH_{m-1}(z)}{H_{m-1}(\Omega_r(y) \cap E)}$$

with the integral extended over $\Omega_r(y) \cap E$. Hence it follows by [DG3, Theorem III] (see also definition of the reduced boundary presented in [DG3, p. 10]) that $E$ is a surface of class $C^1$.

Remark. The main results of this paper (such as Theorem 1.13 or Theorem 3.6) are expressed in terms of the quantity $v_r(y)$. In the definition of $v_r(y)$ one considers all half-lines issuing at $y$, i.e., orthogonal trajectories of the level surfaces of the Green function with a fixed pole at $y$. This suggests the possibility of generalizing these results to the case of a Green space in the sense of [BC].


Notation. We shall keep the notation and assumptions introduced in §3. Besides that we always assume here that $m > 2$ (see Remark 4.10 below dealing with $m = 2$). We shall start with investigation of solutions of the equations

(4.1) \hspace{1cm} \left(\frac{1}{2}AI + \bar{W}\right)f = 0 \hspace{0.5cm} \text{over } C(B),

(4.2) \hspace{1cm} \left(\frac{1}{2}AI + \bar{W}\right)^* \mu = 0 \hspace{0.5cm} \text{over } C^*(B).

$C_0(B)$ will denote the class of all $f \in C(B)$ satisfying (4.1) and $C^*(B)$ will stand for
the set of all \( \mu \in C^*(B) \) satisfying (4.2). We agree to use \( M \) as a generic notation for a Borel set. If \( \mu \) is a signed Borel measure in \( R^m \) and \( R \subseteq R^m \) is a fixed Borel set, we define \( \mu \cap R \) by

\[
\mu \cap R(M) = \mu(M \cap R), \quad M \subseteq R^m.
\]

Recalling the definition of \( \bar{v}_y \) presented in (3.8) we obtain from (3.9) that, for every \( \mu \in C^*(B) \),

\[
(W^* \mu)(M) = \int_B \bar{v}_y(M) \, d\mu(y), \quad M \subseteq B.
\]

It follows from (3.10) that

\[
\|W^*\| \leq \frac{1}{2} A + V,
\]

where \( V = V^\circ \) has been defined in 2.13.

4.1. Lemma. If \( \mu \in C^0_B(B) \) then \( |\mu|(I_B) = 0 \) (see 3.6 for notation).

Proof. Let \( \mu \in C^0_B(B) \), \( z \in I_B \) and denote by \( f \) the characteristic function of \( \{z\} \). We have by 3.4, 1.13, and (3.3)

\[
0 = NU\mu(\{z\}) = \int_B (Ad_\phi(y)f(y) + Wcf(y)) \, d\mu(y).
\]

It follows from (2.12) that \( Wcf = 0 \), so that \( Ad_\phi(y)f(y) + Wcf(y) = Af(y) \) for all \( y \in B \). Hence \( \mu(\{z\}) = 0 \).

Remark. A refinement of the preceding argument may be used to show that, for every \( \mu \in C^0_B(B) \), \( \mu \cap M \) is absolutely continuous with respect to \( H_{m-1} \cap \hat{B} \) provided \( d_\phi(y) > 0 \) for all \( y \in M \).

As it follows from 4.1, \( C^0_B(B) \) contains only trivial measure in case \( B = I_B \). In what follows we always exclude the trivial case of a finite \( B \).

4.2. Lemma. Fix \( z \in B \), \( \mu \in C^*(B) \) and put for \( t > 0 \)

\[
R_t = B \cap \Omega_t(z), \quad \alpha(t) = H_{m-1}(R_t \cap \hat{B}), \quad \beta(t) = |\mu|(R_t).
\]

Let \( 0 < \rho < \delta < \Delta \) and suppose that

\[
\mu \cap (R_{\Delta} - R_{\rho}) = \mu.
\]

Then

\[
|W^* \mu|(R_\rho) \leq \frac{\alpha(\rho)\beta(\Delta)}{(\Delta - \rho)^{m-1}} + (m-1)\alpha(\rho) \int_\rho^\Delta \frac{\beta(t) \, dt}{(t-\rho)^m}.
\]
Proof. Let $g$ denote the characteristic function of $B \cap R_\rho$. By 2.12 we obtain for $y \in R_\Delta - R_\epsilon$ and $M \subset R_\rho$

$$|\nu_y(M)| = |\nu_y(M)| \leq \int_M g(x)|y-x|^{1-m} \, dH_{m-1}(x),$$

whence it follows easily by (4.3)

(4.9) \quad |W^*_\mu|(R) \leq \int_{B \times B} g(x)|y-x|^{1-m} \, dH_{m-1}(x) \, d\mu(y).

Since

$$\int_B |y-x|^{1-m} \, d|\mu|(y) \leq \int_0^\Delta (t-|x|)^{1-m} \, d\beta(t)$$

$$\leq \frac{\beta(\Delta)}{(\Delta-|x|)^{n-1} + (m-1) \int_0^\Delta \frac{\beta(t)}{(t-|x|)^m}}$$

(4.9) implies (4.8).

4.3. Lemma. Fix $z \in B$, $r > 0$ and put, with the notation from 4.2,

$$R = R_r(= \Omega_r(z) \cap B),$$

$$V(R) = \sup \{|\nu_y|(R) : y \in R\},$$

$$Q(R) = \sup \{\rho^{1-m}\alpha(\rho) : 0 < \rho < r\},$$

$$K(R) = \inf \left\{V(R)k^{m-2} + Q(R)\left(\frac{k}{k-1}\right)^{m-1} - 1 : k > 1\right\}.$$

Define

$$W^*_\mu = (W^*_{-\mu}) \cap R, \quad \mu \in C^*(B).$$

Let $C^*_R$ denote the set of all $\mu \in C^*(B)$ enjoying

$$J(\mu) = \int_0^\gamma \rho^{1-m}|\mu|(R_\rho) \, d\rho < \infty$$

and

(4.10) \quad |\mu|(B - R) = 0

and put

$$\|\mu\|_R = \frac{1}{m-2} r^{2-m} \|\mu\| + J(\mu), \quad \mu \in C^*_R.$$

Then $\mu \in C^*_R$ implies $W^*_\mu \in C^*_R$ and

$$\|W^*_\mu\|_R \leq K(R)\|\mu\|_R.$$

Proof. Fix $\mu \in C^*_R$ and $k > 1$. We have with the notation from (4.7)

(4.11) \quad J(\mu) = \int_0^\gamma \rho^{1-m}\beta(\rho) \, d\rho.$
Let now $0 < \rho < r/k$ and define

$$\mu_{\rho} = \mu \cap R_{k\rho}, \quad \mu^\rho = \mu - \mu_{\rho}.$$ 

In view of (4.3)

$$\|W^s_{\#} \mu^\rho\| \leq V(R) \beta(k\rho).$$

Employing 4.2 we obtain

$$\|W^s_{\#} \mu\|_{(R_{\rho})} \leq \frac{\alpha(\rho)\beta(r)}{(r-\rho)^{m-1} + (m-1)\alpha(\rho) \int_{k\rho}^{r} \frac{\beta(t) \, dt}{(t-\rho)^m}}$$

On account of (4.12), (4.13) we get for $0 < \rho < r/k$

$$\rho^{1-m}\|W^s_{\#} \mu\|_{(R_{\rho})} \leq V(R)\rho^{1-m}\beta(k\rho) + Q(R) \frac{\beta(r)}{(r-\rho)^{m-1} + (m-1)Q(R) \int_{k\rho}^{r} \frac{\beta(t) \, dt}{(t-\rho)^m}},$$

while, by (4.3),

$$\rho^{1-m}\|W^s_{\#} \mu\|_{(R_{\rho})} \leq V(R)\rho^{1-m}\beta(r) \quad \text{for } r/k \leq \rho < r.$$ 

Using (4.11) we obtain after simple calculation

$$J(W^s_{\#} \mu) = \int_0^{r/k} \rho^{1-m} \|W^s_{\#} \mu\|_{(R_{\rho})} \, d\rho + \int_{r/k}^{r} \rho^{1-m} \|W^s_{\#} \mu\|_{(R_{\rho})} \, d\rho$$

$$\leq \frac{\beta(r)}{m-2} r^{2-m} \left( V(R)(k^{m-2} - 1) + Q(R) \left[ \left( \frac{k}{k-1} \right)^{m-2} - 1 \right] \right)$$

$$+ J(\mu) \left( V(R)k^{m-2} + Q(R) \left[ \left( \frac{k}{k-1} \right)^{m-1} - 1 \right] \right).$$

Since, by virtue of (4.3),

$$\|W^s_{\#} \mu\| \leq V(R)\beta(r),$$

we get finally

$$\|W^s_{\#} \mu\|_{R} = \frac{1}{m-2} r^{2-m} \|W^s_{\#} \mu\| + J(W^s_{\#} \mu)$$

$$\leq \|\mu\|_{R} \left( V(R)k^{m-2} + Q(R) \left[ \left( \frac{k}{k-1} \right)^{m-1} - 1 \right] \right).$$

4.4. Notation. Let

$$Q_r = \sup \{\rho^{1-m}H_{m-1}(\Omega_\rho(z) \cap B) : z \in B, 0 < \rho < r\}, \quad r > 0,$$

$$Q_0 = \lim_{r \to 0^+} Q_r.$$ 

Further define $V_0$ as in 3.6 and put

$$K_0 = \inf \left\{ V_0 k^{m-2} + Q_0 \left[ \left( \frac{k}{k-1} \right)^{m-1} - 1 \right] : k > 1 \right\}.$$
In what follows we shall always assume that

\[(4.15) \quad K_0 < \frac{1}{4}A.\]

4.5. Remark. The inequality (4.15) implies

\[(4.16) \quad V_0 < \frac{1}{4}A.\]

Indeed, since \(B\) is infinite and \(V_0 < \frac{1}{4}A\), (3.7) secures \(H_{m-1}(B) > 0\). It is known from [DG2], [F3] that for \((H_{m-1})\) almost all \(y \in \mathcal{B}\)

\[\lim_{\rho \to 0^+} \rho^{1-m}H_{m-1}(\Omega_\rho(y) \cap \mathcal{B}) = b_{m-1},\]

where \(b_{m-1}\) denotes the volume of the unit ball in \(R^{m-1}\). Hence \(Q_0 \geq b_{m-1}\) and \(k\) minimizing

\[V_0k^{m-2} + Q_0\left[\left(\frac{k}{k-1}\right)^{m-1} - 1\right]\]

must satisfy

\[\frac{1}{4}A > Q_0\left[\left(\frac{k}{k-1}\right)^{m-1} - 1\right] \geq b_{m-1}\frac{m-1}{k-1},\]

which guarantees \(k^{m-2} > 2\).

On the other hand, if (4.16) holds, then 3.14 provides an estimate for \(Q_0\) in terms of \(V_0\). Clearly, (4.15) is fulfilled whenever \(V_0\) is sufficiently small.

In view of (4.16) and 3.6, the Fredholm theory applies to the pair of adjoint equations

\[(\frac{1}{4}AI + \overline{W})f = g,\]

\[(\frac{1}{4}AI + \overline{W})^*\mu = \nu.\]

4.6. Lemma. If \(\mu \in C_0^s(B)\) then \(U|\mu|\) (see the introduction for notation) is bounded on \(B\).

Proof. Define \(V_r\) and \(E\) as in 3.6 and fix \(r > 0\) and \(k > 1\) such that

\[K = V_rk^{m-2} + Q_r\left[\left(\frac{k}{k-1}\right)^{m-1} - 1\right] < \frac{1}{4}A,\]

\[r < \text{dist}(E, B - E).\]

Fix an arbitrary \(z \in E\) and define \(R = \Omega_(z) \cap B\). We have then with the notation from 4.3

\[V(R) \leq V_{2r}, \quad Q(R) \leq Q_r,\]

\[(4.17) \quad K(R) \leq K < \frac{1}{4}A.\]

Let \(\mu \in C_0^s(B), \|\mu\| \leq 1\) and put

\[\mu_0 = \mu \cap R, \quad \mu^0 = \mu - \mu_0.\]
In view of (4.2)

\[(4.18) \quad \frac{1}{2}A\mu_0 + \overline{W}^*\mu_0 = -\frac{1}{2}A\mu_0 - \overline{W}^*\mu_0.\]

Restricting all measures occurring in (4.18) to \(R\) we obtain

\[(4.19) \quad (I + 2A^{-1}\overline{W}_R^*)\mu_0 = -2A^{-1}\overline{W}_R^*\mu_0\]

where, of course, \(I\) is the identity operator. Employing 4.2 with \(\delta = r\) and \(\Delta = r + \text{diam } B\) we obtain easily for \(0 < r \leq r/2\)

\[\rho^{1-m}\|\overline{W}^*\mu^0\| (R_\rho) \leq Q_r 2^{m-1-r}\|\mu^0\| \leq Q_r 2^{m-1-r}.\]

On the other hand, we have for \(r > r/2\)

\[\rho^{1-m}\|\overline{W}^*\mu^0\| (R_\rho) \leq 2^{m-1-r}\|\overline{W}^*\| \cdot \|\mu^0\| \leq 2^{m-1-r}(\frac{1}{2}A + V),\]

so that

\[J(\overline{W}_R^*\mu_0) \leq 2^{m-2}2^{-m}(2Q_r + \frac{1}{2}A + V).\]

Since

\[\|\overline{W}_R^*\mu_0\| \leq \|\overline{W}^*\mu_0\| \leq \frac{1}{2}A + V\]

we arrive at

\[\|\overline{W}_R^*\mu_0\|_R \leq \gamma_r,\]

where

\[\gamma_r = \frac{1}{m-2}2^{m-1-r}(\frac{1}{2}A + V) + 2^{m-2}2^{-m}(2Q_r + \frac{1}{2}A + V).\]

We see that \(\overline{W}_R^*\mu_0 \in C^*_R\). It is easily seen that \(C^*_R\), equipped with the norm \(\|\cdot\|_R\), is a Banach space. In view of (4.3) and (4.17)

\[\|\overline{W}_R^*\|_R \leq K < \frac{1}{2}A.\]

Hence we conclude by virtue of (4.19) that \(\mu_0 \in C^*_R\) and

\[\|\mu_0\|_R \leq \left(1 - \frac{2K}{A}\right)^{-1}2A^{-1}\gamma_r = a_r.\]

Since \(a_r\) is independent of \(z \in E\), we have, in particular,

\[\sup_{z \in E} \int_0^r \rho^{1-m}\|\mu\|((\Omega_\rho(z) \cap B) \, d\rho < \infty,\]

whence it follows easily

\[\sup_{z \in E} \int_0^\infty \rho^{1-m}\|\mu\|((\Omega_\rho(z) \cap B) \, d\rho < \infty.\]
Noting that

\[ U|\mu|(z) = \frac{1}{m-2} \int_B |x-z|^{2-m} d|\mu|(x) \]

\[ = \frac{1}{m-2} \int_0^\infty |\mu|(B \cap \{x : |x-z|^{2-m} > t\}) dt \]

\[ = \int_0^\infty t^{1-m}|\mu|(\Omega(z) \cap B) d\rho \]

we see that \( U|\mu| \) is bounded on \( E \). Since, by 4.1, \( \text{spt} \mu \subset E \) and \( B-E \) has a positive distance from \( E \), \( U|\mu| \) is bounded on \( B \) as well.

4.7. Notation. It follows easily from (4.16) and 3.13 that \( G \) has only a finite number of components; their closures are mutually disjoint. We shall denote by \( q(0 \leq q < \infty) \) the number of bounded components of \( G \). \( G_0 \) will stand for the unbounded component of \( G \) (if any); the bounded components of \( G \) will be denoted by \( G_1, \ldots, G_q \).

Employing 4.6 we obtain by standard reasoning the following.

4.8. Lemma. The dimension of \( C^\infty_0(B) \) does not exceed \( q \).

Proof. Let \( \mu \in C^\infty_0(B) \). By 4.6, \( U|\mu| \) is bounded on \( B \). Hence it follows that \( \mu \) has finite energy \([B, p. 122]\) and

\[ \int_B |\text{grad} U\mu(x)|^2 dx = A \int_B U\mu(y) d\mu(y) < \infty \]

(see \([B, pp. 131, 132]\)). In particular, there are \( \phi_k \in D \) such that

\[ \int_B |\text{grad} \phi_k(x) - \text{grad} U\mu(x)|^2 dx \to 0 \quad \text{as} \quad k \to \infty. \]

(4.2) means that \( NU\mu = 0 \) (see 3.4), so that

\[ \int_G \text{grad} \phi_k(x) \cdot \text{grad} U\mu(x) dx = 0 \]

for each \( k \); making \( k \to \infty \) we obtain

\[ \int_G |\text{grad} U\mu(x)|^2 dx = 0. \]

We see that \( U\mu \) is constant on each \( G_i \) and vanishes on \( G_0 \). Next we prove the following assertion:

(a) If \( U\mu = 0 \) on \( G \) then \( \mu = 0 \).

Indeed, let \( \mu = \mu_1 - \mu_2 \) be the Jordan decomposition of \( \mu \) and assume that \( U\mu_1 \) and \( U\mu_2 \) coincide on \( G \). Since \( G \) has a positive \( m \)-dimensional density at any \( z \in B \), every fine neighborhood of \( z \) (in the Cartan topology) meets \( G \) (compare
and we conclude from the Cartan Theorem \([B, p. 86; see also p. 84]\) that \(U_{\mu_1}(z) = U_{\mu_2}(z)\). Since \(U_{\mu_1}\) and \(U_{\mu_2}\) coincide on \(B\), they must coincide on \(R^m\), by the domination principle \([B, p. 123]\). We have thus \(U_{\mu} = 0\) on \(R^m\), whence \(\mu = 0\) \([B, p. 122]\).

If \(q = 0\) then (a) completes the proof of 4.8. Assume now \(q > 0\). With every \(\mu \in C_0^q(B)\) we may associate the \(q\)-tuple \(c(\mu) = [c_1(\mu), \ldots, c_q(\mu)]\), where \(c_j(\mu)\) is the value taken on by \(U_{\mu}\) in \(G_j\). The map

\[c: \mu \mapsto c(\mu)\]

is an injection of \(C_0^q(B)\) into \(R^q\). Indeed, \(c(\mu) = 0\) means that \(U_{\mu} = 0\) on \(G\) and (a) implies \(\mu = 0\).

4.9. **Proposition.** Let \(f_j\) denote the characteristic function of \(fr G_j\) \((1 \leq j \leq q)\). Then \(\{f_1, \ldots, f_q\}\) is a basis in \(C_0(B)\).

**Proof.** Let us fix \(j \in \langle 1, q \rangle\) and put \(H = R^m - G_j\). Employing 2.12 and 2.6 we obtain for any \(z \in R^m - \text{cl } G \subset \text{int } H\)

\[Wc f_j(z) = v_j^q (fr G_j) = 0,\]

whence it follows by (3.6)

\[(1A + W)f_j = 0,\]

so that \(f_j \in C_0(B)\). Since the dimension of \(C_0(B)\) coincides with the dimension of \(C_0^q(B)\) which is known to be \(\leq q\) and \(f_1, \ldots, f_q\) are linearly independent, the proof is complete.

4.10. **Remark.** Combining the above proposition and Fredholm’s theorems one obtains Theorems 4.11-4.13 below.

If \(m = 2\) then 4.9 holds under more general assumptions on \(B\). It is sufficient to require that \(E\) (see 3.6) consists of mutually disjoint simple closed curves and \(V_0 < A\) (compare \([K3]\), where further references may be found).

4.11. **Theorem.** Let \(v \in C^*(B)\). Then \(v = N U_{\mu}\) for some \(\mu \in C^*(B)\) if and only if \(v(fr G_j) = 0, \quad j = 1, \ldots, q\).

**Proof.** This follows at once from 4.9 and the Fredholm Theorem.

4.12. **Theorem.** Let \(\{f_1, \ldots, f_q\}\) be a basis in \(C_0(B)\). Given \(g \in C(B)\) there are \(f \in C(B)\) and constants \(\alpha_j\) \((j = 1, \ldots, q)\) such that, for every \(y \in B\), \(Wf(x)\) tends to

\[g(y) - \sum_{j=1}^{q} \alpha_j f_j(y)\]

as \(x \to y, \quad x \in \text{int } C\). The constants \(\alpha_j\) are uniquely determined and \(f\) is determined modulo \(C_0(B)\).
Proof. Let \( \{ \mu_1, \ldots, \mu_q \} \) and \( \{ f_1, \ldots, f_q \} \) be dual bases in \( C^*(B) \) and \( C_0(B) \), respectively. Given \( g \in C(B) \) we can find \( \alpha_k \) so that
\[
\langle g - \sum_{k=1}^{q} \alpha_k f_k, \mu_j \rangle = 0
\]
for all \( j \); clearly, \( \alpha_k = \langle g, \mu_k \rangle \). Then
\[
\langle g - \sum_{k=1}^{q} \alpha_k f_k, C^*(B) \rangle = 0
\]
and the Fredholm Theorem yields an \( f \in C(B) \) such that
\[
(\frac{1}{2}AI + W)f = g - \sum_{k=1}^{q} \alpha_k f_k.
\]
The rest follows from (3.6).

Standard reasoning yields also the following.

**4.13. Theorem.** Fix \( x_j \in G_j \) (\( j = 1, \ldots, q \)). Given \( g \in C(B) \) there are \( f \in C(B) \) (determined modulo \( C_0(B) \)) and uniquely determined constants \( a_j \) such that, for every \( y \in B \),
\[
Wf(x) + \sum_{i=1}^{q} a_i |x - x_i|^{2-m}
\]
tends to \( g(y) \) as \( x \to y \), \( x \in \text{int } C \).

**Proof.** Define \( g_k \) by
\[
g_k(x) = \frac{1}{m-2} |x - x_k|^{2-m}.
\]
Then \( \langle g_k, \mu \rangle = U_\mu(x_k) \) for every \( \mu \in C^*(B) \). It follows from (3.6) that
\[
Wf + \sum_{j=1}^{q} a_j g_j \quad (f \in C(B), \ a_j \in R^1)
\]
represents a solution of the Dirichlet problem for \( C \) and the boundary condition \( g \) if and only if
\[
(\frac{1}{2}AI + W)f = g - \sum_{j=1}^{q} a_j g_j
\]
on \( B \). For the existence of an \( f \in C(B) \) satisfying (4.20) it is necessary and sufficient that
\[
\langle g - \sum_{j=1}^{q} a_j g_j, C_0^*(B) \rangle = 0,
\]
i.e.,
\[
\sum_{j=1}^{q} a_j U_\mu(x_j) = \langle g, \mu \rangle, \quad \mu \in C_0^*(B).
\]
We know from the proof of 4.8 (note also that $C^r(B)$ has dimension $q$) that

$$\mu \mapsto [U\mu(x_1), \ldots, U\mu(x_q)]$$

is an isomorphism of $C^r(B)$ onto $R^q$. Consequently, (4.21) determines $a_j$ uniquely. The rest is obvious.

**Remark.** Results related to some of those proved in the present paper were announced without proofs in [K1] (for the plane), [BMS] and [MS] (for a domain bounded by a simple closed surface in 3-space), [K2] (for a domain bounded by a hyper-surface in $m$-space) and in Abstract 630-197, (Theorem 1.13), Notices Amer. Math. Soc. 13 (1966).

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