TOPOLOGICAL PROPERTIES OF THE HILBERT CUBE
AND THE INFINITE PRODUCT OF OPEN INTERVALS

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1. For each \( i > 0 \), let \( I_i \) denote the closed interval \( 0 \leq x \leq 1 \) and let \( O_i \) denote the open interval \( 0 < x < 1 \). Let \( I^\infty = \prod_{i>0} I_i \) and \( O^\infty = \prod_{i>0} O_i \). \( I^\infty \) is the Hilbert cube or parallelotope sometimes denoted by \( Q \). \( O^\infty \) is homeomorphic to the space sometimes called \( s \), the countable infinite product of lines.

The principal theorems of this paper are found in §§5, 7, 8, and 9. In §5 it is shown as a special case of a somewhat more general theorem that for any countable set \( G \) of compact subsets of \( O^\infty \), \( O^\infty \setminus G^* \) is homeomorphic to \( O^\infty \) (where \( G^* \) denotes the union of the elements of \( G \)).

In §7, it is shown that a great many homeomorphisms of closed subsets of \( I^\infty \) into \( I^\infty \) can be extended to homeomorphisms of \( I^\infty \) onto itself. The conditions are in terms of the way in which the sets are coordinatewise imbedded in \( I^\infty \). A corollary is the known fact (Keller [6], Klee [7], and Fort [5]) that if \( X \) is a countable closed subset of \( I^\infty \), then every homeomorphism of \( X \) into \( I^\infty \) can be extended to a homeomorphism of \( I^\infty \) onto itself. In a further paper based on the results and methods of this paper, the author will give a topological characterization in terms of imbeddings of those closed subsets \( X \) of \( I^\infty \) for which homeomorphisms of \( X \) into \( W_1 = \{ p \mid p \in I^\infty \text{ and the first coordinate of } p \text{ is zero} \} \) can be extended to homeomorphisms of \( X \) onto itself. In his recent dissertation, Raymond Wong has settled a question of Blankinship [4] by showing that there do exist two Cantor Sets in \( I^\infty \) such that no homeomorphism of one onto the other can be extended to a homeomorphism of \( I^\infty \) onto itself.

In §8 the results of §7 are used to give conditions under which the union of two Hilbert cubes can be seen to be homeomorphic to \( I^\infty \).

In §9 it is proved that many countable infinite products not obviously homeomorphic to \( O^\infty \) are, in fact, homeomorphic to \( O^\infty \). A theorem (Theorem 9.5) equivalent to the following is proved: "For \( i = 1, 2, \ldots \), let \( C_i \) be a closed \( n_i \)-cell
(0 < t < \infty) and let Y_t be a subset of C, which contains the interior of C_t. Then the product of the Y_t's is homeomorphic with \( I^\infty \) if and only if each Y_t is a G_\delta set and \( Y_t \neq C_t \) for infinitely many t.

The arguments of this paper use purely set-theoretic topological methods.

A REMARK CONCERNING STABLE HOMEOMORPHISMS. It will be shown in a further paper that every homeomorphism of \( I^\infty \) onto itself is stable in the sense of Brown-Gluck, i.e., every homeomorphism is the finite product of homeomorphisms each the identity on some nonempty open set. However, it follows easily (as was pointed out to the author by Raymond Wong) that all the homeomorphisms of \( I^\infty \) onto itself used in this paper can be specified to be stable. Specifically, the results of §§3–7 on existence of homeomorphisms, can be strengthened by adding the condition that the homeomorphisms asserted to exist also be stable.

2. Definitions, notation, and preliminary lemmas. Let \( Z \) denote the set of positive integers. For \( \alpha \subseteq Z \), let \( I_\alpha \) denote \( \prod_{i \in \alpha} I_i \) and let \( \partial I_\alpha \) denote \( \prod_{i \in \alpha} \partial I_i \). For \( \alpha \subseteq \beta \subseteq Z \), let \( \tau_\alpha \) denote the projection of \( I^\infty \) (or of \( I_\beta \) where appropriate) onto \( I_\alpha \). For \( \alpha \) the set whose only element is \( i \), \( \tau_\alpha \) may be written as \( \tau_i \).

For a nonnull proper subset of \( Z \), \( \alpha' \) will denote \( Z \setminus \alpha \).

The collection \( \{ \alpha_i \}_{i \geq 0} \) is said to be a partition of \( Z \) provided

1. For each \( i, j > 0 \), \( \alpha_i \cap \alpha_j = \emptyset \) if and only if \( i \neq j \) and
2. \( \bigcup_{i > 0} \alpha_i = Z \). A collection satisfying condition (1) is called a subpartition. A partition (or subpartition) is said to be simple if each element is finite.

For any infinite \( \alpha \subseteq Z \) we consider \( I_\alpha = \prod_{i \in \alpha} I_i \) to be endowed with metric \( \rho_\alpha \) defined by

\[
\rho_\alpha(x, y) = \sqrt{\sum_{i \in \alpha} \frac{1}{2^i} (x_i - y_i)^2}
\]

where \( x = \{x_i\} \) and \( y = \{y_i\} \), \( x_i, y_i \in I_i \). For \( \alpha = Z \) we let \( \rho \) be \( \rho_\alpha \). As \( \partial I_\alpha \subseteq I_\alpha \), the above metric is inherited by \( \partial I_\alpha \). For finite \( \alpha \), we consider \( \rho_\alpha \) to be the ordinary Euclidean metric with each \( I_i, j \in \alpha \), being of length 1. The set \( \partial I_\alpha \) is dense in \( I_\alpha \) and is called the pseudo-interior of \( I_\alpha \). Let \( B(I_\alpha) = \partial I_\alpha \) and let \( B(I_\alpha) \) be called the pseudo-boundary of \( I_\alpha \). For \( \alpha \) finite, \( B(I_\alpha) \) is the boundary of the finite-dimensional cell \( I_\alpha \).

For \( \alpha \) infinite, \( B(I_\alpha) \) is dense in \( I_\alpha^\infty \).

REMARK. \( B(I^\infty) \) and \( \partial I^\infty \) are clearly not homeomorphic to each other. Note that both \( B(I^\infty) \) and \( \partial I^\infty \) are dense in \( I^\infty \) and \( B(I^\infty) \) is an \( F_\sigma \) set. Since \( I^\infty \) is compact and hence of the second category, \( \partial I^\infty \) cannot also be an \( F_\sigma \) set.

By a \( \beta^*(I_\alpha) \)-homeomorphism we shall mean a homeomorphism of \( I_\alpha \) onto itself which carries \( B(I_\alpha) \) onto \( B(I_\alpha) \). By a \( \beta(I_\alpha) \)-homeomorphism we shall mean a homeomorphism of \( I_\alpha \) onto itself carrying \( B(I_\alpha) \) into \( B(I_\alpha) \). For \( \alpha = Z \), we simplify the notation to \( \beta^* \) and \( \beta \)-homeomorphisms.

Let \( G(I^\infty) \) denote the group of all homeomorphisms of \( I^\infty \) onto itself. For \( K \subseteq I^\infty \) and \( f \in G(I^\infty) \), let \( f \mid K \) be the homeomorphism \( f \) restricted to the domain \( K \). For \( M \subseteq I^\infty \), \( f \) is said to be supported on \( M \) if \( f \mid (I^\infty \setminus M) \) is the identity.
Many of the homeomorphisms with which we shall be concerned may be described in something like the following manner. Let $\alpha = \{1, 2\}$. Let $h$ be a homeomorphism of $I_\alpha$ onto itself such that (1) $h|B(I_\alpha)$ is the identity, (2) for some $p \in I_\alpha$, $\tau_1(p) \neq \tau_1(h(p))$ and (3) for each $p \in I_\alpha$, $\tau_2(p) = \tau_2(h(p))$. Let $I^\infty$ be regarded as $I_\alpha \times I_\alpha$ and for $q \in I^\infty$, let $\gamma = (\tau_2(q), \tau_\alpha(q))$. Then define $f(q) = (h(\tau_\alpha(q)), \tau_\alpha(q))$. (Incidentally, $f$ is a $\beta^*$-homeomorphism.) Associated with $f$ are three subsets of $Z$ which in the definition to follow we shall label $\alpha(f)$, $\beta(f)$, and $\gamma(f)$. In our example $\alpha(f) = \{1\}$, $\beta(f) = \alpha^\prime$ and $\gamma(f) = \{2\}$. In effect, $\alpha(f)$ represents the set of directions in which $f$ acts, $\beta(f)$ represents the set of directions which can be ignored or factored out when defining $f$ and $\gamma(f)$ represents the other directions.

For $f \in G(I^\infty)$, let $\alpha(f)$ denote the set of all elements $i$ of $Z$ such that for some $p \in I^\infty$, $\tau_i(p) \neq \tau_i(f(p))$. Clearly $\alpha(f) = \emptyset$ if and only if $f$ is the identity. For $j \in Z$, $f$ is said to be independent of $j$ provided that $j \notin \alpha(f)$ and for any $p, q \in I^\infty$ for which $\tau_{2i}(p) = \tau_{2i}(q)$, $\tau_{2i}(f(p)) = \tau_{2i}(f(q))$. Let $\beta(f)$ denote the set of all $j \in Z$ for which $f$ is independent of $j$. Let $\gamma(f) = Z \setminus (\alpha(f) \cup \beta(f))$.

For any $\alpha \subseteq Z$ and any $i \in \alpha$, let $W_i(I_\alpha)$ be the set of all points of $I_\alpha$ with $i$-coordinate 0 and let $\partial W_i(I_\alpha)$ be the set of all points of $W_i(I_\alpha)$ whose other coordinates are properly between 0 and 1. For $\alpha = Z$ we let $W_i(I^\infty)$ and $\partial W_i(I^\infty)$ be denoted by $W_i$ and $\partial W_i$ respectively.

Let, for each $i, f_i \in G(I^\infty)$. We shall be concerned with the infinite product (composition) of the $\{f_i\}_{i>0}$ in the form $\ldots \cdot f_3 f_2 f_1$ in cases where such composition is, in fact, a homeomorphism. To this end we give the following definition. The (formal) left product $L \prod_{i>0} f_i$ of $\{f_i\}$ is the transformation $\mu$ of $I^\infty$ into the space $C$ of closed subsets of $I^\infty$ defined by $\mu(p) = \lim_{i \to 0} \sup \{f_i \cdot \ldots \cdot f_2 f_1(p)\}$. For a double sequence $\{f_{i,j}\}_{i,j>0}$ we define $L \prod_{i,j>0} f_{i,j}$ by ordering $\{f_{i,j}\}$ as a simple sequence, $i, j < i', j'$ if $i+j < i'+j'$ or if $i+j = i'+j'$ and $i < i'$. We are interested in conditions under which (1) $\mu$ is a transformation of $I^\infty$ into $I^\infty$ as a subset of $C$, (2) $\mu$ is onto $I^\infty$, (3) $\mu$ is continuous, and (4) $\mu$ is 1-1.

**Lemma 2.1.** If $\{\alpha(f_i)\}_{i>0}$ is a subpartition, then $\mu = L \prod_{i>0} f_i$ is a continuous transformation of $I^\infty$ onto $I^\infty$.

**Proof.** Consider $p \in I^\infty$ and $k \in Z$. If $k \notin \bigcup_{i>0} \alpha(f_i)$ then $\tau_k(\mu(p)) = \tau_k(p)$ and if, for some $i, k \in \alpha(f_i)$, then $\tau_k(\mu(p)) = \tau_k(f_i(p))$. Hence $\mu$ carries $I^\infty$ into $I^\infty$. Also since, for each $i, f_i$ is continuous and $\tau_k$ is continuous then $\mu$ is continuous in each coordinate and hence is continuous. Finally we verify that $\mu$ carries $I^\infty$ onto $I^\infty$. We suppose it does not. Since $\mu(I^\infty)$ is closed there is some $j$ such that for $\alpha = \{1, 2, \ldots, j\}$, $\tau_\alpha(\mu(I^\infty)) \neq I_\alpha$. But there is a $k$ such that for $k' > k$, $\alpha \cap \alpha(f_{k'}) = \emptyset$. Hence $\tau_\alpha(\mu(I^\infty)) = \tau_\alpha(f_k \cdot \ldots \cdot f_2 f_1(I^\infty)) = I_\alpha$, a contradiction.

**Remark.** Under the hypotheses of Lemma 2.1 $\mu$ need not be 1-1. For example, if $\alpha(f) = i$ and $\gamma(f) = i+1$, for each $i$, then for some $p \neq q$ we can define $\{f_i\}$ so as successively to make $\tau_k(\mu(p)) = \tau_k(\mu(q))$ for all $k$.

There are many conditions other than those of Lemma 2.1 under which $\mu$ is a
mapping and even a homeomorphism. In the next lemma, we observe some other conditions for \( \mu \) to be a mapping, such conditions being designed to fit applications in §§5 and 6.

**Lemma 2.2.** Suppose that for each \( i, f_i \in G(I^\infty) \) and for each \( i, j > 0 \), \( a(f_i) \cap a(f_j) \) is the single element \( t \in Z \). And suppose that \( \{ \tau_i(f_n \cdots f_1(p)) \}_{n>0} \) converges for each \( p \in I^\infty \) and the mapping thus defined is continuous from \( I^\infty \) onto \( I_t \). Then \( \mu = L \prod_{i>0} f_i \) is a continuous transformation of \( I^\infty \) onto \( I^\infty \).

**Proof.** The proof of this lemma is like that for Lemma 2.1 except that the properties of \( \mu \) with respect to the \( r \)th direction are implied by the special hypotheses.

**Lemma 2.3.** If \( \{ a(f_i) \}_{i>0} \) is a subpartition and if
\[
\left( \bigcup_{i>0} \gamma(f_i) \right) \cap \left( \bigcup_{i>0} a(f_i) \right) = \emptyset,
\]
then \( \mu = L \prod_{i>0} f_i \) is a homeomorphism onto \( I^\infty \).

**Proof.** Using Lemma 2.1, it suffices to show that \( \mu = 1-1 \). Suppose \( p \neq q \) and let \( k \) be an element of \( Z \) such that \( \tau_k(p) \neq \tau_k(q) \). Then
\[
k \notin \bigcup_{i>0} a(f_i), \quad \tau_k(\mu(p)) = \tau_k(p) \neq \tau_k(q) = \tau_k(\mu(q)).
\]
If for every \( k \notin \bigcup_{i>0} a(f_i) \), \( \tau_i(p) = \tau_i(q) \) then for some \( i \) and some \( j, i \in a(f_j) \) and \( \tau_i(p) \neq \tau_i(q) \). Since \( f_j \in G(I^\infty) \) and \( \gamma(f_i) \subset Z \setminus \bigcup_{i>0} a(f_i) \), for some \( i' \in a(f_i) \),
\[
\tau_{i'}(\mu(p)) = \tau_{i'}(f_{i'}(p)) \neq \tau_{i'}(f_{i'}(q)) = \tau_{i'}(\mu(q))
\]
as was to be shown.

In the following lemma we give an easy metric condition on \( \{ f_i \}_{i>0} \), using uniform continuity of \( (f_1 \cdots f_2 \cdots f_1)^{-1} \) so that \( L \prod_{i>0} f_i \) must be a homeomorphism.

**Lemma 2.4.** Let \( \{ f_i \}_{i>0} \) be a collection of elements of \( G(I^\infty) \) such that \( \{ a(f_i) \}_{i>0} \) is a subpartition. Let \( \{ e_i \}_{i>0} \) be a sequence of positive numbers with, for each \( i \), \( e_{i+1} < \frac{1}{2} e_i \). For each \( i \), let \( \delta_i \) be a positive number such that \( \rho(p, q) < e_i / 2 \) whenever \( p, q \in I^\infty \) with \( \rho(f_1 \cdots f_i(p), f_1 \cdots f_i(q)) < \delta_i \). Suppose, for each \( i \), the distance between \( f_{i+1} \) and the identity is less than \( \min (\delta_i, \delta_{i-1}/2, \delta_{i-2}/2^2, \ldots, \delta_0/2^{i-1}) \). Then \( \mu = L \prod_{i>0} f_i \) is a homeomorphism.

**Proof.** By Lemma 2.1, \( \mu \) is a mapping. By the conditions of the theorem, if two points are at a distance \( > e_a \) from each other, then \( f_a \cdots f_2 f_1 \) keeps the points separated by a distance of \( \delta_a \) and the introduction of the additional factors cannot bring them together.

We next introduce four lemmas giving conditions under which homeomorphisms can be asserted to be \( \beta \) or \( \beta^* \).

**Lemma 2.5.** For \( f \in G(I^\infty) \) with \( a(f) \) finite, \( f \) is \( \beta^* \).
Proof. Let \( x \in I^\infty \) be expressed as \((p, q)\) where \( p \in I_{a(f)} \) and \( q \in I_{b(f) \cup a(f)} \). For each \( q \in I_{a(f) \cup b(f)} \), \( f((p, q)) = (p', q) \) and since \( I_{a(f)} \) is a finite dimensional cube, \( p' \) is on the boundary of \( I_{a(f)} \) if and only if \( p \) is on the boundary of \( I_{a(f)} \). But then \( f((p, q)) \in B(I^\infty) \) if and only if \((p, q) \in B(I^\infty) \) and thus \( f \) is \( \beta^* \).

Lemma 2.6. Any finite product of \( \beta \) (or \( \beta^* \)) homeomorphisms is \( \beta \) (or \( \beta^* \)).

Proof. Obvious.

Lemma 2.7. If, for each \( i > 0 \), \( h_i \in G(I^\infty) \), if \( h = \bigcup_{i > 0} h_i \in G(I^\infty) \), if \( \{a(h_i)\}_{i > 0} \) is a simple subpartition, and if, for each \( i > 0 \), \( h_i \) is \( \beta \) (or \( \beta^* \)), then \( h \) is \( \beta \) (or \( \beta^* \)).

Proof. Suppose \( p \in B(I^\infty) \). Then for some \( j \), \( \tau_j(p) = 0, 1 \). If \( j \notin \bigcup_{i > 0} a(h_i) \), then \( \tau_j(h(p)) = \tau_j(p) \). If \( j \in a(h_i) \) then since \( h_i \) is \( \beta \) (or \( \beta^* \)) and \( I_{a(h_i)} \) is a finite cube, there exists \( k \in a(h_i) \) such that \( \tau_k(h(p)) = 0, 1 \) and hence \( \tau_k(h(p)) = \tau_k(h_i(p)) = 0, 1 \). Therefore \( h \) is \( \beta \).

Suppose each \( h_i \) is \( \beta^* \). Then, as above, \( h \) is \( \beta \). Let \( q \in oI^\infty \). For each \( j \notin \bigcup_{i > 0} a(h_i) \), \( \tau_j(q) = \tau_j(q) \) and hence cannot be 0 or 1. For each \( j \in a(h_i) \), \( \tau_j(q) = \tau_j(q) \) and \( \tau_j(q) \neq 0, 1 \) since \( h_i \) is \( \beta^* \). Therefore \( h \) is \( \beta^* \).

Lemma 2.8. If, for each \( i > 0 \), \( h_i \in G(I^\infty) \), if \( h = \bigcup_{i > 0} h_i \in G(I^\infty) \), if \( \{a(h_i)\}_{i > 0} \) is a subpartition, if \( \bigcup_{i > 0} a(h_i) \cap \bigcup_{i > 0} \gamma(h_i) = \emptyset \), and if for each \( i > 0 \), \( h_i \) is \( \beta \) (or \( \beta^* \)), then \( h \) is \( \beta \) (or \( \beta^* \)).

Proof. Suppose \( p \in B(I^\infty) \). Then, for some \( j \), \( \tau_j(p) = 0, 1 \). If \( j \notin \bigcup_{i > 0} a(h_i) \), then \( \tau_j(h(p)) = \tau_j(p) \). If \( j \in a(h_i) \), then since \( h_i \) is \( \beta \) (or \( \beta^* \)), either (1) there exists a \( j' \in \gamma(h_i) \) for which \( \tau_j(p) = 0, 1 \) or (2) there exists a \( k \in a(h_i) \) such that \( \tau_k(h(p)) = 0, 1 \) and hence \( \tau_k(h(p)) = \tau_k(h_i(p)) = 0, 1 \). Therefore \( h \) is \( \beta \). The argument that \( h \) is \( \beta^* \) if each \( h_i \) is \( \beta^* \) is like that of Lemma 2.7.

Lemma 2.9. Let \( \{a_i\}_{i > 0} \) be a collection of subsets of \( Z \) such that for each \( i \), \( a_i \) is infinite. Let, for each \( i > 0 \), \( \{a_{ij}\}_{j > 0} \) be a simple subpartition the union of whose elements is contained in \( a_i \). Then there exists a subpartition \( \{\beta_i\}_{i > 0} \) such that (1) for each \( i > 0 \), \( \beta_i \subset a_i \) and (2) for each \( i > 0 \), \( \beta_i \) is the union of infinitely many sets \( a_{ij} \).

Proof. This lemma is a standard set-theoretic proposition. It may be proved by an inductive constructive argument. Let \( \{\gamma_i\} \) be any partition of \( Z \) with, for each \( i \), \( \gamma_i \) infinite. For each \( k \), let \( b_k \) be inductively selected so that, if \( k \in \gamma_i \), then \( b_k \) is the \( a_{ij} \) with least index \( j \) for which \( b_k \cap b_j = \emptyset \) for each \( j < k \). Let \( \beta_i \) be the union of all elements \( b_k \) for which \( b_k = a_{ij} \) for some \( j \) and for which \( k \in \gamma_i \).

3. Straightening weakly thin compact sets.

Definitions. A set \( K \subset I^\infty \) is said to be weakly thin provided that (1) \( K \) is closed and (2) there exists a simple subpartition \( \{a_i\}_{i > 0} \) such that, for each \( i \), \( \tau_{a_i}(K) \neq I_{a_i} \). A set \( K \subset I^\infty \) is said to be thin provided that (1) \( K \) is closed and (2) there exists a simple subpartition \( \{a_i\}_{i > 0} \) such that, for each \( i \), \( a_i \) consists of a single integer and \( \tau_{a_i}(K) \subset ^\circ I_{a_i} \). We say that \( K \) is weakly thin (or thin as appropriate) with respect to \( \{a_i\} \).
Remark. Examples of closed sets which are not weakly thin are closed sets which contain nonempty open sets or closed sets whose complements in $I^\alpha$ are not homotopically trivial. For finite $\alpha$, let $p \in \partial I_\alpha$. Then $\tau_\alpha^{-1}(p)$ is not weakly thin nor is its complement homotopically trivial.

Lemma 3.1. Let $\alpha$ be a subset of $Z$. Let $\{K_\alpha\}_{\alpha > 0}$ be a collection of closed sets each weakly thin with respect to a subpartition whose elements are disjoint from $\alpha$. There exists a $\beta^*$-homeomorphism $g$ such that $\alpha(g) = \alpha'$ and, for each $i > 0$, $g(K_i)$ is thin with respect to a subset of $\alpha'$.

Proof. By Lemma 2.9 there exists a subpartition $\{\beta_i\}_{\alpha > 0}$ such that $\bigcup_{\alpha > 0} \beta_\alpha \cap \alpha = \emptyset$ and for each $i$, there exists a simple subpartition $\{\beta_i, \gamma_i\}_{\alpha > 0}$ with respect to which $K_i$ is weakly thin and with for each $j$, $\beta_j \subseteq \beta_i$. If for two disjoint finite sets $\gamma_1, \gamma_2 \subseteq Z$, and for any $i > 0$, $\gamma_1(K_i) \neq I_\gamma_1$ and $\gamma_2(K_i) \neq I_\gamma_2$, then $\gamma_1(K_i) \neq I_\gamma_1$ and $\gamma_2(K_i) \neq I_\gamma_2$. Thus for each $i$ and each $j$ we may let $\gamma_1 = \beta_i \cup \beta_i$ and, for each $i$, $\{\gamma_i\}_{\alpha > 0}$ is a simple subpartition with respect to which $K_i$ is weakly thin, with for each $j$, $\gamma_j \subseteq \beta_i$ and with $\gamma_i(K_j) \neq B(I_{\gamma_i})$. For each $i$ and each $j$, let $g_{ij}$ be a $\beta^*$-homeomorphism such that $\alpha(g_{ij}) = \gamma_i$, $\gamma(g_{ij}) = \emptyset$, and for some $n_j \in \gamma_i, \gamma_j(g_{ij}(K_j)) \subseteq B(I_{\gamma_j})$. By Lemma 2.3, $L \bigcap_{\alpha > 0} g_{ij}$ is an element of $G(I^\alpha)$ and by Lemmas 2.5 and either 2.7 or 2.8, $g = L \bigcap_{\alpha > 0} g_{ij}$ is $\beta^*$. By construction, for each $i$, since $G(K_i)$ is compact, $g(K_i)$ is thin and the lemma is proved.

Lemma 3.2. Let $i \in Z$, let $t$ be a number, $0 < t < 1$, and let $\alpha \subseteq Z$ with $i \notin \alpha$. Let $K$ be a closed subset of $I^\alpha$ such that (1) $0 < \tau_\alpha(K) < 1$ and (2) for $p, q \in K$ with $p \neq q$, $\tau_\alpha(p) \neq \tau_\alpha(q)$. Then there exists a $\beta^*$-homeomorphism $f$ such that $\alpha(f) = \alpha$ and $\gamma(f(K)) = t \in I_\gamma$.

Proof. This lemma is a simple consequence of the Tietze Extension Theorem and is similar to lemmas in the literature. Let $K'$ be the set of points of $\tau_\alpha^{-1}(t)$ for which $\tau_\alpha(K') = \tau_\alpha(K)$. Let $g$ be the map of $K'$ into $\partial I_\alpha$ defined as follows: for $p \in K'$, let $q$ be the point of $K$ for which $\tau_\alpha(p) = \tau_\alpha(q)$. Since $K$ is compact, $\tau_\alpha(K)$ is compact and there exist $t_1, t_2$ such that $0 < t_1 < t < t_2 < 1$ and such that $\tau_\alpha(K) \subseteq [t_1, t_2] \subseteq I_\gamma$. Let $g^*$ be the map from $\tau_\alpha^{-1}(K')$ onto $\tau_\alpha^{-1}(K)$ induced by $g$; for $p \in K'$, $g^*(\tau_\alpha^{-1}(p)) = \tau_\alpha^{-1}(g(p))$.

Let $g^*$ be extended to a map $g'$ of $\tau_\alpha^{-1}(t)$ into $[t_1, t_2]$. Then for each $p \in \tau_\alpha^{-1}(t)$ the interval $[0, g'(p)]$ be mapped linearly onto $[0, p]$ and let $[g'(p), 1]$ be mapped linearly onto $[p, 1]$, all intervals being orthogonal to $\tau_\alpha^{-1}(t)$. The resultant map $f$ of $\tau_\alpha^{-1}(t)$ is a homeomorphism. Let $I^\alpha = I_{\alpha}(\alpha \times U_{\alpha}^{-1}(\gamma))$ and for $p = (p_1, p_2)$ coordinates of these factors let $f(p) = (\tilde{f}(p_1), p_1)$. Then $f$ is the desired homeomorphism as it is easy to verify.

A closed set $K \subseteq I^\alpha$ is said to be straight with respect to the set $\{n_i\}_{i > 0}$ of integers provided that, for any $i$, $\tau_n(K)$ is a single point of $\partial I_{n_i}$.

Lemma 3.3. Let $K$ be a thin set and let $\{n_i\}_{i > 0}$ be a set of integers such that, for each $j$, $\tau_{n_j}(K) \subseteq \partial I_{n_j}$. For any infinite subset $\{m_i\}_{i > 0}$ of $\{n_i\}_{i > 0}$ there exists a
\(\beta^*-\)homeomorphism \(f\) such that \(f(K)\) is straight with respect to some infinite subset of \(\{m_s\}\) and \(\alpha(f) \subseteq \{m_s\}\).

**Proof.** If, for infinitely many \(s\), \(\tau_m(K)\) is a single point, \(f\) can be taken as the identity. Alternatively, let \(\{m_s\}_{s>0}\) be the infinite subset of \(\{m_s\}\) for which \(\tau_m(K)\) is nondegenerate. Let \(\alpha_i\) be a subpartition with \(\bigcup_{i>0} \alpha_i = \{m_s\}_{s>0}\) and with each \(\alpha_i\) infinite. We shall select \(f\) so that \(f(K)\) is straight with respect to \(\alpha_i\) and so that \(\alpha(f) = \{m_s\}_{s>0}\).

Let \(\alpha_0 = \{i\}\), and, for each \(i\), let \(\alpha_{i+1} = \{v_{ij}\}_{j>0}\). Then \(I_{\delta i}\) is a square and since \(K\) is compact, \(\tau_{\alpha_i}(K)\) is a compact subset of \(\delta I_{\delta i}\). Let \(\sigma_i\) be a homeomorphism of \(I_{\delta i}\) onto itself such that (1) \(\sigma_i\) is the identity on \(B(I_{\delta i})\), (2) for each \(p \in I_{\delta i}\), \(\tau_{\alpha_i}(p) = \tau_{\alpha_i}(\sigma_i(p))\), and (3) for \(p, q \in \tau_{\alpha_i}(K)\) with \(|\tau_{\alpha_i}(p) - \tau_{\alpha_i}(q)| > 1/2^i\), \(|\tau_{\alpha_i}(\sigma_i(p)) - \tau_{\alpha_i}(\sigma_i(q))| \neq 0\). We may simply distort \(\delta I_{\delta i}\) in the \(v_{ij}\) direction to produce such a \(\sigma_i\). Letting, for each \(i, j > 0\), \(I^\infty = I_{\delta i} \times I_{\delta j}\) and for \(q \in I^\infty\) letting \(a = (\tau_{\alpha_i}(q), \tau_{\alpha_j}(q))\), we define \(\phi_{ij}(q) = (\alpha_i(\tau_{\alpha_i}(q)), \tau_{\alpha_j}(q))\). Then \(\phi_{ij}\) is a \(\beta^*-\)homeomorphism with \(\alpha(\phi_{ij})\) the set consisting of \(v_{ij}\) itself and with \(\gamma(\phi_{ij})\) the set consisting of \(t_{ij}\) itself.

By Lemma 2.3, 2.5 and 2.7, \(f = L \bigcap_{i,j>0} \phi_{ij}\) is a \(\beta^*-\)homeomorphism with \(\alpha(\phi) = \bigcup_{i>0} \alpha_{i+1}\) and \(\gamma(\phi) = \alpha_{i+1}\). Also, for any \(p, q \in K\) and any \(i > 0\), if \(\tau_{\alpha_i}(\phi(p)) \neq \tau_{\alpha_i}(\phi(q))\), then for some \(j \in \alpha_{i+1}\), \(\tau_{\alpha_j}(\phi(p)) \neq \tau_{\alpha_j}(\phi(q))\).

Now by Lemma 3.2, for each \(i > 0\), there exists a \(\beta^*-\)homeomorphism \(f_i\) with \(\alpha(f_i)\) the set consisting of \(t_i\) itself, \(\gamma(f_i) \subseteq \alpha_{i+1}\), and \(\tau_{\alpha_i}(f_i(K)) = \frac{1}{i} \in I_i\). By Lemmas 2.3 and 2.7, we may let \(f^* = L \bigcap_{i>0} f_i\) and \(f = f^*\phi\) is the desired homeomorphism.

**Lemma 3.4.** Let \(a\) be a subset of \(Z\). Let \(\{K_i\}_{i>0}\) be any collection of closed sets each thin with respect to a subset of \(a\). Then there exists a \(\beta^*-\)homeomorphism \(h\) such that \(\alpha(h) \subseteq a'\) and for each \(i, h(K_i)\) is straight with respect to some infinite subset of \(a\).

**Proof.** By Lemma 2.9, we may let \(\{\beta_i\}_{i>0}\) be a subpartition such that, for each \(i\), \(\beta_i \cap a = \emptyset\), \(\beta_i\) is infinite, and \(K_i\) is thin with respect to the collection of all single element subsets of \(\beta_i\). Let, for each \(i, h_i\) be a \(\beta^*-\)homeomorphism as in Lemma 3.3 with \(\alpha(h_i) \cup \gamma(h_i) \subseteq \beta_i\). Then by Lemmas 2.3 and 2.8 \(L \bigcap_{i>0} h_i\) is the desired homeomorphism \(h\).

**Theorem 3.5.** Let \(a\) be a subset of \(Z\). Let \(\{K_i\}_{i>0}\) be any collection of closed sets such that each is weakly thin with respect to a subpartition whose elements are disjoint from \(a\). Then there exists a \(\beta^*-\)homeomorphism \(h\) such that \(\alpha(h) \subseteq a'\) and, for each \(i, h(K_i)\) is straight with respect to some infinite subset of \(a\).

**Proof.** Theorem 3.5 is an immediate consequence of Lemmas 3.1 and 3.4.

4. Extending homeomorphisms—Klee's method. The proof of the following two theorems is by a method due to Klee [7]. The setting is a bit different from Klee's.
Theorem 4.1. Let \( \alpha \) be a subset of \( \mathbb{Z} \) such that \( \alpha \) and \( \alpha' \) are nonnull. Let \( K \) and \( K' \) be closed sets in \( I^\infty \) such that, for each \( i \in \alpha \), \( \tau_i(K') \) is a single point and, for each \( j \in \alpha' \), \( \tau_j(K) \) is a single point. Then any homeomorphism \( f \) of \( K \) onto \( K' \) can be extended to a \( \beta^* \)-homeomorphism \( F \).

Proof. We consider \( I^\infty \) as \( I_\alpha \times I_{\alpha'} \) and let \( K_f \) be the graph of \( f \), i.e.,

\[
K_f = \{(\tau_\alpha(p), \tau_{\alpha'}(f(p))) \mid p \in K\}.
\]

Let \( g \) be the homeomorphism of \( K \) onto \( K_f \) defined by \( g(p) = (\tau_\alpha(p), \tau_{\alpha'}(f(p))) \) for \( p \in K \) and let \( g' \) be the homeomorphism of \( K' \) onto \( K_f \) defined by

\[
g'(p) = (\tau_\alpha(f^{-1}(p)), \tau_{\alpha'}(p))
\]
for \( p \in K' \).

We shall define a \( \beta^* \)-homeomorphism \( h \) such that \( h|K = g|K \) and another \( \beta^* \)-homeomorphism \( h' \) such that \( h'|K' = g'|K' \). Then \( (h')^{-1}h \) is the desired \( f \).

It suffices simply to define \( h \) since \( h' \) may be defined analogously. For each \( i \in \alpha' \), Lemma 3.2 asserts the existence of a \( \beta^* \)-homeomorphism \( h_i \) with \( \alpha(h_i) \) the set whose only element is \( i \) and with \( \gamma(h_i) \subset \alpha \) such that for \( p \in K \), \( h_i(p) \) is the point whose \( i \)'th coordinate is \( \gamma_i(g(p)) \).

Let \( h = L \prod_{i \in \alpha} h_i \) and by Lemmas 2.3 and 2.7, \( h \) is the desired homeomorphism.

Theorem 4.2. Let \( K \) and \( K' \) be subsets of \( I^\infty \) closed in \( I^\infty \) and let \( f \) be any homeomorphism of \( K \) onto \( K' \). Then \( f \) can be extended to a \( \beta^* \)-homeomorphism.

Proof. Since \( K \) and \( K' \) are closed subsets of \( I^\infty \) they are compact and as they are subsets of \( I^\infty \) they are (weakly) thin. By Theorem 3.5 there is a \( \beta^* \)-homeomorphism \( h \) such that \( h|K = g|K \) and another \( \beta^* \)-homeomorphism \( h' \) which \( h'|K' = g'|K' \). Then \((h')^{-1}h \) is the desired \( f \).

The following two propositions are immediate consequences of Theorems 4.1 and 4.2 respectively, obtained by identifying \( \prod_{i > 1} I_i \) coordinatewise with \( I^\infty \).

Corollary 4.3. Let \( K \) and \( K' \) be closed sets in \( I^\infty \) such that, for each \( i \in \beta \), \( \tau_i(K') \) is a single point and, for each \( j \in \beta' \), \( \tau_j(K) \) is a single point. Then any homeomorphism \( f \) of \( K \) onto \( K' \) can be extended to a \( \beta^* \)-homeomorphism \( F \) such that \( 1 \in \beta(F) \).

Theorem 4.4. Let \( K \) and \( K' \) be closed sets in \( I^W_1 \). Then any homeomorphism \( f \) of \( K \) onto \( K' \) can be extended to a \( \beta^* \)-homeomorphism \( F \) such that \( 1 \in \beta(F) \).
5. Pushing weakly thin sets to $B(I^\infty)$. In this section we begin by proving the key lemma establishing a certain $\beta$-homeomorphism.

**Lemma 5.1.** Let $a = \{a_i\}_{i>0}$ be any infinite set of positive integers with, for each $i$, $a_i < a_{i+1}$ and with $a$ being nonnull. Let $K \subseteq I^\infty$ be any closed set which is straight with respect to $a$. There exists a $\beta$-homeomorphism $h$ such that (1) $a(h) = a$, (2) for $p \in o^{I^\infty}$, $h(p)$ is an element of $B(I^\infty)$ if and only if $p \in K$, (3) for $p \in K$, $\tau_n(h(p)) = 0$ and for $k \neq n$, $\tau_k(p) = \tau_k(h(p))$ and (4) for any $p \in B(I^\infty)$ and any $i$ for which $\tau_i(p) = 0, 1$, there is a $k \leq i$ for which $\tau_k(h(p)) = 0, 1$.

**Proof.** We shall exhibit $h$ as an infinite left product of finitely elementary homeomorphisms $h_i$ where, for each $i$, $a(h_i)$ consists of $n_i$ and $n_{i+1}$ and $\gamma(h_i)$ is null. Each $h_i$ is to move $K$ (or really $A_{i-1}/h_{i-2} \cdots h_1(K)$) a step closer to $W_{n_i} = \tau_{n_i}^{-1}(0)$. Without loss of generality we may assume that, for each $i$, $\tau_{n_i}(K)$ is the single point $1/2^i$ in $T_{n_i}$ (for we could have earlier re coordinatized each factor of $I^\infty$ to achieve this). Let $\omega$ be the point of $I_\omega$ whose $i$th coordinate is $1/2^i$.

Let $q_1, q_2, q_3, q_4$, and $q_5$ denote the points of $C_i = I_{n_i} \times I_{n_{i+1}}$ whose coordinates are $(0, 0)$, $(0, 1/2^i+1)$, $(1/2^i, 1/2^i+1)$, $(1/2^i, 0)$, and $((1/2^i+1)-(1/2^i+1))$ respectively. Let $M_i$ denote the closed rectangular region in $C_i$ whose vertices are $q_1, q_2, q_3$, and $q_4$. To define $h_i$, for $i > 1$, we use four sets:

- $S_i$ is the infinite product $I_{n_i+2} \times I_{n_i+3} \times \cdots$
- $T_i$ is a neighborhood of $(1/2^{2i}, 1/2^{2i+1}, \ldots, 1/2^i)$ in $I_{n_{i+2}} \times I_{n_{i+3}} \times \cdots \times I_{n_i}$
- $U_i$ is a neighborhood of $M_i$ in $C_i$; and
- $V_i$ is a neighborhood of $\tau_{n_i}(K)$ in $T_{n_i}(I^\infty)$.

In the case of $h_1$, as $T_1$ is vacuous, we use only $S_1$, $U_1$, and $V_1$.

The homeomorphism $h_i$ is to be supported on $S_i \times T_i \times U_i \times V_i$ (or on $S_1 \times U_1 \times V_1$ in the case of $h_1$). The sizes of the neighborhoods are to be selected (as specified later) in terms of $h_{i-1} \cdots h_1(K)$ so that the infinite left product $h$ of the $\{h_i\}$ will be defined and will move exactly $K$ to $W_{n_i}$. In fact $h$ will move $K$ onto the projection of $K$ in $W_{n_i}$ and will move exactly $K \cap o^{I^\infty}$ from $o^{I^\infty}$. Also for any $p \notin K$, $h(p)$ will be $h_nh_{n-1} \cdots h_1(p)$ for all sufficiently large $n$ and for any $p$ not in the projection of $K$ on $W_{n_i}$, $h^{-1}(p)$ will be $h_nh_{n-1} \cdots h_1^{-1}(p)$ for all sufficiently large $n$.

Let $\lambda_i$ be a homeomorphism of $C_i$ onto itself such that $\lambda_i$ is supported on $U_i$, is isotopic to the identity with support on $U_i$, carries the interval $[q_0, q_4]$ onto the interval $[q_5, q_2]$ and, for each $p \in C_i$, $\tau_{n_i}(\lambda(p)) = \tau_{n_i}(p)$. Let $\Lambda_i^t$ be an isotopy of $\lambda_i$ to the identity $e$ with $\Lambda_i^t = \lambda_t$ and $\Lambda_i^1 = e$, with, for each $t$, $\Lambda_i^t$ supported on $U_i$, and with, for each $t$ and each $p \in C_i$, $\tau_{n_i}^t(\Lambda_i^t(p)) \leq \tau_{n_i}(p)$. This last condition helps imply condition (4) of the lemma.

Let $Y_i = \overline{V_i} \times \overline{T_i}$ and, for $y \in Y_i$, let $y = (y_V, y_T)$ with $y_V \in \overline{V_i}$ and $y_T \in \overline{T_i}$. By the Urysohn Lemma we may let $\phi_i$ be a continuous map of $Y_i$ onto $[0, 1]$ such that (1) $\phi_i(y_V, y_T) = 0$ if and only if $y_V \in \tau_{n_i}(K)$ and $y_T = (1/2^i, 1/2^{i+1}, \ldots, 1/2^i)$ and (2) $\phi_i(y_V, y_T) = 1$ if $y_V \in \overline{V_i} \setminus V_i$ or $y_T \in \overline{T_i} \setminus T_i$.

For any $p \in S_i \times \overline{U_i} \times Y_i$ let $p$ be expressed as $(p_1, p_2, p_3)$ with $p_1, p_2, p_3$ points of
the factors above. For any $t, 0 \leq t \leq 1$, any $p_1 \in S_t$, any $p_2 \in \overline{U}_t$ and any $p_3 \in \phi_1^{-1}(t)$, let $h_i(p) = (p_1, \Lambda_i(p_2), p_3)$. Clearly under the given conditions since $\phi_1$ is continuous and $\Lambda_i$ is an isotopy, $h_i$ so defined is a homeomorphism of $S_t \times \overline{U}_t \times Y_t$ onto itself and $h_i$ is the identity on the boundary of $S_t \times \overline{U}_t \times Y_t$ in $I^\infty$. We let $h_i$ be defined as the identity outside of $S_t \times \overline{U}_t \times Y_t$ and, thus, this extended $h_i$ is by Lemma 2.5, a $\beta$-homeomorphism.

We define $h_1$ as above but with $Y_1 = \overline{Y}_1$. Since $\lambda_i$ carries the interval $[q_3, q_4]$ to the interval $[q_5, q_6]$ in $C_i$, then, for any $j$, $\tau_n(h_j \cdot \cdots \cdot h_2 h_1(K))$ is the point $1/2^{j+1}$ of $I_{n+1}$, and $\tau_n(h_j \cdot \cdots \cdot h_2 h_1(K)) = \tau_n(K)$ is the point $1/2^j$ of $I_n$.

We now ask for conditions on $T_i$, $U_i$, and $V_i$ so that $h$ will be the desired homeomorphism.

Let $U_i$ be the $(1/10)$-neighborhood of $M_i$ in $C_i$ and let $V_i = \tau_{\alpha}(I^\infty)$. Suppose we have given $h_1, \ldots, h_{i-1}$. Let $\delta_i$ be a positive number $< 1/10'$ such that for any two points $p, q \in I^\infty$ with $\rho(h_{i-1} \cdot \cdots \cdot h_1(p), h_{i-1} \cdot \cdots \cdot h_1(q)) > \delta_i$, then $\rho(p, q) < 1/10^i$.

Let $V_i$ be the $(\delta_i/3)$-neighborhood of $\tau\alpha(K)$ in $\tau\alpha(I^\infty)$. Let $T_i$ be the $(\delta_i/3)$-neighborhood of $(1/2^2, \ldots, 1/2^i)$ in $I_{n+2} \times I_{n+3} \times \cdots \times I_{n+1}$. Let $U_i$ be the $(\delta_i/3)$-neighborhood of $M_i$ in $C_i$.

We note that the metric $\rho$ on infinite products is a scaled down version of the metric $d$ on finite products. Thus a $\delta$-neighborhood of a point in a finite product corresponds to an open subset of the $\delta$-neighborhood of the point in the finite product regarded as a projection of the infinite product.

We consider the difference in action between $h_n = h_n h_{n-1} \cdot \cdots \cdot h_1$ and $\bar{h}_n = h_n \cdot \cdots \cdot h_1$. Since $h_n$ is supported on $S_n \times T_n \times U_n \times V_n$, then $\bar{h}_n$ differs from $\bar{h}_{n-1}$ in domain $\bar{h}_{n-1}^{-1}(S_n \times T_n \times U_n \times V_n)$ and in range $S_n \times T_n \times U_n \times V_n$. In effect, this range is a small neighborhood of the projection of $K$ on $W_{n+1}$ whereas the domain is a small neighborhood of $K$ itself. Furthermore the set which is the projection of $\bar{h}_n(K)$ on $W_{n+1}$ is the projection of $\bar{h}_{n-1}(K)$ on $W_{n}$ and such a set is moved by $h_n$ through the action of $\lambda_n$ to a nearby set on $W_{n+1}$ and then is eventually left alone. Since $\tau\alpha$ carries $K$ homeomorphically onto $\tau\alpha(K)$ and no $h_n$ affects any $\alpha'$ coordinate, then no two points of $K$ are brought close together. From these considerations and Lemma 2.2 it follows that $h = L \prod_{i > 0} h_i$ exists and is the desired homeomorphism.

**Lemma 5.2.** Let $\alpha$ be a subset of $Z$ and let $\{K_i\}_{i > 0}$ and $\{M_i\}_{i > 0}$ be any collection of closed sets in $I^\infty$ each straight with respect to an infinite set of integers disjoint from $\alpha$. Then there exists a $\beta$-homeomorphism $h$ such that

1. for $p \in \partial I^\infty \bigcap_{i > 0} K_i$, $h(p) \in \partial I^\infty$,
2. for any $i$, there is a $j_i$ such that $h(K_i) \subset W_{j_i}$,
3. for each $i$, $h(M_i)$ is straight with respect to some infinite set of integers, and
4. $\alpha(h) \subset \alpha'$.

**Proof.** By Lemma 2.9, there exists a subpartition $\{a_i\}$ of infinite sets such that for each $i$, $\alpha \cap a_i = \emptyset$, $K_i$ is straight with respect to $\alpha_{2i-1}$ and $M_i$ is straight with
respect to \( \alpha_{2i} \). We shall define a set \( \{h_i\}_{i>0} \) of \( \beta \)-homeomorphisms inductively by repeated use of Lemma 5.1. Thus for some \( j_i \in \alpha_{2i-1} \), \( h_i \) is to move \( K_i \) to \( W_{ji} \) with 
\[
\alpha(h_i) \leq \alpha_{2i-1}.
\]

Since \( (h_{i-1} \cdots h_i)^{-1} \) is a homeomorphism of a compact set we let \( \delta_i \) be a positive number \( < 1/10^4 \) such that if \( p, q \in I^\infty \) and 
\[
\rho((h_{i-1} \cdots h_i)(p), (h_{i-1} \cdots h_i)(q)) < \delta_i,
\]
then 
\[
\rho(p, q) < 1/10^4.
\]
By use of the product metric on \( I_{a_{2i} + 2_j - 1} \) we may require that for each \( j \geq 0 \), \( h_{i+j} \) move no point more than \( (1/2^j + 2) \delta_i \). Since for \( p, q \in I^\infty \), \( \rho(p, q) > 0 \), it follows as in Lemma 2.4 that 
\[
h = L \prod_{i>0} h_i
\]
is an element of \( G(I^\infty) \). By condition (4) of Lemma 5.1, it follows that \( h \) is a \( \beta \)-homeomorphism.

**Theorem 5.3.** Let \( \alpha \) be a collection of integers. Let \( \{K_i\}_{i>0} \) be any collection of closed sets such that each is weakly thin with respect to a subpartition whose elements are disjoint from \( \alpha \). Then there exists a \( \beta \)-homeomorphism \( h \) such that

1. for \( p \in (I^\infty \setminus \bigcup_{i>0} K_i) \), \( h(p) \in \partial I^\infty \),
2. for any \( i > 0 \), there is a \( j_i \) such that \( h(K_i) \subseteq W_{ji} \) and
3. \( \alpha(h) \leq \alpha' \).

**Proof.** This theorem follows immediately from Theorem 3.5 and Lemma 5.2.

**Corollary 5.4.** Let \( \{K_i\}_{i>0} \) be any collection of weakly thin subsets of \( I^\infty \). Then there exists a \( \beta \)-homeomorphism \( h \) such that for \( p \in \partial I^\infty \), \( h(p) \in B(I^\infty) \) if and only if \( p \in \bigcup_{i>0} K_i \).

**Corollary 5.5.** Let \( \partial I^\infty \) be regarded as the product of lines and as space and let \( \{K_i\}_{i>0} \) be any collection of closed sets in \( \partial I^\infty \) such that for each \( i \), \( K_i \) is bounded above (or below) in infinitely many directions. Then \( \partial I^\infty \setminus \bigcup_{i>0} K_i \) is homeomorphic to \( \partial I^\infty \).

**Proof.** We imbed \( \partial I^\infty \) in \( I^\infty \) in the natural way and for each \( i \), the closure of \( K_i \) is weakly thin. Hence Corollary 5.5 follows from Theorem 5.3.

We may regard this corollary as a theorem giving conditions under which a subset of \( \partial I^\infty \) is homeomorphic to \( \partial I^\infty \). As a special case of Corollary 5.5 we may assert:

**Corollary 5.6.** Suppose \( M \) is a subset of \( \partial I^\infty \) and \( M \) is the countable union of compact sets. Then \( \partial I^\infty \setminus M \) is homeomorphic to \( \partial I^\infty \).

Our last corollary applies the theorem to a more general setting.

**Corollary 5.7.** For any separable metric space \( X \) and for any countable collection \( \{K_i\}_{i>0} \) of compact subsets of \( X \times \partial I^\infty \), \((X \times \partial I^\infty) \setminus \bigcup_{i>0} K_i \) is homeomorphic to \( X \times \partial I^\infty \).

**Proof.** Let \( I^\infty = I_a^\infty \times I_a^\infty \) where \( \alpha \) and \( \alpha' \) are each infinite. Let \( f \) be an imbedding of \( X \times \partial I^\infty \) into \( I^\infty \) defined by imbedding \( X \) in \( I_a^\infty \) and \( \partial I^\infty \) in \( I_a^\infty \) the latter imbedding carrying \( \partial I^\infty \) onto \( I_a^\infty \). Now \( \{f(K_i)\}_{i>0} \) is a collection of closed sets as in Theorem 5.3 with \( \alpha \), above, regarded as \( \alpha \) of the theorem. The homeomorphism \( h \) of the
theorem carries \(f((X \times I^\infty) \setminus \bigcup_{n \geq 0} K_n)\) onto \(f(X \times I^\infty)\). Thus \(f^{-1}hf\) (with suitable restrictions of \(f\) and \(h\) understood) is the desired homeomorphism.

Finally we state a theorem concerning the pushing of compact sets in \(B(I^\infty)\) into \(I^\infty\).

**Theorem 5.7.** Let \(M\) be any compact subset of \(^oW_1\) and let \(\alpha\) be any infinite subset of \(Z\) with \(\alpha'\) infinite and \(1 \in \alpha'\). Then there exists a \(\beta\)-homeomorphism \(f\) such that

1. for \(p \in B(I^\infty)\), \(f^{-1}(p) \in I^\infty\) if and only if \(p \in M\) and
2. \(f^{-1}(M)\) is straight with respect to \(\alpha\).

**Proof.** Let \(\beta \supseteq Z\) be such that \(\beta \supseteq (\alpha \cup \{1\})\), \(\beta'\) is infinite and \(\beta'\) is infinite. Let \(K\) be a subset of \(I^\infty\) such that \(K\) is homeomorphic to \(M\) and \(K\) is straight with respect to \(\beta\). By Lemma 5.1 let \(h\) be a \(\beta\)-homeomorphism such that

1. \(\alpha(h) \subseteq \beta|\alpha\),
2. for \(p \in I^\infty\), \(h(p)\) is an element of \(B(I^\infty)\) if and only if \(p \in K\) and
3. for each \(p \in K\), \(\tau_1(h(p)) = 0\) and for \(k \in \beta(\alpha \cup \{1\})\), \(\tau_1(p) = \tau_1(h(p))\). Hence \(h(K) \subseteq ^oW_1\). By Theorem 4.4, we may let \(g\) be a \(\beta^*\)-homeomorphism carrying \(h(K)\) onto \(M\) such that \(1 \in \beta(g)\). Then \(gh\) is the desired homeomorphism.

6. The contraction theorem.

**Theorem 6.1.** There exists a \(\beta^*\)-homeomorphism, \(h\), such that for \(p \in I^\infty\) with

\(\tau_1(p) = 0\), \(\tau_1(h(p)) = 0\) and, for each \(j > 1\), \(0 < \tau_j(h(p)) < 1\).

**Proof.** We shall inductively construct a sequence \(\{h_i\}_{i=0}^\infty\) of finitely elementary homeomorphisms such that \(\alpha(h_i) = \{i, i+1\}\), \(\gamma(h_i) \subseteq \{2, 3, \ldots, i\}\) and the left product of the \(h_i\)'s exists as the desired homeomorphism \(h\). As before, \(W_1 = \tau_1^{-1}(0)\) and \(^oW_1 = \{0\} \times I_2 \times I_3 \times \cdots\). Let, for \(i > 1\), \(W_i(i)\) denote the set of all points \(p\) of \(W_i\) for which, for each \(1 < j \leq i\), \(0 < \tau_j(p) < 1\).

Let \(U_1\) be the \((1/10)-neighborhood of \(^oW_1\) in \(I^\infty\). Let \(h_1\) be defined so that

1. \(h_1\) is supported on \(U_1\), \(2. h_1(W_1) \subseteq W_1(2)\), \(3. \) for each \(j > 2\), \(j \in \beta(h_1)\) and \(4. \) for each \(p \in I^\infty\), \(\tau_1(p) \geq \tau_1(h_1(p))\). Inductively, let \(U_i\) be the \((1/10^i)-neighborhood of \(^oW_1\) in \(I^\infty\). Let \(h_i\) be defined so that

\(h_i[h_{i-1} \cdots h_1(W_1)] \subseteq W_i(i+1)\),

(3) for each \(j > i + 1\), \(j \in \beta(h_i)\), \(4. \) \(\alpha(h_i) = \{1, i+1\}\), \(5. \) for each \(p \in I^\infty\) with \(\tau_1(p) > 1/10^i\), \(h_{i-1} \cdots h_2 h_1(p) = h_{i-1} \cdots h_1(p)\), and \(6. \) for each \(p \in I^\infty\), \(\tau_1(p) \geq \tau_1(h_i(p))\). Note that condition (5) may be achieved since \(h_{i-1} \cdots h_2 h_1\) is uniformly continuous and \(h_i\) could be supported on a small neighborhood of \(h_{i-1} \cdots h_2 h_1(W_1)\) inside \(U_i\).

We wish to verify that, in fact, the left product of \(\{h_i\}_{i=0}^\infty\) is a \(\beta^*\)-homeomorphism satisfying the conditions of the theorem. Since with \(t = 1\), the conditions of Lemma 2.2 with respect to \(\tau_1\) are clearly satisfied using particularly conditions (1), (2), and (5) above, and since conditions (3) and (4) imply that the other hypotheses of Lemma 2.2 are satisfied, then the left product of \(\{h_i\}_{i=0}^\infty\) is a mapping \(h\) of \(I^\infty\) onto itself. To verify that \(h\) is a homeomorphism it suffices to note that no two points
\( p \) and \( q \) can be mapped to the same point by \( h \). From conditions (5) and (6) it follows that for some \( n \) and all \( m > n \)

\[
\tau_1(h_n \cdots h_1(p)) = \tau_1(h_m \cdots h_1(p)) = \tau_1(h(p))
\]
and

\[
\tau_1(h_n \cdots h_1(q)) = \tau_1(h_m \cdots h_1(q)) = \tau_1(h(q)).
\]

But then, as in Lemma 2.3, if \( p \neq q \), there is some \( k > 0 \) such that \( \tau_k(h(p)) \neq \tau_k(h(q)) \) for otherwise, for some \( j, h_1 \cdots h_1 \) could not be a homeomorphism. Conditions (2) and (4) (principally) imply that \( h(W_1) \subset W_1 \). To see that \( h \) is a \( \beta^* \)-homeomorphism we observe using conditions (3), (4), and (5), that no point of \( \delta I^\infty \) is moved to \( W_1 \), or, in fact, to \( B(I^\infty) \), and that no point of \( B(I^\infty) \) is moved to \( \delta I^\infty \) since, for \( p \in B(I^\infty) \), if \( \tau_1(p) = 0, 1 \), then either \( \tau_1(h(p)) = 0, 1 \) or \( \tau_1(h(p)) = 0 \) as implied by condition (6) in the presence of the other conditions.

We now reformulate the contraction theorem in a more general setting for use in §9. Let \( \alpha = \{n_i \} \) be any infinite subset of \( Z \).

For each \( i \), let \( f_i \) be an order-preserving homeomorphism on \( I_{n_i} \) onto \( I_i \). For \( p = \{p_{n_i} \} \in I_\alpha \), let \( f(p) = \{f_i(p_{n_i})\} = \{f(p_{n_i})\} \) and \( f \) so defined is a homeomorphism of \( I_\alpha \) onto \( I^\infty \). Each \( \beta \)- or \( \beta^* \)-homeomorphism \( g \) of \( I^\infty \) produces a \( \beta(I_\alpha) \) or \( \beta^*(I_\alpha) \)-homeomorphism of \( I_\alpha \), namely \( f^{-1}gf \). The set \( W_1 \in I^\infty \) is \( f(W_1(I_\alpha)) \) and \( \delta W_1 = f(\delta W_1(I_\alpha)) \).

These considerations lead us immediately to the desired corollary.

**Corollary 6.2.** Let \( \alpha = \{n_i \} \geq 0 \) be any infinite subset of \( Z \). There exists a \( \beta^*(I_\alpha) \)-homeomorphism, \( h \), such that, for \( p \in I_\alpha \), with \( \tau_{n_i}(p) = 0, \tau_{n_i}(h(p)) = 0 \) and, for each \( j > 1, 0 < \tau_{n_j}(h(p)) < 1 \).

**Note.** We can let any element of \( \alpha \) be listed as \( n_i \).

**Remark 6.3.** The lemmas, theorems, and definitions of the preceding sections can be similarly reformulated to refer to \( I_\alpha \) instead of \( I^\infty \).

7. The principal extension theorem.

**Definition.** A closed set \( K \) is said to be normally imbedded in \( B(I^\infty) \) if for some finite subset \( \alpha \subset Z \), \( \tau_\alpha(K) \) is a subset of the boundary of \( \tau_\alpha(I^\infty) \).

**Definition.** A closed set \( K \) is said to be calm in \( I^\infty \) if \( K \) is the finite union of closed sets each of which is either weakly thin in \( I^\infty \) or normally imbedded in \( B(I^\infty) \).

**Theorem 7.1.** Let \( M \) be a closed subset of \( I^\infty \) and let \( f \) be a homeomorphism of \( M \) into \( I^\infty \). If \( M \cup f(M) \) is calm, then \( f \) can be extended to an element of \( G(I^\infty) \).

**Proof.** Since \( M \cup f(M) \) is calm, let \( \delta \) be a finite subset of \( Z \) such that \( M \cup f(M) \) is the union of closed sets \( N_1 \) and \( N_2 \) with the property that \( \tau_\delta(N_1) \) is a subset of the boundary of \( \tau_\delta(I^\infty) \) and \( N_2 \) is the union of finitely many closed sets each weakly thin with respect to a subpartition whose elements are in \( Z \setminus \delta \). By Theorem 5.3 there exist a \( \beta \)-homeomorphism \( h \) and a finite set \( \alpha \subset Z \) such that \( \tau_\alpha(h(M \cup f(M))) \)
is a closed subset of $B(\tau_n(I^\omega))$. Let $g$ be a $\beta^*$-homeomorphism such that $\alpha(g) = \alpha \cup \{1\}$ and $g(h(M \cup f(M))) \in W_1$. Clearly $g$ can be produced from a homeomorphism of the finite cell $I_{\alpha(1)}$.

By Theorem 6.1 there exists a $\beta^*$-homeomorphism $\phi$ such that $\phi gh(M \cup f(M))$ is a (compact) subset of $W_1$. Hence by Theorem 4.4 there exists a $\beta^*$-homeomorphism $\eta$ extending $\phi ghf^{-1}g^{-1}\phi^{-1}$ (with $h$, $g$, and $\phi$ cut down appropriately) from the set $\phi gh(M)$ onto $\phi ghf(M)$. But then $h^{-1}g^{-1}\phi^{-1}\eta\phi gh$ extends $f$ as was to be shown.

Theorem 7.1 gives a rather strong homogeneity property of the Hilbert cube: namely, for many closed subsets of $I^\omega$ any homeomorphism from one to the other may be extended to an element of $G(I^\omega)$. (For ordinary 1-point homogeneity, $M$ and $f(M)$ may be regarded as single points.) In a paper entitled On topological infinite deficiency, a definitive theorem will be given.

8. Unions of two Hilbert cubes. In this section we use the result of §7 to get conditions under which the union of two Hilbert cubes is homeomorphic to a Hilbert cube.

Let $A$ and $A'$ be disjoint metric spaces and let $K$ and $K'$ be closed subsets of $A$ and $A'$ respectively. Suppose there exists a homeomorphism $f$ of $K$ onto $K'$. Let $A \cup f A'$ denote the space whose points are (1) the points of $A \setminus K$, (2) the points of $A' \setminus K'$, and (3) the pairs $\{k, f(k)\}$ for $k \in K$. Let $g$ and $g'$ be the canonical 1-1 transformations of $A$ and $A'$ respectively into $A \cup f A'$. A set $U$ in $A \cup f A'$ is open if and only if $g^{-1}(U)$ and $g'^{-1}(U)$ are open in $A$ and $A'$ respectively.

**Theorem 8.1.** Suppose $A$, $A'$, $K$, and $K'$ are all homeomorphic to $I^\omega$ and suppose $K$ and $K'$ are calm subsets of $A$ and $A'$ respectively. Then for any homeomorphism $h$ of $K$ onto $K'$, $A \cup f A'$ is homeomorphic to $I^\omega$.

**Proof.** This theorem follows readily from Theorem 7.1. Let $g$ and $g'$ be as in the explanation of $A \cup f A'$. Let $f$ be a homeomorphism of $g(A)$ onto itself carrying the set of points of $g(A)$ with first coordinate 0 onto $g(K)$. Let $f'$ be a similar homeomorphism of $g(A')$ onto itself. We give the desired coordinatization of $A \cup f A'$ as a Hilbert cube. For any $x \in g(A)$ let every coordinate of $x$ after the first be the similar coordinate of $f^{-1}(x)$ and let the first coordinate of $x$ be $\frac{1}{2}(1-x_1)$ where $x_1$ is the first coordinate of $f^{-1}(x)$. For any $x \in g'(A')$, let the first coordinate of $x$ be $\frac{1}{2} + \frac{1}{2}(x_1)$ where $x_1$ is the first coordinate of $f'^{-1}(x)$ and let every other coordinate be that already assigned to the point of $g'(K')$ whose other coordinates under $f'^{-1}$ agree with those of $f^{-1}(x)$.

**Conjecture.** If the intersection of two Hilbert cubes is a Hilbert cube then their union is a Hilbert cube.

In a paper to be published separately the author has shown that the product of any dendron and $I^\omega$ is homeomorphic to $I^\omega$. It is easy to see that the above conjecture could not be true unless, for example, the product of a triod and $I^\omega$ were
homeomorphic to \( I^\infty \). But the conjecture is, in fact, much stronger than this latter statement.

**Question.** If the union of two Hilbert cubes is a Hilbert cube must their intersection be a Hilbert cube? It seems likely that the answer is in the negative but the author does not know how to prove it. Clearly such intersection must be an infinite-dimensional Cantorian manifold. If there exists (as seems likely) an infinite-dimensional Cantorian manifold which is an absolute retract and is not homeomorphic to \( I^\infty \), then it might be possible to slice a Hilbert cube into two Hilbert cubes by such a set.

9. **Products homeomorphic to \( 0(I)^\infty \).** In this section we apply the procedures of the preceding sections to show that many infinite products not obviously homeomorphic to \( 0I^\infty \) are, in fact, homeomorphic to this set.

**Definitions.** A set \( Y \) is said to be a near \( n \)-cell, \( n > 0 \), if \( Y \) is a subset of a closed \( n \)-cell \( V \) such that \( Y \) contains \( \text{Int} V \). We say that \( V \) carries \( Y \). A near \( n \)-cell \( Y \) for which \( V \setminus Y \neq \emptyset \) is called proper. A near \( n \)-cell \( Y \) for which \( Y \setminus \text{Int} V \) is a \( G_\delta \) subset of \( B(V) \) is called a \( G_\delta \) near \( n \)-cell.

The following lemma is almost obvious and is given without proof.

**Lemma 9.1.** If \( Y_1 \) and \( Y_2 \) are \( G_\delta \) near \( n \)-cells, \( i = 1, 2 \), and \( Y_1 \) is proper, then \( Y_1 \times Y_2 \) is a proper \( G_\delta \) near \( (n_1 + n_2) \)-cell.

**Lemma 9.2.** Let \( Y \) be a proper near \( n \)-cell. Then for any set \( \alpha = \{j_1, \ldots, j_n\} \) of distinct positive integers, there exists a homeomorphism \( \phi \) of \( Y \) into \( I_\alpha \) such that (i) \( I_\alpha \) carries \( \phi(Y) \), (ii) \( \tau_{j_1}(\phi(Y)) \) is the point \( 0 \in I_{j_1} \), and (iii) for each \( k \), \( 1 \leq k \leq n \), \( 0 < \tau_{j_k}(\phi(Y)) < 1 \).

**Proof.** Let \( V \) be an \( n \)-cell which carries \( Y \) and let \( p \) be a point of \( V \setminus Y \). Let \( \sigma \) be a map of \( I_\alpha \) onto \( V \) such that for each \( q \in V \) with \( q \neq p \), \( \sigma^{-1}(q) \) is a single point of \( I_\alpha \) and such that \( \sigma^{-1}(p) \) is the closure of the set of all points of the boundary of \( I_\alpha \) with \( j_1 \) coordinate \( > 0 \). Then \( \sigma^{-1}|Y \) is the desired homeomorphism \( \phi \).

We call \( I_{j_1} \) of the above lemma the end-factor and \( j_1 \) the end-index of \( I_\alpha \).

**Theorem 9.3.** Let, for each \( \alpha = (j_1, \ldots, j_n) \), \( Y_\alpha \) be a \( G_\delta \) near \( n \)-cell and let, for infinitely many \( i \), \( Y_i \) be proper. Then \( Y = \prod_{i>0} Y_i \) is homeomorphic to \( 0I^\infty \).

**Proof.** By Lemma 9.1 we may take products of pairs of \( Y_i \)'s so that each product is a proper near \( m_k \)-cell for some \( m_k \geq 2 \). Without loss of generality, we assume the \( Y_i \)'s themselves to be proper and each \( n_i \) to be \( \geq 2 \). By Lemma 9.2 we may regard \( I^\infty \) as \( V_1 \times V_2 \cdots \) where, for each \( j \), \( V_j \) is a finite product \( I_{j_1}, \ldots, I_{j_{n_j}} \) and \( V_j \) carries \( Y_j \) in the manner of Lemma 9.2. Let \( T \) be the collection of all end-indices of the various \( V_j \)'s.

For each \( j \), let \( X_j \) be the set of all points of \( V_j \) with end-factor coordinate \( \neq 1 \) and other coordinates neither 0 nor 1. Thus we have \( 0V_j \subset Y_j \subset X_j \subset V_j \). Letting \( X = \prod_{i>0} X_i \) we also have \( 0I^\infty \subset Y \subset X \subset I^\infty \).
Our strategy will be first to exhibit a homeomorphism \( f \) of \( I^\infty \) onto itself such that \( f(X) = I^\infty \). Then we find a \( \beta \)-homeomorphism \( h \) deleting a countable union of straight sets (relatively closed in \( I^\infty \)) from \( I^\infty \) so that \( hf \) carries \( Y \) onto \( I^\infty \), i.e., we use \( h \) to eliminate the points of \( f(X\setminus Y) \) from \( I^\infty \).

Let \( \{a_j\} \) be a partition of \( Z \) such that for each \( j > 0 \), \( a_j \) is infinite and \( a_j \cap T \) is a single element.

Let \( j^* \) denote the element of \( T \) in \( a_j \). Let \( W'_j \) denote the set of all points of \( I_{a_j} \) with \( j^* \) coordinate 0. Let \( \tilde{W}'_j \) denote the subset of \( W'_j \) consisting of all points of \( W'_j \) each of whose other coordinates is neither 0 nor 1.

We note that \( X = \prod_{j > 0} (I_{a_j} \cup \tilde{W}'_j) \) since \( I_{a_j} \cup \tilde{W}'_j \) is the product of an appropriate half-open interval by infinitely many open intervals. Thus \( \prod_{j > 0} (I_{a_j} \cup \tilde{W}'_j) \) can be refactored to produce \( X \).

By Corollary 6.2, there exists a \( \beta^*(I_{a_j}) \)-homeomorphism \( \eta_j \) such that \( \eta_j(W'_j) \subset \tilde{W}'_j \). By Theorem 5.7 and Remark 6.3 there exists a \( \beta(I_{a_j}) \)-homeomorphism \( \phi_j \) such that \( \phi_j^{-1}\eta_j(W'_j) \) is a straight set in \( I_{a_j} \) and

\[
\phi_j(I_{a_j}) = I_{a_j} \cup \tilde{W}'_j.
\]

For each \( j \), consider \( I^\infty = I_{a_j} \times I_{a_j} \). For each \( j \), \( k \in \alpha_j \), with \( k \neq j^* \), let \( C_k(j) \) denote the (closed) set of points of \( W'_j \) with \( k \)-coordinate 0 or 1. Then \( W'_j = \tilde{W}'_j \cup \bigcup_{k \neq j^*} C_k(j) \). Also since \( \phi_j^{-1}\eta_j(W'_j) \) is straight in \( I_{a_j} \) then so is \( \phi^{-1}\eta_j(C_k(j)) \) for each \( k \). Therefore by Lemma 5.2, there is a \( \beta(I_{a_j}) \)-homeomorphism \( g_k \) such that \( g_k \) carries \( I_{a_j} \cup \bigcup_{k \neq j^*} \phi_k^{-1}\eta_j(C_k(j)) \) onto \( I_{a_j} \), and \( g \phi_j^{-1}\eta_j(W'_j) \) is straight in \( I_{a_j} \). Let \( \tilde{g}_j \) be defined with respect to \( g_k \) as were \( \tilde{\eta}_j \) and \( \phi_j \) with respect to \( \eta_j \) and \( \phi_j \). Let \( g = L \prod_{j < \infty} \tilde{g}_j \), and \( g \), by Lemmas 2.3 and 2.8 is a \( \beta \)-homeomorphism. Hence \( g \phi_j^{-1}\eta \) is the desired homeomorphism \( f \) of our strategy and \( g \phi_j^{-1}(\tau^{-1}(0)) \) is straight for each \( t \in T \).

For any \( i \), we consider \( V_i, X_i, \) and \( Y_i \). By definition of near \( n \)-cell and the construction of Lemma 9.2, for each \( i \), there exists a countable collection \( R_n(i), \lambda > 0 \), of closed subsets of the closure of \( X_i \setminus I^\infty \) such that \( X_i \setminus I^\infty = Y_i \). For \( j^* \) the end-index of \( V_i \) and \( j^* \in \alpha_n \), the closure of \( X_i \setminus I^\infty \) crossed the factors of \( I^\infty \) not factors of \( V_i \), is precisely the set \( W'_j \) crossed the factors of \( I^\infty \) not in \( a_j \) and is precisely the set \( \tau^{-1}(0) \). Let \( M \) be the collection of all sets \( R_n(i) \) crossed with the factors of \( I^\infty \) not factors of \( V_i \). Since for \( t \in T \), \( g \phi_j^{-1}(\tau^{-1}(0)) \) is straight in \( I^\infty \), then so is each set \( g \phi_j^{-1}(M) \) for \( M \in M \). The collection of all such sets \( g \phi_j^{-1}(M) \) is a countable collection of straight sets and by Lemma 5.2 there is a \( \beta \)-homeomorphism \( h \) of \( I^\infty \) onto itself such that for \( p \in \partial I^\infty \), \( h(p) \in B(I^\infty) \) if and only if \( p \) is an element of some such set \( g \phi_j^{-1}(M) \). But \( X \setminus \bigcup_{M \in M} M \) must be the set \( Y \) since
in order for a point \( p \) to be in \( X \setminus Y \), it is necessary and sufficient that there be an \( i \) such that the projection of \( p \) on \( V_i \) is in \( X_i \setminus Y_i \).

Thus Theorem 9.3 is proved.

**Remark.** The question as to whether a given set \( M, \beta^I \subset M \subset I^\infty \), is homeomorphic to \( I^\infty \) is partially answered by the results of §5: the images of \( I^\infty \) under \( \beta \)-homeomorphisms are such sets. Also §5 explicitly gives us conditions under which certain subsets of \( I^\infty \) are homeomorphic to \( I^\infty \).

**Corollary 9.4.** In order that the countable infinite product of intervals (each open, closed, or half-open) be homeomorphic to \( I^\infty \) it is necessary and sufficient that infinitely many of the factors be open or half-open.

**Proof.** The sufficiency follows explicitly from Theorem 9.3. The necessity follows from the fact that \( I^\infty \) is not locally compact whereas such a product with only finitely many open or half-open factors would be locally compact.

**Corollary 9.5.** \( I^\infty \times I^\infty \) is homeomorphic to \( I^\infty \).

In [3] Bessaga and Klee show that Corollary 9.4 above is a corollary of one of their theorems dealing with infinite (not necessarily countable) products.

Sierpiński [9] has shown that the absolute \( G_\delta \)'s are the spaces which are topologically complete. Thus the \( G_\delta \) requirement of the definition of near \( n \)-cell cannot be weakened at all if Theorem 9.3 is to be true. In fact, from Sierpiński's result and Theorem 9.3 we get our final theorem.

**Theorem 9.5.** In order that a countable infinite product of near \( n \)-cells be homeomorphic to \( I^\infty \) it is necessary and sufficient that (1) each factor be a \( G_\delta \) near \( n \)-cell and (2) infinitely many of the factors be proper near \( n \)-cells.

**References**