0. Introduction. In this note we shall describe how to adapt the methods of [1] to studying the group of pseudo-isotopy classes of diffeomorphisms of 1-connected manifolds. We shall use the notation and results of [1] freely. In the first four sections we shall discuss smooth manifolds and orientation preserving diffeomorphisms, but the results may be proved for piecewise linear (p.l.) manifolds and p.l. equivalences using the technique of [3] to adapt these proofs to the p.l. case. I am indebted to W.-C. Hsiang and J. Milnor for helpful comments.

In §1 we consider the problem of when a given homotopy equivalence is homotopic to a diffeomorphism. We get a condition in terms of the normal bundle. Our theorem refines a result of Novikov [12]. In §2 we consider the uniqueness question, and get results on pseudo-isotopy of diffeomorphisms. In §3 we use the results of §1 to construct diffeomorphisms which are homotopic to the identity but are not pseudo-isotopic to the identity, even piecewise linearly or topologically. The lowest dimensional example is on a 5-manifold, e.g., $S^3 \times S^3$. Some similar examples have been obtained by Hodgson [9]. In §4 we consider the group of diffeomorphisms $\mathcal{D}(M)$ (up to pseudo-isotopy) and deduce an exact sequence relating it to the group of tangential equivalences $\mathcal{F}(M)$. It follows that the kernel of the homomorphism of $\mathcal{D}(M)$ into $\mathcal{F}(M)$ is finite up to conjugacy. In §5 we deduce some facts about automorphisms of $S^n \times S^1$. We show in particular that the group of pseudo-isotopy classes of p.l. automorphisms of $S^n \times S^1$ is $Z_2 + Z_2 + Z_2$ for $n \geq 5$ (cf. [6]).

If $\xi^k$ is a $k$-plane bundle we denote by $T(\xi)$ the Thom complex of $\xi$, $T(\xi) = E(\xi)/E_0(\xi)$, where $E(\xi)$ is the total space of the closed disk bundle of $\xi$, $E_0(\xi)$ is its boundary sphere bundle.

We recall that if $M^n$ is a smooth manifold embedded in $S^{n+k}$ with normal bundle $\nu^k$, $k$ very large, then the natural collapsing map of $S^{n+k}$ onto the Thom complex $T(\nu) = E(\nu)/E_0(\nu)$ defines an element in $\pi_{n+k}(T(\nu))$. Any bundle automorphism sends this element into another one obtained in this way. We will call the set of such elements the normal invariants of $M^n$, and denote the set by $C_M \subset \pi_{n+k}(T(\nu))$.

If $Z = X \times I \cup_f Y$, where $f: X \times \{0, 1\} \to Y$ and if $\xi$ is a bundle over $Z$ then there is a natural collapsing map $q: T(\xi) \to T(\xi)/T(\xi|Y)$ and $T(\xi)/T(\xi|Y) = T((\xi|X) + e^1)$ where $e^1$ is the trivial line bundle.
1. When is a map homotopic to a diffeomorphism? Let $M^n$ be a smooth manifold. We first consider the question of when a given homotopy equivalence $h: M^n \to M^n$ is homotopic to a diffeomorphism. We give a refinement of a theorem of Novikov [12] which describes the situation.

Define a space $M_h$ called the mapping torus of $h$ by $M_h = M \times I$ with identifications $(x, 0) = (h(x), 1)$ for all $x \in M$. If $h$ is a diffeomorphism, $M_h$ is a smooth manifold. If $h$ is a homotopy equivalence, $M_h$ is homotopy equivalent to a fiber space over $S^1$, with fiber homotopy equivalent to $M$.

If $h$ is orientation preserving(2), then as was remarked in [1], if $M$ is closed then $M_h$ has a splitting satisfying Poincaré duality, and if $\partial M \neq \emptyset$, then the pair $(M_h, \partial M_h \cap M)$ has a splitting satisfying Poincaré duality. We recall that if $\dim M = n = 4q - 1$, then it was proved in [2, Theorem 2.2] that the index $\sigma(M_h) = 0$. (Note that in [2], although $h$ was assumed a homeomorphism, it is sufficient to assume that $h$ is a homotopy equivalence.)

**Theorem 1.** Let $M^n$ be a closed 1-connected smooth manifold of dimension $n \geq 5$, $n \not\equiv 1$ mod 4, or $n = 5, 13$. An orientation preserving homotopy equivalence $h: M \to M$ is homotopic to a diffeomorphism if and only if the following conditions are satisfied:

1. There is a $k$-plane bundle $\xi^k$ over $M_h$ such that $\xi|M = \nu^k$, the normal bundle to $M$ embedded in $S^{n+k}$ ($k$ very large).

2. There is an element $\alpha \in \pi_{n+k+1}(T(\xi))$ such that $q^k(\alpha) = C_M \subset \pi_{n+k+1}(T(\nu^k + e^1))$, (see §0).

3. If $n = 4q - 1$, then $L_q (\bar{p}_1 (\xi), \ldots, \bar{p}_Q (\xi)) = 0$, where $\bar{p}(\xi)$ are the dual Pontrjagin classes of $\xi$, $L_q$ is the Hirzebruch $L$ polynomial.

Further, the bundle $\xi^k$ will be the normal bundle of the mapping torus of the diffeomorphism and if $n \neq 5$ or 13, then $\alpha$ will be a normal invariant for it.

**Proof.** We apply Theorem 1.1 of [1]. We get a smooth manifold $W^{n+1}$ with $M^n \subset W^{n+1}$ and a homotopy equivalence $f: W^{n+1} \to M_h$ with $f | M \times [0, \frac{1}{2}] = \text{identity}$, and $f | W - M \times (0, \frac{1}{2})$ a homotopy equivalence with $M_h - M \times (0, \frac{1}{2}) = M \times [\frac{1}{2}, 1]$. It follows that $W - M \times (0, \frac{1}{2})$ is a $h$-cobordism between $M \times 0$ and $M \times \frac{1}{2}$ and so by the $h$-cobordism theorem [15], $W - M \times (0, \frac{1}{2})$ is diffeomorphic to $M \times [0, 1]$, with the diffeomorphism being the identity on $M \times 0$. It follows that the diffeomorphism of $M \times \frac{1}{2}$ with $M \times 1$ is homotopic to $h$ and $W$ is the mapping torus

(2) In this paper we always restrict our attention to orientation preserving maps, just as in [1] we considered only oriented manifolds. However, Dennis Sullivan has pointed out to me that all the arguments of [1] carry over to nonorientable manifolds with $\pi_1 = \mathbb{Z}$, if one changes the definition of a splitting satisfying Poincaré duality to be (in the notation of [1]):

A splitting $A \times I \cup I Y$ satisfies Poincaré duality if the pair $(Y, A \times 0 \cup A \times 1)$ satisfies relative Poincaré duality (thinking of $f$ as an inclusion, so that $A \times 0 \cup A \times 1 \subset Y$). Hence, the results of this paper may be carried over to orientation reversing diffeomorphisms, using this strengthening of the results of [1].
of the diffeomorphism. The remaining statements follow from the analogous ones in the statement of Theorem (1.1) of [1].

Considering the case of manifolds with boundary, we have

**Theorem 2.** Let \((W^n, \partial W)\) be a 1-connected \(n\)-dimensional oriented smooth manifold with 1-connected boundary, \(n \geq 6\). An orientation preserving homotopy equivalence of pairs \(h : (W, \partial W) \to (\mathcal{W}, \partial \mathcal{W})\) is homotopic to a diffeomorphism of \(W\) if and only if there is an oriented \(k\)-plane bundle \(\xi^k\) over \(W\) such that \(\xi^k|W = v^k\), the normal bundle to \(W\) for an embedding in \(\mathcal{D}^{n+k}\) (with \(\partial W \subset \partial \mathcal{D}^{n+k}\)) and an element \(\alpha \in \pi_{n+k+1}(T(\xi), T(\xi|\partial W)_h)\) such that \(q_4(\alpha) \in \pi_{n+k+1}(T(v \oplus e^k), T(v \oplus e^k|\partial W))\) is a normal invariant for \(W\) (where \(q\) is the natural collapsing as in \(\S 0\)). Further \(\xi\) will be the normal bundle of the mapping torus of the diffeomorphism and \(\alpha\) will be its normal invariant.

This follows from Theorem 2.1 of [1], in a way similar to Theorem 1.

A bundle \(\xi\) over \(M_h\) (or \(W_h\)) such that \(\xi|M \ (or \(\xi|W\))\) is \(v\), is given by a bundle equivalence \(b : v \to v\) which covers \(h\). Two such bundles \(\xi_1, \xi_2\) over \(M_h\) (or \(W_h\)) are isomorphic if and only if the bundle maps \(b_1\) and \(b_2\) are homotopic (as bundle maps). Thus the first condition for \(h\) to be conjugate to a diffeomorphism is simply the obvious condition that \(h\) is covered by an equivalence of stable normal bundles.

Similarly, the second condition on \(h\) reduces to the statement that the equivalence of stable tangent bundles carries a normal invariant for \(M\) (or \(W\)) into itself.

2. Pseudo-isotopy of diffeomorphisms. Two diffeomorphisms \(f_0\) and \(f_1\) of \(M\) are called pseudo-isotopic if there is a diffeomorphism \(F : M \times [0, 1] \to M \times [0, 1]\) such that \(F(x, 0) = (f_0(x), 0), F(x, 1) = (f_1(x), 1)\), for all \(x \in M\). We will say \(f_0\) is conjugate to \(f_1\) if \(f_0\) is pseudo-isotopic to \(g f_1 g^{-1}\) for some diffeomorphism \(g\). It follows that for \(n \geq 5\), \(M^n\) closed and 1-connected, \(f_0\) is conjugate to \(f_1\) if and only if there is an \(h\)-cobordism of \(M\) with itself and a diffeomorphism \(F : U \to U\) which agrees with \(f_0\) on one end and \(f_1\) at the other. Note that \(f\) conjugate to the identity implies \(f\) is pseudo-isotopic to the identity. The set of pseudo-isotopy classes of orientation preserving diffeomorphisms of \(M\) forms a group \(\mathcal{P}(M)\).

If \(f\) is an orientation preserving diffeomorphism of a smooth \(n\)-manifold \(M\), then we may change \(f\) by an isotopy so that \(f|D^n = \text{identity}\) for an embedded closed disk \(D^n \subset M^n\), (see [14]). If \(g\) is an orientation preserving diffeomorphism of \(S^n\), then we may define a diffeomorphism \(\psi g\) of \(M\) as follows: By an isotopy, make \(g|D^n = \text{identity}\), and define \(\psi g(x) = x\) if \(x \in M - D^n\) and \(\psi g|D^n = g|D^n\). The pseudo-isotopy class of \(\psi g\) depends only on the pseudo-isotopy class of \(g\), and it is easy to see that \(\psi\) defines a homomorphism \(\psi : \mathcal{P}(S^n) \to \mathcal{P}(M^n)\), and that image \(\psi \subset \text{center of } \mathcal{P}(M^n)\). Now the group \(\mathcal{P}(S^n)\) is isomorphic to \(\Gamma^{n+1}\), the group of differential structures on \(S^{n+1}\), which, if \(n \geq 4\), is the same as \(\theta^{n+1}\), the group of homotopy \((n+1)\)-spheres (see [10]).
**Lemma 1.** Let $f$ be a diffeomorphism of a smooth closed manifold $M$. If $\Sigma \in \Gamma^{n+1}$, then $M_f \# \Sigma$ is diffeomorphic to $M_f (\# \psi \phi)$, where $g \in \mathcal{D}(S^n)$ is the element corresponding to $\Sigma$ under the isomorphism of $\mathcal{D}(S^n)$ with $\Gamma^{n+1}$.

**Proof.** $M_f \# \Sigma = (M_f - D^{n+1}_g) \cup (\Sigma - D^{n+1}_g)$ where $D^{n+1}_g$ is an $(n+1)$-cell, $D^{n+1}$ its interior. Now $\Sigma = D^{n+1}_g \cup g\ D^{n+1}_g$, where $g \in \mathcal{D}(S^n)$. Hence $M_f \# \Sigma = (M_f - D^{n+1}_g) \cup g\ D^{n+1}_g$. Now $g$ may be assumed to leave a disk $D^{n}_g \subset S^n$ fixed and we may assume that $D^{n+1}_g \cap M^{n} \times D^n = D^n_g$ where $f|D^n_g$ = identity. Hence $M_f \# \Sigma = (M_f - D^{n+1}_g) \cup g\ D^{n+1}_g = M_f \# \psi \phi$.

**Theorem 3.** Let $M$ be a closed oriented $1$-connected $n$-manifold, $f_i: M \rightarrow M$, $i = 1, 2$, two orientation preserving diffeomorphisms, $v_i$ their normal bundles, so $v_i$ is a bundle over $M_f$, such that $v_i|M = v$ the normal bundle of $M$. Suppose $f_i$ is homotopic to $f_2$ so that $M_f$ is homotopy equivalent to $M_{f_2}$, and the homotopy equivalence is covered by a bundle map $b: v_1 \rightarrow v_2$ such that $T(b)(a) = a_2 \in \pi_{n+k+1}(T(v_2))$, where $a_i$ is some normal invariant for $M_i$. If $n$ is even, $n \geq 6$, then $f_1$ is conjugate to $f_2: \psi g$ for some $g \in \mathcal{D}(S^n)$ corresponding to some element $\Sigma \in \theta^{n+1}(\partial \Sigma) \subset \theta^{n+1} = \Gamma^{n+1}$.

This follows from Lemma 1 and [1, (1.3)]. The case of $n$ odd is slightly more complicated.

If $f: M^n \rightarrow M^n$ is a diffeomorphism fixed on a cell $D^n \subset M^n$, then $f$ defines in an obvious way a diffeomorphism $f': M \# \Sigma \rightarrow M \# \Sigma$ for any $\Sigma \in \theta^n$. If $\Sigma^n$ is diffeomorphic to $S^n$ then $f' = f$. Any orientation preserving diffeomorphism of $M^n$ is isotopic to one which is fixed on a disk [14]. On the other hand, if $g: M \rightarrow M \# \Sigma$ is a diffeomorphism, then any diffeomorphism $d$ of $M \# \Sigma$ induces a diffeomorphism, namely $g^{-1} dg$, on $M$. Hence given a diffeomorphism $f$ on $M$, we may change it by an isotopy to $f'$, which is fixed on a disk, and if $g: M \rightarrow M \# \Sigma$ is a diffeomorphism, we shall say that $f$ and $g^{-1} f' g^{-1}$ are weakly conjugate, mod $\Sigma$. Note that if $\Sigma^n = S^n$ (or if we are in the p.l. category) weakly conjugate is the same as conjugate. Also, $f$ weakly conjugate to the identity implies $f$ is pseudo-isotopic to the identity.

**Theorem 4.** Let $f_i: M^n \rightarrow M^n$, $i = 1, 2$, be diffeomorphisms satisfying all the conditions of Theorem 3, but with $n$ odd, $n \geq 5$. Then $f_1$ is weakly conjugate to $f_2$, mod $\Sigma$, for some $\Sigma \in \theta^n(\partial \Sigma)$.

(In particular, if $M$ has the property:

\[ M \# \Sigma \text{ diffeomorphic to } M, \quad \Sigma \in \theta^n(\partial \Sigma) \text{ implies } \Sigma = S^n, \]

(cf. [2]), then in Theorem 4, $f_1$ is conjugate to $f_2$.)

**Proof of Theorem 4.** From [1, Theorem 1.4] we deduce that $M_{f_1}$ is diffeomorphic to $M_{f_2} \#_\beta \Sigma^n$, for some $\Sigma^n \in \theta^n(\partial \Sigma)$. That is, there is an embedding $\beta: S^1 \times D^n \subset M_{f_2}$ such that $\beta(S \times 0)$ represents a generator of $\pi_1$ such that $M_{f_1}$ is diffeomorphic to $(M_{f_2} - \beta(S^1 \times \text{int } D^n)) \cup (S^1 \times \Sigma^n - S^1 \times \text{int } D^n)$ with the obvious identification along $S^1 \times S^{n-1}$, where $\Sigma^n = \theta^n(\partial \Sigma)$. By changing $f_2$ by an isotopy,
we may assume that $f_2$ is the identity on $D^m \subset M^n$, so that $M_{f_2} \#_g \Sigma$ is obviously the mapping torus of the diffeomorphism $f_2'$ of $M \#_g \Sigma$. Now the result follows from:

**Lemma 2.** Let $M_f$ and $M'_g$ be differentiable fiber bundles over $S^1$ with projections $p$ and $p'$, and with fibers $M^n$ and $M'^n$ which are 1-connected closed manifolds of dimension $n \geq 5$. If $h: M_f \to M'_g$ is a diffeomorphism such that $p'h$ is homotopic to $p$, then $h$ is pseudo-isotopic to $h'$ such that $p'h'=p$, so that $h'$ restricts to a diffeomorphism of $M$ with $M'$.

**Proof of Lemma 2.** We consider the fibration of $M_f \times \{0,1\}$ given by $p$ on $M_f \times 0$ and $p'h$ on $M_f \times 1$. By the result of [4], this fibration extends to a fibration of all of $M_f \times [0,1]$, so that the fiber is an $h$-cobordism between $M \times 0$ and $h^{-1}(M') \times 1$. Using the $h$-cobordism theorem of Smale on this $h$-cobordism we see that we have an embedding of $M \times I$ in $M_f \times I$ with $M \times 1 = h^{-1}(M') \times 1$, so that it follows using the $h$-cobordism theorem again that there is a pseudo-isotopy of the identity of $M_f$ carrying $M$ into $h^{-1}(M')$. Then the composite of this and $h$ is a pseudo-isotopy of $h$ carrying $M$ into $M'$.

In the bounded case we have the following

**Theorem 5.** Let $W^n$ be a 1-connected $n$-manifold with nonempty 1-connected boundary, $\partial W$, $n \geq 6$. Let $f_i: W \to W$, $i=1,2$ be two orientation preserving diffeomorphisms which are homotopic, such that the induced homotopy equivalence of $M_{f_i}$ with $M_{f_2}$ is covered by a bundle equivalence $b: \nu_1 \to \nu_2$, and such that $T(b)\alpha_1 = \alpha_2$ where $\alpha_1 \in \pi_{n+k+1}(T(\nu_1))$ is a normal invariant for $f_1$ (for $M_{f_1}$) and $\alpha_2$ is a normal invariant for $f_2$ (for $M_{f_2}$), ($T(b)$ is the induced map of Thom complexes). Then $f_1$ is conjugate to $f_2$.

3. Examples of diffeomorphisms. Now we apply Theorem 1 to create some examples of diffeomorphisms.

**Theorem 6.** Let $M^n$ be a 1-connected manifold, closed or with 1-connected boundary, and suppose $H^{4k-1}(M; \mathbb{Q}) \neq 0$ for some $k$, and $p_\ast(M) = 0$. If $M$ is closed, suppose $n$ is even $\geq 6$ or $n=5,13$. Then there is a diffeomorphism $f: M \to M$ which is homotopic to the identity, but such that $f^q$ is not pseudo-isotopic to the identity for any $q > 0$.

In fact we shall construct an $f$ using Theorem 1, such that $t(f^q)$ induces a tangential equivalence different from the identity for all $q > 0$.

Since $H^{4k-1}(M; \mathbb{Q}) = H^{4k}(\Sigma M^+; \mathbb{Q}) \neq 0$, it follows (using the Pontrjagin character) that there is a bundle $\eta$ over $\Sigma M^+$ with nonzero $p_\ast(\eta)$. Now the group of homotopy classes $[\Sigma M^+, B_{\mathbb{Q}}]$ is finite since $\Sigma M^+$ is a finite polyhedron and $\pi_i(B_{\mathbb{Q}}) = \pi_{i+n-1}(S^\infty)$ (for large $n$) is finite for all $i$. Hence some multiple $m\eta$ is fiber homotopy trivial (see [5]). By the Whitney sum formula $p_\ast(m\eta) = mp_\ast(\eta)$, since cup products are zero in $\Sigma M^+$.
Now we apply Theorem 1 or 2 using \( h = \text{identity} \) so that \( M_h = M \times S^1 \), and 
\[
\xi^e = (\nu \times e') + c^*(m_\eta),
\]
where \( c: M \times S^1 \to M \times S^1 \text{}/\text{M x t = } \Sigma M \). Since \( m_\eta \) is fiber homotopy trivial, \( (\nu \times e') + c^*(m_\eta) \) is fiber homotopy equivalent by a map \( g \) to \( \nu \times e' \) for some \( r \), i.e., to the normal bundle of the identity diffeomorphism. Then a normal invariant \( s \) for the identity map is sent by the induced map of Thom complexes \( T(g) \) to an element \( T(g)_n(r) \in \pi_{n+2+1}(T(\xi)) \) which satisfies the hypothesis of Theorem 1 or 2. Hence there is a diffeomorphism \( f \) homotopic to the identity with normal bundle \( \xi \).

Now we define an invariant \( p_k \in H^{4k}(\Sigma M^+; \mathbb{Q}) \) for diffeomorphisms of \( M \) homotopic to the identity as follows:

Consider the exact sequence

\[
0 \to H^{4k}(\Sigma M^+; \mathbb{Q}) \xrightarrow{j^*} H^{4k}(M; \mathbb{Q}) \xrightarrow{i^*} H^{4k}(M; \mathbb{Q}) \to 0
\]

(recall that \( M \cong M \times S^1 \)). Then \( i^*p_k(M) = 0 \) so \( p_k(M) = j^*(x) \) and we define \( p_k(d) = x \). Then \( p_k(d) \) is uniquely defined and is nonzero for \( d = f^q \) for \( q > 0 \), since \( p_k(M) \neq 0 \) for \( d = f^q \).

This proves Theorem 5. Note that \( p_k \) is a p.l. and topological invariant (see [16] and [13]) so that \( f^q \) is not p.l. or topologically pseudo-isotopic to the identity.

If \( n = 4(k + 1) - 1 \), and \( M \) is closed, then an additional condition on \( L(\tau M) \) will yield a similar result, namely we must assume also that the annihilator of \( L(\tau M) \) in \( H^{4k-1}(M; \mathbb{Q}) \) is nonzero.

As an example let us consider \( M^n = S^2 \times S^3 \). Then by Theorem 6 there exist diffeomorphisms homotopic to the identity, which are not pseudo-isotopic to the identity (in the differential, p.l., or topological sense). W.-C. Hsiang has pointed out to me that this also follows from results of Haefliger [7] and Montgomery and Yang [11]. Similar examples of diffeomorphisms of \( S^n \times S^q \) have been found by J. Hodgson [9].

In the case of \( S^3 \times S^3 \) such a diffeomorphism \( f \) may be constructed as follows: Let \( \alpha: S^3 \to SO(4) \) be such that the induced bundle over \( S^4 \) has a nonzero Pontrjagin class \( p_1 \) but is fiber homotopically trivial. Then define \( f(x, y) = (x, \alpha(x)y) \).

4. Group properties. Denote by \( \mathcal{T}(M) \) the group of tangential equivalences of \( M \), i.e., bundle maps of the stable normal bundle of \( M \), up to homotopy as bundle maps.

**Lemma 3.** The elements of \( \mathcal{T}(M) \) which cover a fixed map of \( M \) to \( M \) are isomorphic to the group \( [M, SO] = KSO^{-1}(M) \).

**Proof.** (Cf. Hirsch-Mazur [8].) Let \( \text{End}(\nu) \) denote the group of bundle maps covering the identity. If \( \nu \) is the bundle and if \( \xi \) is its inverse, define a map

\[
i: \text{End}(\nu) \to \text{End}(\nu + \xi) \text{ by } i(f) = f + \text{id}.
\]

Then the composite \( \text{End}(\nu) \to \text{End}(\nu + \xi) \to \text{End}(\nu + \xi + \nu) \) is the same as \( j: \text{End}(\nu) \to \text{End}(\nu + \text{trivial bundle}) \) defined by \( j(f) = f + \text{id} \). If dimension \( \nu \) is much larger than
dimension $M$, it follows that $j$ is an isomorphism. On the other hand $\text{End}(\nu + \xi) = \text{End}(\text{trivial bundle}) = [M, SO]$, so that $\text{End}(\nu)$ is a direct summand of $[M, SO]$. A similar argument starting from $\text{End}(\text{trivial})$ shows that $\text{End}(\nu) \cong [M, SO]$.

If $a, b$ are two bundle maps covering the same map of base spaces, then $ab^{-1}$ covers the identity, so that Lemma 3 follows.

If $\mathcal{H}(M)$ denotes the group of homotopy equivalences of $M$, Lemma 2 gives us an exact sequence

$$0 \rightarrow [M, SO] \rightarrow \mathcal{I}(M) \rightarrow \mathcal{H}(M).$$

The image of $\mathcal{I}(M)$ consists of those $h \in \mathcal{H}(M)$ such that $h^*(\nu) = \nu$, or in other words those $h$ which commute with the classifying map of $\nu$.

Now suppose $M$ and $\partial M$ are 1-connected, $\dim M \geq 5$, if $\partial M = \emptyset$, $\dim M > 5$, if $\partial M \neq \emptyset$.

Theorems 1 and 2 characterize the image of $\mathcal{B}(M)$ in $\mathcal{I}(M)$, provided $n \neq 1 \mod 4$ or $n = 5, 13$ when $\partial M = \emptyset$.

If $f \in \mathcal{D}(M)$ is in the kernel, i.e., if $f$ induces the identity on the stable normal $k$-plane bundle $\nu^k$ of $M$, then $M_f \cong M \times S^1$ and $\nu_f = \nu \times (0) = p_f^*(\nu)$. Since $\nu_f$ is induced by the projection map $p_f$ from $\nu$ over $M$, it follows that $T(\nu_f)$ has $T(\nu)$ as a retract. Hence we get the exact sequence

$$0 \rightarrow \pi_{n+k+1}(T(\nu^k)) \rightarrow \pi_{n+k+1}(T(\nu_f)) \xrightarrow{q_*} \pi_{n+k+1}(T(\nu^k + e^1)) \rightarrow 0$$

(we are in stable homotopy so the sequence is exact). It follows that the different possible normal invariants for diffeomorphisms homotopic to $f$ are in 1-1 correspondence with kernel $q_* = \pi_{n+k+1}(T(\nu^k))$. It follows from standard homotopy theory that this group is finite, since $H_{n+k+1}(T(\nu^k)) = 0$. Then it follows from Theorems 3, 4, and 5 that each of these elements of $\pi_{n+k+1}(T(\nu))$ corresponds to only a finite number of conjugacy classes of diffeomorphisms, so that the kernel $\mathcal{D}(M) \rightarrow \mathcal{I}(M)$ is finite up to conjugacy.

5. Automorphisms of $S^n \times S^1$. In this section we point out some easy consequences of Lemma 2 in §2. By automorphism we will mean diffeomorphism or p.l. equivalence, depending on the context.

**Proof.** By Lemma 2 if $h: M_f \rightarrow M_f$ is an automorphism (Lemma 2 holds in both the smooth and p.l. categories), then $h$ is pseudo-isotopic to $h'$ such that $h'(M) = M$. Then clearly $h' \mid M \in \mathcal{G}$ and $h$ is pseudo-isotopic to $\eta(h' \mid M)$.

**Corollary 1.** If $\mathcal{D}(S^n \times S^1)$ is the group of pseudo-isotopy classes of diffeomorphisms of $S^n \times S^1$, then the sequence

$$\Gamma^{n+2} + \Gamma^{n+1} \rightarrow \mathcal{D}(S^n \times S^1) \xrightarrow{\gamma} \mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \rightarrow 0$$

is exact for $n \geq 5$. 
Here $\Gamma_{n+1} = \mathcal{D}(S^n)$ pseudo-isotopy classes of diffeomorphisms of $S^n$ of degree $+1$, and acts by operating only on the fibers $S^n \times t$ in $S^n \times S^1$, and $\Gamma_{n+2}$ acts by leaving the complement of an $(n+1)$-cell in $S^n \times S^1$ fixed (cf. §2). (Recall that $\Gamma_{n+2}$ goes into the center of $\mathcal{D}(M)$ for any $(n+1)$-manifold $M$.)

The first $\mathbb{Z}_2 + \mathbb{Z}_2$ on the right is simply the automorphism group of $H_2(S^n \times S^1)$ if $n > 1$, and the map is simply $f \to f_*$. The map into the third $\mathbb{Z}_2$ is obtained by taking $f \to w_2(M, f)$. There is a subgroup $\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \subseteq \mathcal{D}(S^n \times S^1)$ which maps isomorphically onto the group on the right, which is described as follows:

The first and second $\mathbb{Z}_2$'s are generated by the orientation reversing diffeomorphisms of $S^1$ and $S^n$, while the third $\mathbb{Z}_2$ is generated by the map $f(x, y) = (\alpha(y)x, y)$ where the map $\alpha: S^1 \to SO(n + 1)$ represents a generator of $\pi_1(SO(n + 1)) = \mathbb{Z}_2$. It follows that by composing with these automorphisms we may get an automorphism $h: S^n \times S^1 \to S^n \times S^1$ which is the identity in homology and leaves fixed a tubular neighborhood of an embedded $S^1 \subseteq S^n \times S^1$. (This procedure generalizes Gluck [6].)

Lemma 2 of §2 implies that if $f_*$ is the identity on $H_1(S^n \times S^1)$, $f$ is pseudo-isotopic to an automorphism sending $S^n \times t$ into itself, (for some $t \in S^1$).

Corollary 1 now follows by showing that if $\gamma(f) = 0$ then we may compose with an automorphism of $S^n$ along all the fibers $S^n \times t$ to get $f'$ which is fixed on a neighborhood of $S^n \times t_0 \cup x_0 \times S^1$, such that the complement of this neighborhood is an $(n+1)$-cell, so that the resultant diffeomorphism comes from $\Gamma_{n+2}$.

Applying the above argument in the p.l. case, where by the theorem of Alexander the p.l. analogues of $\Gamma_{n+1}$ and $\Gamma_{n+2}$ are zero we get:

**Corollary 2.** The group of pseudo-isotopy classes of p.l. automorphisms of $S^n \times S^1$ for $n \geq 5$, is isomorphic to $\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2$.

This is a higher dimensional analogue of the theorem of Gluck [6] for $S^2 \times S^1$.

Following Gluck, we get:

**Corollary 3.** There are at most two different smooth or p.l. knots $(S^{n+2}, S^n)$ $n \geq 5$, which have equivalent complements.

Here p.l. knot is taken to mean that $S^n$ has a product neighborhood $S^n \times D^2 \subset S^{n+2}$.

The different knots with equivalent complements correspond to the different ways of attaching $S^n \times D^2$ to the complement along $S^n \times S^1$, so that these knots correspond to equivalences of $S^n \times S^1$. But it follows easily that the image of $\Gamma_{n+1}$ in $\mathcal{D}(S^n \times S^1)$ is extendable to $S^n \times D^2$ so that these yield the same knot. The image of $\Gamma_{n+2}$ in $\mathcal{D}(S^n \times S^1)$ changes a knot $(S^{n+2}, S^n)$ to a knot $(\Sigma^{n+2}, S^n)$ where $\Sigma^{n+2}$ is the corresponding homotopy $(n+2)$-sphere. Hence two knots in $S^{n+2}$ cannot differ by such a diffeomorphism. Finally in the quotient $\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2$ (which is the whole thing in the p.l. case), the only element which does not extend to $S^n \times D^2$ is the element arising from $\pi_1(SO(n+1))$, and the corollary follows. (I am indebted to the referee for pointing out the smooth part of this corollary.)
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