SOME PROPERTIES OF $p(n)$ AND $c(n)$ MODULO POWERS OF 13

BY

A. O. L. ATKIN AND J. N. O'BRIEN

1. **Introduction.** Let

$$x = e^{2\pi i\tau}, \quad \operatorname{Im} \tau > 0, \quad |x| < 1,$$

$$f(x) = \prod_{r=1}^{\infty} (1-x^r),$$

$$\eta(\tau) = e^{\pi i\tau/12} f(x),$$

$$1/f(x) = \sum_{n=0}^{\infty} p(n)x^n,$$

$$j(\tau) = \sum_{n=-1}^{\infty} c(n)x^n = \left(1+240 \sum_{n=1}^{\infty} \sigma_3(n)x^n\right)^3 \Big/ xf^{24}(x) - 744,$$

where

$$\sigma_3(n) = \sum_{d|n} d^3.$$

Then $p(n)$ is just the number of unrestricted partitions of n, and $c(n)$ is the Fourier coefficient of Klein's modular invariant. For primes $q \leqq 11$, there exist congruence properties of the type first found by Ramanujan. In fact

(1) *if* $n \equiv 0 \pmod{2^a 3^b 5^c 7^d 11^e}$, *then* $c(n) \equiv 0 \pmod{2^{3a+8} 3^{2b+3} 5^{c+1} 7^d 11^e}$,

and

(2) *if* $24n \equiv 1 \pmod{5^c 7^d 11^e}$, *then* $p(n) \equiv 0 \pmod{5^c 7^{[(d+2)/2]} 11^e}$.

These are proved in Watson [2], Lehner [3], [4], Atkin [5]. For primes $q > 11$ this type[1] of congruence does not persist.

Newman [6], in the context of $p(n)$, has proposed the question:

(3) *Given* a, m, *is* $p(n) \equiv a \pmod{m}$ *soluble for an infinity of* n?

We may ask further:

(4) *Given* a, m, *is* $p(n) \equiv a \pmod{m}$ *soluble for values of* n *with positive density?*

The best hope of establishing (3) seems to be that one may exhibit *explicit* congruences of the form

(5) $$p(h(n)) \equiv a \pmod{m},$$

Received by the editors May 3, 1966.

[1] I.e., $p(kn+l) \equiv 0 \pmod{m}$ where no primes divide k which do not also divide m.

and in particular if $h(n)=bn+c$ is a linear function of n, we have (4) also. The same questions, of course, arise for $c(n)$, and indeed for the Fourier coefficients of other suitable modular forms and functions. In this paper we make some contribution to the solution of these problems when m is a power of 13. We confine ourselves to $p(n)$ and $c(n)$, both for simplicity and because these coefficients have been most extensively studied, but the reader will see that the methods are applicable to the Fourier coefficients of any negative power of $\eta(\tau)$ or any entire function on $\Gamma_0(13)$.

1.1. *Congruence properties of* $c(n)$. Newman [7] shows that

$$c(13^2n) \equiv 8c(13n) \pmod{13}, \tag{6}$$

and, with

$$\begin{aligned} t(n) &= -c(13n) \equiv c(13n)/c(13) \pmod{13}, \\ t(np)-t(n)t(p)+p^{-1}t(n/p) &\equiv 0 \pmod{13}, \end{aligned} \tag{7}$$

where $p\neq 13$ is prime[2].

Now, by using the "accident" that $c(91)\equiv 0 \pmod{13}$ he infers that

$$c(7.13^\alpha) \equiv 0 \pmod{13} \qquad \text{if } \alpha \geqq 1, \tag{8}$$

$$c(91n) \equiv 0 \pmod{13} \qquad \text{if } (n, 7) = 1. \tag{9}$$

Both these results prove that $c(n)\equiv 0 \pmod{13}$ infinitely often; the second also shows that $c(n)\equiv 0 \pmod{13}$ in positive density. One can now with a little calculation show also that $c(n)$ fills *all* residue classes (mod 13) infinitely often, but 'positive density' seems more intractable. We may regard (7) as expressing a 'multiplicative' property of $c(13n)$ (mod 13), which takes the form (6) in the ramified case $p=13$. Indeed Newman's proof is based on the fact that $c(13n)\equiv -\tau(n) \pmod{13}$, where $\tau(n)$ is defined by

$$\sum_{n=1}^{\infty} \tau(n)x^n = xf^{24}(x),$$

and satisfies

$$\tau(np)-\tau(n)\tau(p)+p^{11}\tau(n/p) = 0,$$

as found by Ramanujan [10] and proved by Mordell [11]. In this paper we prove a generalization of (6):

THEOREM 1. *For all* $\alpha\geqq 1$ *there exists a constant* k_α *not divisible by* 13 *such that for all n*

$$c(13^{\alpha+1}n) \equiv k_\alpha c(13^\alpha n) \pmod{13^\alpha}.$$

---

(2) Here and later we use the reciprocal of a (mod 13^α) freely when $(a, 13)=1$. We also write, for any number-theoretic coefficient $t(n)$, $t(n)=0$ if n is nonintegral.

We have also overwhelming evidence for the following generalization of (7):

CONJECTURE 1. *Let* $\alpha \geqq 1$ *be integral, and write*

$$t(n) \equiv c(13^\alpha n)/c(13^\alpha) \pmod{13^\alpha}.$$

Then if $p \neq 13$ *is prime we have*

$$t(np) - t(n)t(p) + p^{-1}t(n/p) \equiv 0 \pmod{13^\alpha}.$$

We observe that the final *congruence* of Conjecture 1 is exactly analogous to the *identities* in Hecke's theory of modular forms of negative dimension. Thus it seems that, in a sense, Newman's congruences (6) and (7) exist in their own right, although it happens that $c(13n) \equiv -\tau(n) \pmod{13}$, and this enables (7) to be proved.

We use Theorem 1 and a good deal of actual computation to establish:

THEOREM 3. $c(n) \equiv 0 \pmod{13^3}$ *for infinitely many n.*

THEOREM 4. *For all* $\alpha \geqq 1$, *and all a with* $(a, 13) = 1$, *there exist infinitely many n such that* $c(n) \equiv a \pmod{13^\alpha}$.

1.2. *Congruence properties of* $p(n)$. These are more involved than those of $c(n)$, as we might expect from the fact that $\eta(\tau)$ is a modular form of half-integral dimension and complicated multiplier system. In order to obtain some simplification, we define

$$(10) \quad \begin{aligned} P(N) &= p(n) \qquad \text{if } N = 24n-1, \\ P(N) &= 0 \qquad \text{if } N < -1 \text{ or } N \not\equiv -1 \pmod{24} \text{ or } N \text{ is nonintegral.} \end{aligned}$$

After subsection 1.1, we shall describe the analogous position here more briefly. We have

THEOREM 2. *For all* $\alpha \geqq 1$ *there exists an integral constant* K_α *not divisible by* 13 *such that for all N*

$$P(13^{\alpha+2}N) \equiv K_\alpha P(13^\alpha N) \pmod{13^\alpha}(^3).$$

We also conjecture

CONJECTURE 2. *Let* $\alpha \geqq 1$, *and* $p \neq 13$ *be a prime* $\geqq 5$. *Then there exists a constant* $k = k(p, \alpha)$ *such that for all N*

$$P(p^2 \cdot 13^\alpha N) - \{k - (-3 \cdot 13^\alpha N/p)p^{-2}\}P(13^\alpha N) + p^{-3}P(13^\alpha N/p^2) \equiv 0 \pmod{13^\alpha}(^4),$$

where (a/b) *is the quadratic reciprocity symbol.*

THEOREM 5. *Conjecture* 2 *is valid if* $\alpha = 1$ *or* $\alpha = 2$.

---

(3) In these forms, of course, the congruences are nugatory unless N is such that $13^\alpha N \equiv -1 \pmod{24}$.

(4) Since the factor multiplying $P(13^\alpha N)$ cannot be written as $P(13^\alpha N_1)$ the use of $t(n)$ as in Conjecture 1 offers no advantage.

Again, we have a kind of multiplicative property modulo 13^α, and Theorem 2 is essentially the case $p=13$ of Conjecture 2.

Newman [8], [9] obtains the case $\alpha=1$ of Theorem 2, and (although he does not state it in our form) of Conjecture 2, using Zuckermann's result

$$p(13n+6) \equiv 11p_{11}(n) \pmod{13},$$

where

(11) $$\sum_{n=0}^{\infty} p_r(n)x^n = f^r(x),$$

and his own results on $p_r(n)$ for r odd, $1 \leqq r \leqq 23$, with $r=11$. He now uses the "accident" $P(2015)=p(84)\equiv 0 \pmod{13}$ to establish

(12) $$P(2015\cdot 13^{2n}) \equiv 0 \pmod{13} \qquad (n \geqq 0),$$

(13) $$P(2015n^2) = p(84n^2-(n^2-1)/24) \equiv 0 \pmod{13} \qquad \text{if } (n, 6) = 1.$$

Note that the result analogous to (9) is (13), which does not give $p(n)\equiv 0 \pmod{13}$ in positive density. If we search for an "accident" involving $k(p, \alpha)$ rather than $P(13N_0)$ we do in fact obtain a linear congruence from Newman's results:

THEOREM 6. $P(59^3\cdot 13N)\equiv 0 \pmod{13}$ *if* $(N, 59)=1$.

Thus, for example, in terms of $p(n)$, with $N=24.59n+1$,

(14) $$p(59^4\cdot 13n+111247) \equiv 0 \pmod{13}.$$

Using Theorem 2, we can prove

THEOREM 7. $p(n)\equiv 0 \pmod{13^4}$ *for infinitely many n.*

THEOREM 8. *For all* $\alpha \geqq 1$, *and all a with* $(a, 13)=1$, *there exist infinitely many n such that* $p(n)\equiv a \pmod{13^\alpha}$.

Finally, using Conjecture 2 in the proved case $\alpha=2$, we obtain

THEOREM 9. $p(3373n^2-(n^2-1)/24)=P(13^2\cdot 479n^2)\equiv 0 \pmod{13^2}$ if $(n, 6)=1$.

THEOREM 10. $P(97^2\cdot 103^2\cdot 13^2N)\equiv 0 \pmod{13^2}$ *if* $(N/97)=(N/103)=-1$.

Thus, for example, in terms of $p(n)$

(15) $$p(168544110546799n-6950975499605) \equiv 0 \pmod{13^2} \qquad \text{if } n \geqq 1.$$

The methods we employ in this paper are essentially elementary, based (as in Watson [2]) on the use of the modular equation between $\eta(169\tau)/\eta(\tau)$ and $\eta^2(169\tau)/\eta^2(13\tau)$, although modular theory was used in [1] to obtain this equation. Although the forms of some of our results can be obtained easily by deeper methods, the precise number-theoretic details cannot, and we have therefore avoided modular notation as far as possible. The only case where we require modular theory is Lemma 5. We may add that Lehner [4] gives the modular equation between $\eta^2(13\tau)/\eta^2(\tau)$ and $\eta^2(169\tau)/\eta^2(13\tau)$. This would have sufficed for our results on $c(n)$, but not for those on $p(n)$. We have been occupied with the work described in this paper at various times since 1962, and would like to thank Dr. Morris Newman of the National Bureau of Standards, Washington, D. C., for his constant advice and encouragement.

2.1. *Sums of powers of roots of the modular equation.* We now write

$$\phi(x) = x^7 f(x^{169})/f(x), \quad g(x) = xf^2(x^{13})/f^2(x). \tag{16}$$

Then, as shown by O'Brien in [1] the equation connecting $\phi(x^{1/13})=t$ and $g(x)=g$ may be written

$$t^{13}+\alpha_1 t^{12}+\cdots+\alpha_{12}t+\alpha_{13} = 0, \tag{17}$$

where

$$\alpha_r = \sum_{\sigma=[(r+2)/2]}^{7} \beta_{r\sigma} g^{\sigma},$$

and the $\beta_{r\sigma}$ are integers. We now define $\pi(a)$, for integral a, by

$$13^{\pi(a)} \mid a, \qquad 13^{\pi(a)+1} \nmid a. \tag{18}$$

If $a=b/c$ is rational, we define $\pi(a)=\pi(b)-\pi(c)$, and finally we write $\pi(0)=\infty$ and regard any inequality $\pi(0)\geqq k$ as valid. We clearly have

$$\pi(ab) = \pi(a)+\pi(b), \tag{19}$$

$$\pi(a+b) \geqq \min(\pi(a), \pi(b)), \tag{20}$$

with equality unless $\pi(a)=\pi(b)$. With this notation, we have

LEMMA 1. $\pi(\beta_{r\sigma})\geqq[(13\sigma-7r+13)/14]$.

The proof consists of direct verification using the calculated values of $\beta_{r\sigma}$ in [1], which we give in Appendix C.

It is possible to deduce some cases of Lemma 1 a priori, as shown in [1], but not all. We now denote by S_r the sum of the rth powers of the roots of (17), regarded as an equation in t. Evidently we have

$$S_r = \sum_{\rho=1}^{\infty} a_{r\rho} g^{\rho}, \tag{21}$$

where in fact([5])

$$a_{r\rho} = 0 \text{ unless } [(7r+12)/13] \leqq \rho \leqq 7r. \tag{22}$$

We now prove

LEMMA 2. $\pi(a_{r\rho})\geqq[(13\rho-7r+13)/14]$.

([5]) Here and in our later inductions it is convenient to write finite sums as $\sum_\rho=\sum_{\rho=1}^{\infty}$, since the actual limits do not in most cases affect the argument. We do however usually state without proof the exact limits, which the reader can easily verify, in lemmas.

This is true for $r=1$ and all ρ, by Lemma 1. Suppose it has been established for all ρ and all $r<R$. Then, by Newton's formula

$$-\sum a_{R\rho}g^{\rho} = \sum_{r=1}^{R-1}\Big(\sum_{\sigma} a_{r\sigma}g^{\sigma}\Big)\Big(\sum_{\tau}\beta_{R-r,\tau}g^{\tau}\Big)+R\sum_{\rho}\beta_{R\rho}g^{\rho},$$

(where $\beta_{R\rho} = 0$ for $R \geqq 14$, all ρ)

so that

$$-a_{R\rho} = \sum_{r=1}^{R-1}\sum_{\sigma} a_{r\sigma}\beta_{R-r,\rho-\sigma}+R\beta_{R\rho}, \tag{23}$$

whence, by (19) and (20),

$$\pi(a_{R\rho}) \geqq \min\{[13\rho-7R+13)/14], \psi(\rho, R)\},$$

where

$$\psi(\rho, R) = \min_{\sigma\geqq 1, 1\leqq r\leqq R-1}\{[(13\sigma-7r+13)/14]+[(13(\rho-\sigma+1)-7(R-r))/14]\}.$$

Now

$$[\lambda/14]+[\mu/14] \geqq [(\lambda+\mu-13)/14], \tag{24}$$

so that

$$\psi(\rho, R) \geqq \min_{\sigma\geqq 1, 1\leqq r\leqq R-1}\{[(13\rho-7R+13)/14]\} = [(13\rho-7R+13)/14].$$

Hence

$$\pi(a_{R\rho}) \geqq [(13\rho-7R+13)/14],$$

and Lemma 2 is proved by induction. We have also

$$\pi(a_{r\rho}) \geqq 1. \tag{25}$$

For $\pi(\beta_{R\rho})\geqq 1$ unless $R=13$, $\rho=7$ and in this case (23) has a term $R\beta_{R\rho}=13\beta_{13,7}$ and $\pi(13\beta_{13,7})=1$. An easy induction now gives (25).

2.2. *The operator U.* We now define, for any power series

$$F(x) = \sum_{n\geqq N}\lambda(n)x^n, \tag{26}$$

the operator U $(=U_{13})$ by

$$UF(x) = \sum_{13n\geqq N}\lambda(13n)x^n. \tag{27}$$

Clearly U is linear so that $U\{a_1F_1(x)+a_2F_2(x)\}=a_1UF_1(x)+a_2UF_2(x)$. Further we have

$$U\{F_1(x^{13})\cdot F_2(x)\} = F_1(x)\cdot UF_2(x). \tag{28}$$

Finally the elementary properties of roots of unity give

LEMMA 3. *Suppose that* $\omega\neq 1$, $\omega^{13}=1$. *Then* $13UF(x)=\sum_{m=0}^{12}F(\omega^m x^{1/13})$.

If we now regard (17) as an equation in t with g fixed, the roots are clearly

$$t = \phi(\omega^m x^{1/13}), \qquad m = 0 \text{ to } 12.$$

Thus, by Lemma 3, we have

$$U\phi^r(x) = 13^{-1}S_r. \tag{29}$$

Moreover $Ug^k(x) = U\{\phi^{2k}(x)\cdot x^{-13k}\cdot f^{2k}(x^{13})\cdot f^{-2k}(x^{169})\} = 13^{-1}g^{-k}(x)\cdot S_{2k}$, by (28) and (29), while similarly $U\{\phi(x)\cdot g^k(x)\} = 13^{-1}g^{-k}(x)\cdot S_{2k+1}$. Using the results of §2.1 we derive

LEMMA 4. *We have, for* $k \geqq 1$,

$$Ug^k(x) = \sum_{r=1}^{\infty} c_{kr}g^r(x), \tag{30}$$

$$U\{\phi(x)g^k(x)\} = \sum_{r=1}^{\infty} d_{kr}g^r(x), \tag{31}$$

where the c_{kr} *and* d_{kr} *are integers satisfying*

$$\pi(c_{kr}) \geqq [(13r-k-1)/14], \tag{32}$$

$$\pi(d_{kr}) \geqq [(13r-k-8)/14], \tag{33}$$

$$\pi(c_{11}) = 0, \ \pi(d_{11}) = 0. \tag{34}$$

Also in fact

$$c_{kr} = 0 \ \textit{unless} \ [(k+12)/13] \leqq r \leqq 13k,$$

$$d_{kr} = 0 \ \textit{unless} \ [(k+19)/13] \leqq r \leqq 13k+7.$$

3. **Proof of congruence properties of $c(n)$.** In order to obtain a start for our induction we require

LEMMA 5. $\sum_{n=1}^{\infty} c(13n)x^n = -g(x) + 13^2U^2g(x)$.

We prove this in Appendix A. It can be seen by repeated application of Lemma 4, equation (30), that we may write

$$J_\alpha(x) = \sum_{n=1}^{\infty} c(13^\alpha n)x^n = \sum_r j_{\alpha r}g^r(x), \qquad (\alpha \geqq 1), \tag{35}$$

where in fact r runs from 1 to $13^{\alpha+1}$. We have

LEMMA 6.

$$\pi(j_{\alpha r}) \geqq [(13r-2)/14] \quad (\alpha \geqq 1), \tag{36}$$

$$\pi(j_{\alpha 1}) = 0 \qquad (\alpha \geqq 1). \tag{37}$$

For

$$U\sum_{n=1}^{\infty} c(13^{\alpha}n)x^n = \sum_{n=1}^{\infty} c(13^{\alpha+1}n)x^n,$$

and hence, by Lemma 4

$$(38)\qquad j_{\alpha+1,r} = \sum_{\rho} j_{\alpha\rho}c_{\rho r}, \qquad \pi(j_{\alpha+1,r}) \geqq \min_{\rho}\{\pi(j_{\alpha\rho})+[(13r-\rho-1)/14]\}.$$

Thus if we assume (36) for some α and all r, we have

$$(39)\qquad \pi(j_{\alpha+1,r}) \geqq \min_{\rho}\{[(13\rho-2)/14]+[(13r-\rho-1)/14]\}.$$

Increasing ρ by 2 increases $[(13\rho-2)/14]$ by at least 1, and decreases $[(13r-\rho-1)/14]$ by at most 1. Hence[6] the $\min_\rho$ in (39) is attained for $\rho=1$ or $\rho=2$, and in fact for $\rho=1$, since $[11/14]=0$ and $[24/14]=1$, whence $\pi(j_{\alpha+1,r})\geqq[(13r-2)/14]$, $(r\geqq 1)$, which is (36) for $\alpha+1$ and all r. Next, assuming (37) for α, we have

$$j_{\alpha+1,1} = \sum_{\rho} j_{\alpha\rho}c_{\rho 1}.$$

Now $\pi(j_{\alpha 1}c_{11})=0$, by hypothesis and (34), and for $\rho\geqq 2$,

$$\pi(j_{\alpha\rho}c_{\rho 1}) \geqq [(13\rho-2)/14]+\pi(c_{\rho 1}) \geqq 1.$$

Hence, by (20), we have $\pi(j_{\alpha+1,1})=0$, which is (37) for $\alpha+1$. Finally, for $\alpha=1$, (37) is immediate from Lemma 5, and (36) for $r\geqq 2$ is easily established by the foregoing argument (with something to spare). Lemma 6 is thus proved by induction.

We now define

$$(40)\qquad \gamma_{rs}^{\alpha} = j_{\alpha+1,r}j_{\alpha s}-j_{\alpha r}j_{\alpha+1,s} \quad (\alpha \geqq 1; r, s \geqq 1),$$

so that

$$(41)\qquad \gamma_{rr}^{\alpha} = 0, \qquad \gamma_{rs}^{\alpha} = -\gamma_{sr}^{\alpha}.$$

LEMMA 7. $\pi(\gamma_{rs}^{\alpha})\geqq\alpha+[(13(r+s)-32)/14]$.

First for $\alpha=1$ we have by Lemma 6

$$\begin{aligned}\pi(\gamma_{rs}^{1}) &\geqq [(13r-2)/14]+[(13s-2)/14]\\ &\geqq [(13(r+s)-17)/14] \geqq 1+[(13(r+s)-32)/14].\end{aligned}$$

We now have, by (38),

$$(42)\qquad \gamma_{rs}^{\alpha+1} = \sum_{\rho}\sum_{\sigma}\gamma_{\rho\sigma}^{\alpha}c_{\rho r}c_{\sigma s} \quad (\alpha \geqq 1),$$

and by (32),

$$\pi(c_{\rho r}c_{\sigma s}) \geqq [(13r-\rho-1)/14]+[(13s-\sigma-1)/14] \geqq [(13(r+s)-(\rho+\sigma)-15)/14].$$

[6] The argument of this paragraph is constantly used hereafter, and in future we shall often omit the details.

Thus assuming Lemma 7 for α and all r, s, we have

$$\pi(\gamma_{rs}^{\alpha+1}) \geqq \min_{\rho,\sigma} \{\alpha+[(13(\rho+\sigma)-32)/14]+[(13(r+s)-(\rho+\sigma)-15)/14]\}.$$

The $\min_{\rho,\sigma}$ is attained for $\rho+\sigma=3$, whence

$$\pi(\gamma_{rs}^{\alpha+1}) \geqq \alpha+[(13(r+s)-18)/14] = \alpha+1+[(13(r+s)-32)/14],$$

which is Lemma 7 for $\alpha+1$ and all r, s. Lemma 7 is thus proved by induction. All we actually require later is

$$\pi(\gamma_{r1}^{\alpha}) \geqq \alpha, \tag{43}$$

but it is not possible to prove (43) alone by induction.

Let now k_α not divisible by 13 be chosen so that

$$j_{\alpha+1,1} \equiv k_\alpha j_{\alpha 1} \pmod{13^\alpha},$$

which is possible by (37) of Lemma 6. Since for all r we have from (43)

$$j_{\alpha+1,r} j_{\alpha 1} - j_{\alpha r} j_{\alpha+1,1} \equiv 0 \pmod{13^\alpha},$$

it follows that for all r

$$j_{\alpha+1,r} \equiv k_\alpha j_{\alpha r} \pmod{13^\alpha}.$$

Hence by (35)

$$J_{\alpha+1}(x) \equiv k_\alpha J_\alpha(x) \pmod{13^\alpha},$$

and equating coefficients of like powers of x we obtain Theorem 1. We may add that Theorem 1 is best possible in the sense that we cannot replace (mod 13^α) by (mod $13^{\alpha+1}$), which follows from

$$\pi(\gamma_{21}^{\alpha}) = \alpha. \tag{44}$$

The proof of (44) is similar to that of (37), if we write (42) in the form

$$\gamma_{rs}^{\alpha+1} = \sum_{\rho>\sigma\geqq 1}\sum \gamma_{\rho\sigma}^{\alpha}(c_{\rho r}c_{\sigma s}-c_{\rho s}c_{\sigma r}),$$

and compute

$$\pi(c_{11}c_{22}-c_{12}c_{21}) = 1, \qquad \pi(c_{12}) = 1.$$

We observe next that $c(5299\cdot 13^3)\equiv 0 \pmod{13^3}$, and hence by Theorem 1

$$c(5299\cdot 13^{m+3}) \equiv 0 \pmod{13^3} \qquad (m \geqq 0), \tag{45}$$

which implies Theorem 3. To prove Theorem 4 we note that $k_\alpha \equiv k_2 \pmod{13^2}$ for $\alpha \geqq 2$, while $k_2=138\equiv 41^3 \pmod{13^2}$ and 41 is a primitive root of 13^2. Thus k_α generates a group of index 3 in the multiplicative group of residues prime to 13^α, and hence by Theorem 1, if $c(13^\alpha n_0)$ equals any residue in one of the cosets, then the numbers $c(13^{\alpha+m}n_0)$ for $m \geqq 0$ fill all residues in that coset infinitely often. But $c(13^{\alpha}n_0)$ belongs to the same coset at $c(13n_0)$, and

$$c(13n_0) \equiv -1, -2, 6 \pmod{13} \qquad \text{for } n_0 = 1, 2, 5.$$

This implies Theorem 4.

4. **Proof of Theorems 2, 7, and 8.** We now write, for $\alpha \geqq 1$,

$$\begin{aligned} L_{2\alpha-1}(x) &= f(x^{13}) \cdot \sum_{n=1}^{\infty} P(13^{2\alpha-1}(24n-13)) \cdot x^n, \\ L_{2\alpha}(x) &= f(x) \cdot \sum_{n=1}^{\infty} P(13^{2\alpha}(24n-1)) \cdot x^n. \end{aligned} \tag{46}$$

LEMMA 8.

$$L_1(x) = U\phi(x), \tag{47}$$

$$L_{2\alpha}(x) = UL_{2\alpha-1}(x) \qquad (\alpha \geqq 1), \tag{48}$$

$$L_{2\alpha+1}(x) = U(\phi(x) \cdot L_{2\alpha}(x)) \quad (\alpha \geqq 1). \tag{49}$$

We have for (47),

$$U\phi(x) = f(x^{13}) \cdot U \sum_{n=1}^{\infty} p(n) x^{n+7} = f(x^{13}) \cdot \sum_{n=1}^{\infty} p(13n-7) \cdot x^n = L_1(x),$$

and (48) is immediate. Also

$$\begin{aligned} U(\phi(x) \cdot L_{2\alpha}(x)) &= f(x^{13}) \cdot U \sum_{n=1}^{\infty} P(13^{2\alpha}(24n-1)) x^{n+7} \\ &= f(x^{13}) \cdot \sum_{n=1}^{\infty} P(13^{2\alpha}(24(13n-7)-1)) x^n = L_{2\alpha+1}(x). \end{aligned}$$

It now follows by repeated application of Lemma 4 that we may write, for $\alpha \geqq 1$,

$$L_{2\alpha-1}(x) = \sum_r k_{\alpha r} g^r(x), \tag{50}$$

$$L_{2\alpha}(x) = \sum_r l_{\alpha r} g^r(x), \tag{51}$$

where

$$l_{\alpha r} = \sum_\rho k_{\alpha\rho} c_{\rho r}, \tag{52}$$

$$k_{\alpha+1,r} = \sum_\rho l_{\alpha\rho} d_{\rho r}. \tag{53}$$

LEMMA 9.

$$\pi(k_{\alpha r}) \geqq [(13r-9)/14] \quad (\alpha \geqq 1), \tag{54}$$

$$\pi(l_{\alpha r}) \geqq [(13r-2)/14] \quad (\alpha \geqq 1), \tag{55}$$

$$\pi(k_{\alpha 1}) = \pi(l_{\alpha 1}) = 0 \qquad (\alpha \geqq 1). \tag{56}$$

First we have by (29)

$$\sum k_{1r}g^r(x) = U\phi(x) = 13^{-1}S_1 = 13^{-1}\sum a_{1r}g^r(x),$$

and

$$\pi(13^{-1}a_{1r}) \geqq -1+[(13r+6)/14] \geqq [(13r-9)/14]$$

by Lemma 2. Next, assuming (54) for α and all r we have by (52) and (32)

$$\pi(l_{\alpha r}) \geqq \min_{\rho}\{[(13\rho-9)/14]+[(13r-\rho-1)/14]\},$$

and the $\min_\rho$ is attained for $\rho=1$, giving (55) for α and all r. Finally assuming (55) for α and all r, we have by (53) and (33)

$$\pi(k_{\alpha+1,r}) \geqq \min_{\rho}\{[(13\rho-2)/14]+[(13r-\rho-8)/14]\},$$

and again the $\min_\rho$ is attained for $\rho=1$, giving (54) for $\alpha+1$ and all r. Thus (54) and (55) are proved for all $\alpha \geqq 1$ by induction. The proof of (56) is analogous to that of (37), if we observe that

$$\pi(c_{11}) = \pi(d_{11}) = \pi(k_{11}) = 0.$$

We now write

$$\begin{aligned} \delta^{\alpha}_{rs} &= k_{\alpha+1,r}k_{\alpha s}-k_{\alpha r}k_{\alpha+1,s}, \\ \varepsilon^{\alpha}_{rs} &= l_{\alpha+1,r}l_{\alpha s}-l_{\alpha r}l_{\alpha+1,s}, \quad (\alpha \geqq 1;\, r, s \geqq 1), \end{aligned} \tag{57}$$

so that

$$\delta^{\alpha}_{rr} = \varepsilon^{\alpha}_{rr} = 0, \quad \delta^{\alpha}_{rs} = -\delta^{\alpha}_{sr}, \quad \varepsilon^{\alpha}_{rs} = -\varepsilon^{\alpha}_{sr}, \tag{58}$$

and

$$\varepsilon^{\alpha}_{rs} = \sum_{\rho}\sum_{\sigma}\delta^{\alpha}_{\rho\sigma}c_{\rho r}c_{\sigma s}, \tag{59}$$

$$\delta^{\alpha+1}_{rs} = \sum_{\rho}\sum_{\sigma}\varepsilon^{\alpha}_{\rho\sigma}d_{\rho r}d_{\sigma s}. \tag{60}$$

LEMMA 10.

$$\pi(\delta^{\alpha}_{rs}) \geqq 2\alpha-1+[(13(r+s)-46)/14]+\theta(r, s), \tag{61}$$

$$\pi(\varepsilon^{\alpha}_{rs}) \geqq 2\alpha+[(13(r+s)-33)/14], \tag{62}$$

where $\theta(r, s)=1$ *if* $r+s=3$, *and* 0 *otherwise*.

First we have by (54)

$$\pi(\delta^{1}_{rs}) \geqq [(13r-9)/14]+[(13s-9)/14] \geqq 1+[(13(r+s)-44)/14],$$

while if $r+s=3$ we have $\pi(\delta^1_{rs}) \geqq 0+1=1$, which proves (61) for $\alpha=1$. Next, assuming (61) for α and all r, s, we have by (59) and (32)

$$\pi(\varepsilon^{\alpha}_{rs}) \geqq 2\alpha-1+\min_{\rho,\sigma}\{[(13(\rho+\sigma)-46)/14]+\theta(\rho, \sigma)+[(13(r+s)-(\rho+\sigma)-15)/14]\},$$

and the $\min_{\rho,\sigma}$ is attained for $\rho+\sigma=3$, 4, or 5, and in fact for $\rho+\sigma=4$, giving (62) for α and all r, s. Finally, assuming (62) for α and all r, s, we have by (60) and (33)

$$\pi(\varepsilon_{rs}^{\alpha+1}) \geqq 2\alpha+\min_{\rho,\sigma}\{[(13(\rho+\sigma)-33)/14]+\psi(\rho,\sigma)\}, \tag{63}$$

where

$$\begin{aligned}\pi(d_{\rho r}d_{\sigma s}) \geqq \psi(\rho,\sigma) &= 1 \text{ if } r+s = \rho+\sigma = 3,\\ &= [(13(r+s)-(\rho+\sigma)-29)/14] \text{ otherwise.}\end{aligned}$$

The $\min_{\rho,\sigma}$ in (63) is attained for $\rho+\sigma=3$ or 4, and in fact for $\rho+\sigma=3$, when $r+s>3$, which gives (61) for $\alpha+1$ and all r, s; while if $r+s=3$ the $\min_{\rho,\sigma}$ in (63) is attained for $\rho+\sigma=3$, 4, or 5, and in fact for $\rho+\sigma=3$ and 4, giving (61) for $\alpha+1$ and $r+s=3$. This completes the proof of Lemma 10. As in §3, we only require

$$\pi(\delta_{rs}^{\alpha}) \geqq 2\alpha-1, \tag{64}$$

$$\pi(\varepsilon_{rs}^{\alpha}) \geqq 2\alpha. \tag{65}$$

We now choose $K_{2\alpha-1}$ not divisible by 13 so that

$$k_{\alpha+1,1} \equiv K_{2\alpha-1}k_{\alpha 1} \pmod{13^{2\alpha-1}},$$

which is possible by (56). Then (64) shows that

$$k_{\alpha+1,r} \equiv K_{2\alpha-1}k_{\alpha r} \pmod{13^{2\alpha-1}} \quad (r \geqq 1),$$

so that by (50)

$$L_{2\alpha+1}(x) \equiv K_{2\alpha-1}L_{2\alpha-1}(x) \pmod{13^{2\alpha-1}},$$

and dividing by $f(x^{13})$ (which is legitimate since the power series of $f(x^{13})$ has leading coefficient unity) and equating coefficients of like powers of x, we obtain

$$P(13^{2\alpha+1}N) \equiv K_{2\alpha-1}P(13^{2\alpha-1}N) \pmod{13^{2\alpha-1}},$$

for all $n\geqq 1$ and $N=24n-13$. This is Theorem 2 for odd α; the proof for even α is analogous. We can also prove, by a detailed examination of δ_{21}^{α} and $\varepsilon_{21}^{\alpha}$, that Theorem 2 is best possible.

We now observe that $P(13^4\cdot 22655)\equiv 0 \pmod{13^4}$ and hence

$$P(13^{4+2m}\cdot 22655) \equiv 0 \pmod{13^4} \qquad (m \geqq 0),$$

which proves Theorem 7. Next, since $K_\alpha\equiv K_2 \pmod{13^2}$ for $\alpha\geqq 2$, and $K_2=45$ is a primitive root of 13^2, it follows that K_α is a primitive root of 13^α. Also, since $k_{\alpha 1}$ and $l_{\alpha 1}$ are not divisible by 13, we have $P(13^{2\alpha-1}\cdot 11)$ and $P(13^{2\alpha}\cdot 23)$ not divisible by 13. Thus $P(13^{2\alpha-1+2m}\cdot 11)$ for $m\geqq 0$ fills all residue classes prime to 13 $(\bmod\ 13^{2\alpha-1})$ infinitely often, and similarly $P(13^{2\alpha+2m}\cdot 23)$ fills all residue classes prime to 13 $(\bmod\ 13^{2\alpha})$ infinitely often. This proves Theorem 8.

The use of a "recurrence formula" like Theorem 2 to prove an "infinitely often" property is due to Newman [8]. If we define a "Newman sequence" $t(m)$ by

$$t(m) = P(13^{2m-1}N_0) \quad (m \geqq 1),$$

then Theorem 2 shows that

$$\pi(t(m+1)) = \pi(t(m)) \quad \text{if} \quad \pi(t(m)) < 2m-1,$$
$$\pi(t(m+1)) \geqq 2m-1 \quad \text{if} \quad \pi(t(m)) \geqq 2m-1.$$

It follows that either

$$\pi(t(m)) \geqq 2m-1 \qquad \text{for all } m$$

or

$$\pi(t(m)) = \pi(t(m_0)) < 2m_0-1 \qquad \text{for all } m \geqq m_0, \text{ and some } m_0.$$

The latter seems more likely.

5. **Proof of Conjecture 2 for** $\alpha=1$ **and** $\alpha=2$. We define

$$\begin{aligned} P_{11}(N) &= p_{11}(n) \text{ if } N = 24n+11,\ n \geqq 0, \\ &= 0 \text{ otherwise,} \end{aligned} \tag{66}$$

and

$$\begin{aligned} P_{23}(N) &= p_{23}(n) \text{ if } N = 24n+23,\ n \geqq 0, \\ &= 0 \text{ otherwise.} \end{aligned} \tag{67}$$

We also let $p \geqq 5$ be prime throughout this section. Then Theorem 1 of Newman [9] shows that for all N

$$P_{11}(Np^2) - \{k_{11}(p) - (-3N/p)p^4\}P_{11}(N) + p^9 P_{11}(N/p^2) = 0, \tag{68}$$

$$P_{23}(Np^2) - \{k_{23}(p) - (-3N/p)p^{10}\}P_{23}(N) + p^{21} P_{23}(N/p^2) = 0, \tag{69}$$

where $k_{11}(p)$ and $k_{23}(p)$ are constants.

Next, by computation from the definitions (46) we find

$$L_1(x) \equiv 11g(x) \pmod{13},$$
$$L_2(x) \equiv 36g(x) + 78g^2(x) \pmod{13^2},$$

so that

$$\sum_{n=1}^{\infty} P(13(24n-13))x^n \equiv 11xf(x^{13})/f^2(x) \equiv 11 \sum_{n=1}^{\infty} p_{11}(n-1)x^n \pmod{13},$$

whence

$$P(13N) \equiv 11P_{11}(N) \pmod{13}, \tag{70}$$

and

$$\sum_{n=1}^{\infty} P(13^2(24n-1))x^n \equiv 36xf^2(x^{13})/f^3(x) + 78x^2f^4(x^{13})/f^5(x) \pmod{13^2}. \tag{71}$$

Now if we write the modular equation (17) as an equation in $1/t$ and compute the sum of the 11th powers of the roots we obtain

$$U\phi^{-11}(x) \equiv 112g^{-5}(x)+65g^{-4}(x) \pmod{13^2},$$

whence

$$(72)\quad \begin{aligned}\sum_{n=1}^{\infty} P_{11}(13(24n-1))x^n &\equiv 112xf(x^{13})f^{10}(x)+65x^2f^3(x^{13})f^8(x) \pmod{13^2}\\ &\equiv 112xf(x^{13})f^{10}(x)+65x^2f^4(x^{13})/f^5(x) \pmod{13^2}.\end{aligned}$$

But $(f(x^{13})-f^{13}(x))^2 \equiv 0 \pmod{13^2}$ so that

$$(73)\qquad f^{23}(x)-2f(x^{13})f^{10}(x)+f^2(x^{13})/f^3(x) \equiv 0 \pmod{13^2}.$$

Combining (71), (72), and (73) we obtain

$$\begin{aligned}\sum_{n=1}^{\infty} P(13^2(24n-1))x^n+4\sum_{n=1}^{\infty} P_{11}(13(24n-1))x^n &\\ &\equiv 224xf^{23}(x)+260x^2f^2(x^{13})/f^3(x)\\ &\equiv 146xf^{23}(x) = 146\sum_{n=1}^{\infty} P_{23}(24n-1)x^n \pmod{13^2},\end{aligned}$$

whence

$$(74)\qquad P(13^2N) \equiv 165P_{11}(13N)+146P_{23}(N) \pmod{13^2}.$$

The coefficients on the RHS of (74) are unique (mod 13) but not (mod 13^2) since from (72) we have

$$(75)\qquad P_{11}(13N) \equiv 8P_{23}(N) \pmod{13}.$$

Now Conjecture 2 for $\alpha=1$ follows from (68) and (70), observing that $p^{12}\equiv 1 \pmod{13}$ and $p^6\equiv(p/13)=(13/p) \pmod{13}$ if $p\neq 13$. Next, if we multiply (68) with $13N$ for N and (69) by 165 and 146 respectively, and add, Conjecture 2 for $\alpha=2$ can be seen to follow from

$$(76)\quad 165(13/p)p^6P_{11}(13N)+146p^{12}P_{23}(N) \equiv 165P_{11}(13N)+146P_{23}(N) \pmod{13^2},$$

and

$$(77)\quad 165p^{12}P_{11}(13N/p^2)+146p^{24}P_{23}(N/p^2) \equiv 165P_{11}(13N/p^2)+146P_{23}(N/p^2) \pmod{13^2}.$$

Now (77) is equivalent to

$$(p^{12}-1)\{165P_{11}(13N/p^2)+146(p^{12}+1)P_{23}(N/p^2)\} \equiv 0 \pmod{13^2},$$

and since $p^{12}-1\equiv 0 \pmod{13}$ we need only consider the expression in curly brackets (mod 13), when it becomes $9P_{11}(13N/p^2)+6P_{23}(N/p^2)\equiv 0 \pmod{13}$ by (75). The proof of (76) is similar, and thus Conjecture 2 is proved for $\alpha=2$.

We can now establish by the method used to prove Theorem 4 in Newman [9]:

LEMMA 11. *If* $P(13^2N_0)\equiv 0 \pmod{13^2}$ *and* N_0 *is squarefree, then*

$$P(13^2n^2N_0) \equiv 0 \pmod{13^2} \text{ if } (n, 6) = 1.$$

The choice $N_0=479$ proves Theorem 9. We were unable to find a *single* "accident" (mod 13^2) for $p\leqq 199$ with p^2k in Conjecture 2 congruent to 0, $\pm 1 \pmod{13^2}$. Accordingly we chose the first two primes (97 and 103) for which $p^2k\equiv \pm 1 \pmod{13^2}$; this makes our explicit linear congruence (15) have $97^3\cdot 103^3\cdot 13^2$ as the multiplier of n. Actually $k\equiv 0 \pmod{13}$ for $p=59$ (as in Theorem 6) and $p=73$ which would give a linear congruence with the multiplier of 13^2n equal to $59^4\cdot 73^4>97^3\cdot 103^3$. In the course of this work we noticed that for small values of N one has the result:

If any (*and hence all*) *of* $P(13^2N)$, $P_{11}(13N)$, *and* $P_{23}(N)$ *is divisible by* 13, *then*

$$P(13^2N) \equiv P_{11}(13N) \equiv -2P_{23}(N) \pmod{13^2}. \tag{78}$$

This would have been a safe conjecture before the advent of computers, for it is a remarkable fact that there are over 200 successful cases of (78) before it fails for $N=24\cdot 1530+23$, a point well beyond the range of hand calculation. We offer an explanation in Appendix B.

6. **Notes on computation.** We obtained the cases $\alpha\leqq 4$ of Theorem 2 in 1963 and communicated our results to Dr. Morris Newman, who asked Dr. Kenneth Kloss of the National Bureau of Standards, Washington, if he could find a suitable zero (mod 13^2) of $P(13^2N)$. Kloss obtained several using the direct definition of $p(n)$ on an IBM 7090, at the same time as the second author found the first two, using (71), the letters crossing in the post. Later the second author obtained by hand the first zero of $P(13^3N) \pmod{13^3}$.

The first author, using the ICT Atlas 1 computer of the Science Research Council at Chilton, computed $P(13^6N)$, $P(13^7N)$, and $c(13^6n) \pmod{13^6}$ up to $N=24\cdot 40960-1$, $N=24\cdot 40960-13$, and $n=40960$ respectively, using (51), (50), and (35) with explicit coefficients. It was rather disappointing that no better than Theorems 3 and 7 was found (13^3 for $c(n)$ and 13^4 for $p(n)$) in view of the surprisingly early appearance of zeros for $P(13^2N)$ and $P(13^3N)$. However we could not have found Conjectures 1 and 2 without the evidence provided by these runs.

We do not regard the results of this paper which require numerical details beyond the capacity of hand calculation as constituting examples of "machine proof." Indeed programmers, and sometimes machines, make mistakes. In so far as a mathematical proof quotes "known" results, the reader who is not prepared to verify directly the whole of relevant previous knowledge can only be sure that the results are highly probable (since the results in published papers can be incorrect). Now all the quoted calculations in this paper have either been verified independently by hand and machine (where possible by different methods), *or an independent check has been found assuming Conjectures* 1 *and* 2. In fact we found that

$$P(13^4 \cdot 22655) \equiv P(13^4 \cdot 5^2 \cdot 22655) \equiv 0 \pmod{13^4},$$

and

$$c(7 \cdot 13^3) \equiv 0 \pmod{13}, \; c(757 \cdot 13^3) \equiv 0 \pmod{13^2}, \; c(5299 \cdot 13^3) \equiv 0 \pmod{13^3}.$$

Thus we claim that Conjectures 1 and 2, by increasing the probability that our calculations are correct, contribute to the proofs of Theorems 3 and 7.

Appendix A

Proof of Lemma 5. We consider $g = g(x)$ as a function $g(\tau)$ of τ where $x \equiv \exp(2\pi i \tau)$, $\operatorname{Im} \tau > 0$. Then $g(\tau)$ is an entire function on $\Gamma_0(13)$, and hence (see for example Atkin [5], Lemma 9)

$$13\{g(-1/13\tau) + 13Ug(\tau)\} = g^{-1}(\tau) + 13^2 Ug(\tau)$$

is an entire function on the full modular group $\Gamma(1)$. But $g^{-1}(\tau)$ has a simple pole, residue 1 at $\tau = i_\infty$ in the local variable x, whence

$$j(\tau) = g^{-1}(\tau) + 13^2 Ug(\tau) + \text{constant}.$$

Next $g^{-1}(-1/13\tau) + 13Ug^{-1}(\tau)$ is an entire function on $\Gamma(1)$, which must be constant since it is regular at i_∞, so that $Ug^{-1}(\tau) = -g(\tau) + \text{constant}$. Thus

$$Uj(\tau) = -g(\tau) + 13^2 U^2 g(\tau) + \text{constant},$$

which implies Lemma 5.

Appendix B

We revert to the use of p- rather than P-notation, and write (78) in the form

If any (and hence all) of $p(13^2n+162)$, $p_{11}(13n+12)$, *and* $p_{23}(n)$ *is divisible by* 13, *then*

$$p(13^2n+162) \equiv p_{11}(13n+12) \equiv -2p_{23}(n) \pmod{13^2}. \tag{79}$$

We remind the reader that (79) is false, and write

$$(a) = \prod_{r=0}^{\infty} (1 - x^{13r+a})(1 - x^{13r+13-a}), \tag{80}$$

and

$$\begin{aligned} A(x) &= \sum_{n=0}^{\infty} \alpha_n x^n = f^3(x^{13}) \cdot (2)(3)(4)(6), \\ B(x) &= \sum_{n=0}^{\infty} \beta_n x^n = f^3(x^{13}) \cdot (1)(5)(4)(6), \\ C(x) &= \sum_{n=0}^{\infty} \gamma_n x^n = x f^3(x^{13}) \cdot (1)(5)(2)(3). \end{aligned} \tag{81}$$

Then

$$A(x) = B(x)+C(x), \tag{82}$$

$$\begin{aligned} \alpha_n &= 0 \text{ if } n \equiv 1, 9, 11, 12 \pmod{13}, \\ \beta_n &= 0 \text{ if } n \equiv 2, 3, 7, 8 \pmod{13}, \\ \gamma_n &= 0 \text{ if } n \equiv 0, 4, 6, 10 \pmod{13}. \end{aligned} \tag{83}$$

Also

$$\begin{aligned} p_{23}(n) &\equiv 4\alpha_n - 3\beta_n \pmod{13}, \\ 2p_{23}(n)+p_{11}(13n+12) &\equiv 27\alpha_n - 82\beta_n \pmod{13^2}. \end{aligned} \tag{84}$$

Suppose now that $p_{23}(n) \equiv 0 \pmod{13}$ and (for example) $n \equiv 1 \pmod{13}$. Then $\alpha_n = 0$ and $\beta_n \equiv 0 \pmod{13}$. Now the coefficients β_n are initially very small numerically, indeed $|\beta_n| \leqq 8$ for $n \leqq 700$, so that $\beta_n \equiv 0 \pmod{13}$ implies that β_n is actually zero in these cases, which now implies (79) by (84) and (74).

This explains the phenomenon when $n \not\equiv 5 \pmod{13}$. If $n \equiv 5 \pmod{13}$ there is an analogous explanation involving $p_{11}(n)$, $p_{23}(13n+5)$, and different α_n, β_n, γ_n. The only cases where (79) fails for $n \leqq 2100$ are $n = 1530$, 1775, 1921, and 1940.

We have given no proofs in this Appendix B, but in fact can prove everything but (83) quite easily. We know in principle how to give a very long proof of (83), but confess that we have not gone through the details.

Appendix C

The modular equation (17) is

$$t^{13} + \sum_{r=1}^{13} \sum_{\sigma=[(r+2)/2]}^{7} \beta_{r\sigma} g^{\sigma} t^{13-r} = 0,$$

and we now give a table of the explicit values of $\beta_{r\sigma}$.

$\sigma =$	1	2	3	4	5	6	7
$r = 1$	-11.13	-36.13^2	-38.13^3	-20.13^4	-6.13^5	-13^6	-13^6
2		204.13	346.13^2	222.13^3	74.13^4	13^6*	13^6
3		-36.13	**-126.13^2**	**-102.13^3**	**-38.13^4**	**-7.13^5**	**-7.13^5**
4			346.13	**422.13^2**	**184.13^3**	**37.13^4**	**3.13^5**
5			-38.13	**-102.13^2**	**-56.13^3**	**-13^5***	**-15.13^4**
6				222.13	**184.13^2**	**51.13^3**	**5.13^4**
7				-20.13	**-38.13^2**	**-13^4***	**-19.13^3**
8					74.13	**37.13^2**	**5.13^3**
9					-6.13	**-7.13^2**	**-15.13^2**
10						13^2*	**3.13^2**
11						-13	-7.13
12							13
13							-1

It is shown in [1] that

$$13^{r-\sigma}\beta_{r\sigma} = \beta_{2\sigma-r,\sigma} \tag{85}$$

so that the symmetry of the columns is provable a priori. Also the inequality

$$\pi(\beta_{r\sigma}) \geqq [(13\sigma - 7r + 13)/14]$$

of Lemma 1 in fact holds with equality except in the cases marked with an asterisk in the table. It is an open question whether Lemma 1 can be proved a priori without calculation; the only method we can find is to use (85), but this is not sufficient to establish Lemma 1 in the cases boldfaced in the table.

References

1. J. N. O'Brien, *Some properties of partitions with special reference to primes other than* 5, 7, *and* 11, Ph.D. Thesis, Univ. of Durham, England, 1966.

2. G. N. Watson, *Ramanujans Vermutung über Zerfällungsanzahlen*, J. Reine Angew. Math. **179** (1938), 97–128.

3. J. Lehner, *Divisibility properties of the Fourier coefficients of the modular invariant* $j(\tau)$, Amer. J. Math. **71** (1949), 136–148.

4. ———, *Further congruence properties of the Fourier coefficients of the modular invariant* $j(\tau)$, Amer. J. Math. **71** (1949), 373–386.

5. A. O. L. Atkin, *Proof of a conjecture of Ramanujan*, Glasgow Math. J. (to appear).

6. M. Newman, *Periodicity modulo m and divisibility properties of the partition function*, Trans. Amer. Math. Soc. **97** (1960), 225–236.

7. ———, *Congruences for the coefficients of modular forms and for the coefficients of* $j(\tau)$, Proc. Amer. Math. Soc. **9** (1958), 609–612.

8. ———, *Congruences for the coefficients of modular forms and some new congruences for the partition function*, Canad. J. Math. **9** (1957), 549–552.

9. ———, *Further identities and congruences for the coefficients of modular forms*, Canad. J. Math. **10** (1958), 577–586.

10. S. Ramanujan, *On certain arithmetical functions*, Trans. Cambridge Philos. Soc. **22** (1916), 159–184.

11. L. J. Mordell, *On Mr. Ramanujan's empirical expansions of modular functions*, Proc. Cambridge Philos. Soc. **19** (1919), 117–124.

The Atlas Computer Laboratory,
Chilton, Didcot, England
The University,
Exeter, England