OSCILLATING SEQUENCES MODULO ONE(1)

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Given a sequence of real numbers $\{a_n\}$ with fractional parts $\langle a_n \rangle$, let

$$N_n(a, b) = \#\{m \leqq n \mid a \leqq \langle a_m \rangle < b\}.$$

(If S is a set, $\#S$ denotes its order.) If $\lim_{n\to\infty} N_n(a, b)/n = b-a$ for all a and b such that $0 \leqq a < b \leqq 1$, then $\{a_n\}$ is said to be *uniformly distributed modulo* 1. LeVeque [3] investigated sequences $\{f(n) \cos g(n)y\}$ for uniform distribution (mod 1) for almost all real numbers y. When either f or g increases as rapidly as the exponential function, rather general results can be obtained. When neither f nor g increases this rapidly, it can be shown that $\{n^\alpha \cos n^\beta y\}$ is u.d. (mod 1) for almost all y when $\alpha \geqq \beta \geqq 1$. So, it is known that $\{n \cos ny\}$ is u.d. (mod 1) for almost all y, and it is easy to show that $\{\cos ny\}$ is not. The question arises: how rapidly must f increase in order to insure that $\{f(n) \cos ny\}$ is u.d. (mod 1) for almost all y? It is the purpose of this paper to show that any sufficiently smooth f which increases to infinity, no matter how slowly, will serve.

In what follows only the following facts about the cosine are used: it is continuous, bounded, twice differentiable, periodic, piecewise monotone, and at no point do both its first and second derivatives vanish. The results below will hold for any function with these properties in place of the cosine.

We will suppose throughout that f increases without bound, f' decreases, and $f(x) = o(x^{1/2-\varepsilon})$ as $x \to \infty$ for some $\varepsilon > 0$. Also, we assume that $y/2\pi$ is irrational.

LEMMA 1. *Suppose that* $\{x_k\}$ *and* $\{y_k\}$ *are such that* $\lim_{k\to\infty} (x_k - y_k) = 0$. *If either sequence is u.d.* (mod 1), *so is the other.*

Proof. We apply Weyl's criterion for uniform distribution (see, e.g., [2, p. 91]), namely, $\{a_k\}$ is u.d. (mod 1) if and only if $S_a(N) = \sum_{k=1}^N \exp(2\pi i m a_k) = o(N)$ for each positive integer m. (In O- and o-symbols, we will always suppose the variable approaches infinity.) If

$$R(N) = \sum_{k=1}^N |1 - \exp 2\pi i m(y_k - x_k)|,$$

then $|S_x(N) - S_y(N)| \leqq R(N)$. But

$$R(N) = \sum_{k=1}^N 2|\sin \pi m(y_k - x_k)| \leqq 2\pi m \sum_{k=1}^N |y_k - x_k| = o(N).$$

It follows that if one of $S_x(N)$ and $S_y(N)$ is $o(N)$, then so is the other.

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We decompose a period of the cosine into its strictly monotone components: $P_1(x)=\cos x$, $0<x\leqq\pi$, and $P_2(x)=\cos x$, $\pi<x\leqq 2\pi$. For $0\leqq\alpha\leqq 1$ and $B>0$, let

$$N(B, n, \alpha) = \#\{m \leqq n \mid \langle B\cos my\rangle < \alpha\}.$$

Let $\omega(B, \alpha)$ denote the distribution function of $\{B\cos ny\}$:

$$\omega(B, \alpha) = \lim_{n\to\infty} N(B, n, \alpha)/n.$$

We later show that the limit exists by computing it. Separate the numbers my, $m=1, 2, \ldots, n$, into two classes: put my in class C_1 if $0<\langle my/2\pi\rangle\leqq 1/2$, otherwise put it in class C_2. Let

$$N_i(B, n, \alpha) = \#\{m \leqq n \mid my \in C_i \text{ and } \langle B\cos my\rangle < \alpha\},$$

$i=1, 2$. Then

$$N(B, n, \alpha) = N_1(B, n, \alpha)+N_2(B, n, \alpha),$$

and

$$N_i(B, n, \alpha) = \#\{m \leqq n \mid \langle BP_i(my)\rangle < \alpha\}. \tag{1}$$

Let

$$\omega_i(B, \alpha) = \lim_{n\to\infty} N_i(B, n, \alpha)/n, \qquad i = 1, 2. \tag{2}$$

LEMMA 2. *With the above notation,*

$$\omega_1(\beta, \alpha) = \sum_{i=-[B]}^{[B]} \frac{1}{2\pi}\left(P_1^{-1}\left(\frac{i}{B}\right)-P_1^{-1}\left(\frac{i+\alpha}{B}\right)\right)+\Gamma(B),$$

where $\lim_{B\to\infty}\Gamma(B)=0$.

Proof. Suppose for the moment that B is an integer. In that case,

$$\begin{aligned} N_1(B, n, \alpha) &= \sum_{i=-B}^{B} \#\{m \leqq n \mid i \leqq BP_1(my) < i+\alpha\} \\ &= \sum_{i=-B}^{B} \#\left\{m \leqq n \,\middle|\, \frac{i}{B} \leqq P_1\left(2\pi\left\langle\frac{my}{2\pi}\right\rangle\right) < \frac{i+\alpha}{B}\right\} \qquad (3) \\ &= \sum_{i=-B}^{B} \#\left\{m \leqq n \,\middle|\, \frac{1}{2\pi}P_1^{-1}\left(\frac{i+\alpha}{B}\right) < \left\langle\frac{my}{2\pi}\right\rangle \leqq \frac{1}{2\pi}P_1^{-1}\left(\frac{i}{B}\right)\right\}. \end{aligned}$$

Since we have assumed that $y/2\pi$ is irrational, it follows that $\{ny/2\pi\}$ is u.d. (mod 1) (see, e.g., [4, p. 24]). Dividing by n in (3) and letting it increase without bound, we have

$$\omega_1(B, \alpha) = \sum_{i=-B}^{B} \frac{1}{2\pi}\left(P_1^{-1}\left(\frac{i}{B}\right)-P_1^{-1}\left(\frac{i+\alpha}{B}\right)\right),$$

the conclusion of the lemma with no error term.

In case B is not an integer, we consider the sum in (3) between $-[B]$ and $[B]$. This introduces two errors, which give rise to $\Gamma(B)$:

$$(4)\quad N_1(B, n, \alpha) = \sum_{i=-[B]}^{[B]} \#\{m \leqq n \mid i \leqq BP_1(my) < i+\alpha\} + E_0(B, n, \alpha) + E_1(B, n, \alpha),$$

where $E_0(B, n, \alpha)$ is the error committed by ignoring in the sum the portion of the graph of $BP_1(x)$ lying between $[B]$ and B. E_1 is the similar error near $-B$. We first consider E_0.

Define $\rho_0 = \rho_0(B)$ to be the smallest positive number such that $BP_1(\rho_0 y) = [B]$. As B increases, the common length of the intervals $(2\pi k, \rho_0 y + 2\pi k)$, $k = 0, 1, \ldots$, decreases, and hence the intervals may be expected to contain fewer integers. We will show that this number of integers is negligible. Taylor's Theorem gives

$$1 - \cos \rho_0 y = \tfrac{1}{2}(\rho_0 y)^2 \cos \xi,$$

where $0 \leqq \xi \leqq \rho_0 y$, so

$$\tfrac{1}{2}(\rho_0 y)^2 = (1 - [B]/B)/\cos \xi,$$

which shows that $\rho_0 = o(1)$ as $B \to \infty$.

We now get an upper bound for E_0 in terms of ρ_0. E_0 is largest when α is 1, so

$$\begin{aligned} E_0(B, n, \alpha) \leqq E_0(B, n) &= \#\{m \leqq n \mid [B] \leqq BP_1(my) \leqq B\} \\ &= \#\{m \leqq n \mid 2\pi t \leqq my \leqq \rho_0 y + 2\pi t \text{ for some integer } t \geqq 0\}. \end{aligned}$$

If we set $z_k = 2\pi k/y$, $k = 0, 1, \ldots$, then

$$(5)\qquad E_0(B, z_n y) \leqq \#\{m \leqq z_n y \mid z_t \leqq m \leqq z_t + \rho_0 \text{ for some } t\}.$$

Since $\rho_0(B) = o(1)$, we may assume that $y\rho_0 < 1$ and $\rho_0 < 1$. In (5), we consider the positive integers less than $z_n y$ and count how many fall in the specified intervals; if instead we consider the intervals (each of which has length less than 1) and count how many contain an integer, we have

$$E_0(B, z_n y) = \#\{t \leqq n \mid z_t \leqq m \leqq z_t + \rho_0 \text{ for some integer } m\}.$$

$\{2\pi k/y\}$ is u.d. (mod 1), so

$$E_0(B, z_n y) = \#\{t \leqq n \mid \langle z_t \rangle \geqq 1 - \rho_0\}$$

and $E_0(B, z_n y)/n \to \rho_0$ as $n \to \infty$. A calculation shows that $E_0(B, n)/n \to \rho_0/2\pi$ as $n \to \infty$.

If we define $\rho_1 = \rho_1(B)$ to be such that $P_1(\pi y - \rho_1 y) = -[B]$, then we can show, as we did for ρ_0, that $\rho_1(B) = o(1)$ and $E_1(B, n)/n \to \rho_1/2\pi$ as $n \to \infty$. It follows that $0 \leqq \Gamma(B) \leqq (\rho_0 + \rho_1)/2\pi$, and the lemma is proved.

LEMMA 3.

$$(6)\qquad \lim_{B \to \infty} \omega_1(B, \alpha) = \alpha/2.$$

Proof. Since P_1^{-1} is differentiable, from Lemma 2 and the mean-value theorem,

$$\omega_1(B, \alpha) = \sum_{i=-[B]}^{[B]} \frac{1}{2\pi} \frac{\alpha}{B} \frac{d}{dt} P_1^{-1}(t)\Big|_{t=\xi_i} + \Gamma(B), \tag{7}$$

where $i/B \leqq \xi_i \leqq (i+\alpha)/B$. But (7) is an approximating sum for a Riemann integral:

$$\lim_{B\to\infty} \omega_1(B, \alpha) = \frac{-\alpha}{2\pi} \int_{-1}^{1} \left(\frac{d}{dt} P_1^{-1}(t)\right) dt + \lim_{B\to\infty} \Gamma(B) = \frac{\alpha}{2}.$$

In the same way as we got (6), we can show that

$$\lim_{B\to\infty} \omega_2(B, \alpha) = \alpha/2.$$

So we have

LEMMA 4. $\lim_{B\to\infty} \omega(B, \alpha) = \alpha$.

For later purposes, we need a quantitative version of (2). If $\{a_k\}$ is a sequence with distribution function $\omega(\alpha)$, the *discrepancy*, $D(n)$, is defined by

$$D(n) = \sup_{0\leqq\alpha\leqq 1} D(n, \alpha),$$

where

$$D(n, \alpha) = (1/n)\#\{m \leqq n \mid \langle a_m \rangle < \alpha\} - \omega(\alpha).$$

We use this in the next lemma.

LEMMA 5. *If $N_j(B, n, \alpha)$ and $\omega_j(B, \alpha)$ are as in* (2) *and* (1), *then*

$$N_j(B, n, \alpha) = n\omega_j(B, \alpha) + O(Bn^{\delta}) + n\Gamma_{1j}(B), \tag{8}$$

for any $\delta > 0$ for almost all y. $\Gamma_{1j}(B) \to 0$ as $B \to \infty$.

Proof. We consider the case $j=1$; the other is similar. For a u.d. (mod 1) sequence $\{a_n\}$, we have for $0 \leqq a < b \leqq 1$,

$$\#\{m \leqq n \mid a \leqq \langle a_m \rangle < b\} = n(b-a) + knD(n),$$

where $|k| < 2$. Applying this to each term of the sum in (4), we have, since $\{ny/2\pi\}$ is u.d. (mod 1),

$$\#\{m \leqq n \mid i \leqq BP_1(my) < i+\alpha\} = \frac{n}{2\pi}\left(P_1^{-1}\left(\frac{i}{B}\right) - P_1^{-1}\left(\frac{i+\alpha}{B}\right)\right) + knD(n), \tag{9}$$

where $D(n)$ is the discrepancy of $\{ny/2\pi\}$.

As in [2, p. 27], given an irrational number θ, let $\eta+1$ be the least upper bound of numbers $\Omega > 0$ for which $|\theta - p/q| < 1/q^{\Omega}$ has infinitely many solutions in integers p and q. If $\eta < \infty$, then θ is said to be of type $I\eta$. A theorem of Khintchine [1] asserts that the inequality $|\theta - p/q| < 1/qF(q)$ has infinitely many or finitely many solutions for almost all θ according as $\int^{\infty} dx/F(x)$ diverges or converges. It follows that almost all irrational numbers are of type I1. It is known [2, p. 103] that for all θ of type I1, the discrepancy of $\{n\theta\}$ satisfies $D(n) = O(n^{-1+\delta})$ for any $\delta > 0$.

Applying this and summing (9) over i from $-[B]$ to $[B]$, we have, from (4) and Lemma 2,

$$N_1(B, n, \alpha) - E_0(B, n, \alpha) - E_1(B, n, \alpha) = n\omega_1(B, \alpha) - n\Gamma(B) + O(Bn^\delta)$$

where $\lim_{B\to\infty} \Gamma(B) = 0$. Since $0 \leqq E_0 + E_1 \leqq n\Gamma(B)$, the lemma follows.

As a corollary, summing (8) over $j = 1, 2$ gives

$$N(B, n, \alpha) = n\omega(B, \alpha) + n\Gamma_2(B) + O(Bn^\delta) \tag{10}$$

for almost all y. $\Gamma_2(B) \to 0$ as $B \to \infty$. The y for which (10) does not hold are those such that $y/2\pi$ is a rational number or is closely approximable by rational numbers (that is, not of type I1).

We now define a function which closely approximates $f(n) \cos ny$ and is u.d. (mod 1).

LEMMA 6. *Let $\{a_k\}$ be such that $a_1 < a_2 < \ldots$, $\lim_{k\to\infty} \Delta a_k = \infty$ $(\Delta a_k = a_{k+1} - a_k)$, and $\lim_{k\to\infty} \Delta a_k / a_k = 0$. Let f be an increasing function with $\lim_{x\to\infty} f(x) = \infty$. Suppose that for $\delta > 0$ sufficiently small,*

$$\lim_{n\to\infty} nf(a_n)/a_n^{1-\delta} = 0.$$

Define g by

$$g(x) = f(a_k) \cos xy \quad \text{when } a_k \leqq x < a_{k+1}, \quad k = 1, 2, \ldots.$$

Then $\{g(n)\}$ is u.d. (mod 1) *for almost all y.*

Proof. Put

$$N'(n, \alpha) = \#\{m \leqq n \mid \langle g(m) \rangle < \alpha\}.$$

We will show that $N'(a_n, \alpha)/a_n \to \alpha$ as $n \to \infty$, and then that $N'(n, \alpha)/n$ has the same limit. If

$$M(a_k, a_{k+1}, \alpha) = \#\{a_k \leqq m < a_{k+1} \mid \langle f(a_k) \cos my \rangle < \alpha\},$$

then

$$N'(a_n, \alpha) = \sum_{k=1}^{n-1} M(a_k, a_{k+1}, \alpha). \tag{11}$$

But $M(a_k, a_{k+1}, \alpha) = N(f(a_k), a_{k+1}, \alpha) - N(f(a_k), a_k, \alpha)$ so from (10),

$$M(a_k, a_{k+1}, \alpha) = \Delta a_k \omega(f(a_k), \alpha) + O(f(a_k) a_{k+1}^\delta) + \Delta a_k \Gamma_2(f(a_k)).$$

From Lemma 4, as $k \to \infty$, $\omega(f(a_k), \alpha) = \alpha + o(1)$. Also, as $k \to \infty$, $\Gamma_2(f(a_k)) = o(1)$. Substituting in (11),

$$N'(a_n, \alpha) = \sum_{k=1}^{n-1} ((\alpha + o(1))\, \Delta a_k) + O(nf(a_n) a_n^\delta)$$

so

$$\frac{1}{a_n} N'(a_n, \alpha) = \alpha + O\left(\frac{1}{a_n}\right) + \frac{1}{a_n} \sum_{k=1}^{n-1} (\Delta a_k) o(1) + O\left(\frac{nf(a_n)}{a_n^{1-\delta}}\right).$$

The error terms are all $o(1)$, so $\lim_{n\to\infty} N'(a_n, \alpha)/a_n = \alpha$. To complete the proof, we need only show that if k is defined by $a_k \leqq n < a_{k+1}$, then

$$\lim_{n\to\infty} \left| \frac{1}{n} N'(n, \alpha) - \frac{1}{a_k} N'(a_k, \alpha) \right| = 0.$$

This follows from a calculation and the fact that $\Delta a_k/a_k = o(1)$.

We combine Lemmas 6 and 1 to get a set of conditions sufficient for the u.d. (mod 1) of $\{f(n) \cos ny\}$ for almost all y.

LEMMA 7. *Let $\{a_k\}$ be as in Lemma 6. Suppose f increases so that $\lim_{x\to\infty} f(x) = \infty$, $\lim_{k\to\infty} kf(a_k)/a_k^{1-\delta} = 0$ for $\delta > 0$ sufficiently small, and $\lim_{k\to\infty} (f(a_{k+1}) - f(a_k)) = 0$. Then $\{f(n) \cos ny\}$ is u.d.* (mod 1) *for almost all y.*

Proof. If g is defined as in Lemma 6, then $\{g(n)\}$ is u.d. (mod 1) for almost all y. We will apply Lemma 1. Let $h(n) = f(n) \cos ny$; since the cosine is bounded,

$$|h(n) - g(n)| \leqq c|f(n) - f(a_k)| \leqq c(f(a_{k+1}) - f(a_k)).$$

Thus $\lim_{n\to\infty} |h(n) - g(n)| = 0$. This proves the lemma.

We now show that to each sufficiently smooth, slowly increasing function f, there corresponds a sequence $\{a_k\}$ which satisfies conditions like those of Lemma 7.

LEMMA 8. *Suppose that f increases, f' decreases,*

$$\lim_{x\to\infty} f(x) = \infty, \quad \textit{and} \quad \lim_{x\to\infty} f(x)/x^{1/2-\varepsilon} = 0$$

for some $\varepsilon > 0$. Then there exists an increasing function $a(x)$ such that

(i) $$\lim_{x\to\infty} a'(x) = \infty,$$

(ii) $$\lim_{x\to\infty} a''(x) = 0,$$

(iii) $$\lim_{x\to\infty} a'(x)/x = 0,$$

(iv) $$\lim_{x\to\infty} (d/dx) f(a(x)) = 0,$$

and

(v) $$\lim_{x\to\infty} xf(a(x))/a^{1-\delta}(x) = 0,$$

for $\delta > 0$ sufficiently small.

Proof. An appropriate function is $a(x) = x^{2-\gamma}$ with $\gamma = \delta/(1-\delta)$. That (i)–(iii) hold is immediate. Since f' decreases and we may assume that $f(0) \geqq 0$, it follows that $f(x)/x \geqq f'(x)$. Hence

$$\frac{d}{dx} f(a(x)) \leqq \frac{a'(x) f(a(x))}{a(x)} = \frac{cf(x^{2-\gamma})}{x} = \frac{cf(y)}{y^{1/(2-\gamma)}}$$

upon putting $y=x^{2-\gamma}$. Since $f(x)=o(x^{1/2})$, the latter quantity tends to zero as $x\to\infty$. Thus (iv) holds. Further,

$$\frac{xf(a(x))}{a^{1-\delta}(x)}=\frac{f(x^{2-\gamma})}{x^{(2-\gamma)(1-\delta)-1}}=\frac{f(y)}{y^{(1-3\delta)(1-\delta)/(2-3\delta)}}$$

upon putting $y=x^{(2-3\delta)/(1-\delta)}$. For $\delta>0$ sufficiently small, the exponent

$$(1-3\delta)(1-\delta)/(2-3\delta)$$

is greater than $1/2-\varepsilon$ for any $\varepsilon>0$, whence (v) holds.

Lemmas 7 and 8 give the result.

THEOREM. *Suppose that f increases without bound, f' decreases, and $f(x)=o(x^{1/2-\varepsilon})$ for some $\varepsilon>0$, as $x\to\infty$. Then $\{f(n)\cos ny\}$ is uniformly distributed* (mod 1) *for almost all y.*

Proof. Take the function $a(x)=x^{2-\gamma}$ of Lemma 8 and choose $a_k=a(k)$, $k=1, 2, \ldots$. To prove the theorem, it suffices to show that the hypotheses of Lemma 7 are satisfied, namely (a) $a_k<a_{k+1}$, (b) $\lim_{k\to\infty}\Delta a_k=\infty$, (c) $\lim_{k\to\infty}\Delta a_k/a_k=0$, (d) $\lim_{k\to\infty}(f(a_{k+1})-f(a_k))=0$, and (e) $\lim_{k\to\infty}kf(a_k)/a_k^{1-\delta}=0$, for $\delta>0$ sufficiently small. (a) and (b) follow from (i) of Lemma 8. Since

$$\Delta a_k \leqq \max_{k\leqq x\leqq k+1} a'(x) \leqq a'(k)+\max_{k\leqq x\leqq k+1}|a''(x)|,$$

(c) follows from (ii) and (iii) of Lemma 8. Also,

$$f(a_{k+1})-f(a_k) \leqq a'(k)f'(a(k))+f'(a(k))\max_{k\leqq x\leqq k+1}|a''(x)|,$$

so (d) follows from (iv). Finally, (e) follows from (v), and the theorem is proved.

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