DIFFEOMORPHISMS OF 1-CONNECTED MANIFOLDS

BY
WILLIAM BROWDER(1)

0. **Introduction.** In this note we shall describe how to adapt the methods of [1] to studying the group of pseudo-isotopy classes of diffeomorphisms of 1-connected manifolds. We shall use the notation and results of [1] freely. In the first four sections we shall discuss smooth manifolds and orientation preserving diffeomorphisms, but the results may be proved for piecewise linear (p.l.) manifolds and p.l. equivalences using the technique of [3] to adapt these proofs to the p.l. case. I am indebted to W.-C. Hsiang and J. Milnor for helpful comments.

In §1 we consider the problem of when a given homotopy equivalence is homotopic to a diffeomorphism. We get a condition in terms of the normal bundle. Our theorem refines a result of Novikov [12]. In §2 we consider the uniqueness question, and get results on pseudo-isotopy of diffeomorphisms. In §3 we use the results of §1 to construct diffeomorphisms which are homotopic to the identity but are not pseudo-isotopic to the identity, even piecewise linearly or topologically. The lowest dimensional example is on a 5-manifold, e.g., $S^2 \times S^3$. Some similar examples have been obtained by Hodgson [9]. In §4 we consider the group of diffeomorphisms $\mathscr{D}(M)$ (up to pseudo-isotopy) and deduce an exact sequence relating it to the group of tangential equivalences $\mathscr{T}(M)$. It follows that the kernel of the homomorphism of $\mathscr{D}(M)$ into $\mathscr{T}(M)$ is finite up to conjugacy. In §5 we deduce some facts about automorphisms of $S^n \times S^1$. We show in particular that the group of pseudo-isotopy classes of p.l. automorphisms of $S^n \times S^1$ is $\boldsymbol{Z}_2 + \boldsymbol{Z}_2 + \boldsymbol{Z}_2$ for $n \geqq 5$ (cf. [6]).

If ξ^k is a k-plane bundle we denote by $T(\xi)$ the Thom complex of ξ, $T(\xi) = E(\xi)/E_0(\xi)$, where $E(\xi)$ is the total space of the closed disk bundle of ξ, $E_0(\xi)$ is its boundary sphere bundle.

We recall that if M^n is a smooth manifold embedded in S^{n+k} with normal bundle ν^k, k very large, then the natural collapsing map of S^{n+k} onto the Thom complex $T(\nu) = E(\nu)/E_0(\nu)$ defines an element in $\pi_{n+k}(T(\nu))$. Any bundle automorphism sends this element into another one obtained in this way. We will call the set of such elements the normal invariants of M^n, and denote the set by $C_M \subset \pi_{n+k}(T(\nu))$.

If $Z = X \times I \cup_f Y$, where $f\colon X \times \{0, 1\} \to Y$ and if ξ is a bundle over Z then there is a natural collapsing map $q\colon T(\xi) \to T(\xi)/T(\xi|Y)$ and $T(\xi)/T(\xi|Y) = T((\xi|X) + \varepsilon^1)$ where ε^1 is the trivial line bundle.

Received by the editors September 23, 1966.

(1) Work supported in part by NSF grant GP2425. Part of this paper was written at the Topology Symposium at Warwick University in 1966.

1. **When is a map homotopic to a diffeomorphism?** Let M^n be a smooth manifold. We first consider the question of when a given homotopy equivalence $h: M^n \to M^n$ is homotopic to a diffeomorphism. We give a refinement of a theorem of Novikov [12] which describes the situation.

Define a space M_h called the *mapping torus* of h by $M_h = M \times I$ with identifications $(x, 0) = (h(x), 1)$ for all $x \in M$. If h is a diffeomorphism, M_h is a smooth manifold. If h is a homotopy equivalence, M_h is homotopy equivalent to a fiber space over S^1, with fiber homotopy equivalent to M.

If h is orientation preserving(2), then as was remarked in [1], if M is closed then M_h has a splitting satisfying Poincaré duality, and if $\partial M \neq \varnothing$, then the pair $(M_h, \partial M_{h|\partial M})$ has a splitting satisfying Poincaré duality. We recall that if $\dim M = n = 4q-1$, then it was proved in [2, Theorem 2.2] that the index $\sigma(M_h) = 0$. (Note that in [2], although h was assumed a homeomorphism, it is sufficient to assume that h is a homotopy equivalence.)

THEOREM 1. *Let M^n be a closed 1-connected smooth manifold of dimension $n \geqq 5$, $n \not\equiv 1 \bmod 4$, or $n = 5$, 13. An orientation preserving homotopy equivalence $h: M \to M$ is homotopic to a diffeomorphism of M if and only if the following conditions are satisfied*:

(1) *There is a k-plane bundle ξ^k over M_k such that $\xi | M = \nu^k$, the normal bundle to M embedded in S^{n+k} (k very large).*

(2) *There is an element $\alpha \in \pi_{n+k+1}(T(\xi))$ such that $q_*(\alpha) \in C_M \subseteq \pi_{n+k+1}(T(\nu^k + \varepsilon^1))$, (see §0).*

(3) *If $n = 4q-1$, then $L_q(\bar{p}_1(\xi), \ldots, \bar{p}_q(\xi)) = 0$, where $\bar{p}_i(\xi)$ are the dual Pontrjagin classes of ξ, L_q is the Hirzebruch L polynomial.*

Further, the bundle ξ^k will be the normal bundle of the mapping torus of the diffeomorphism and if $n \neq 5$ or 13, then α will be a normal invariant for it.

Proof. We apply Theorem 1.1 of [1]. We get a smooth manifold W^{n+1} with $M^n \subset W^{n+1}$ and a homotopy equivalence $f: W^{n+1} \to M_h$ with $f \mid M \times [0, \frac{1}{2}] =$ identity, and $f \mid W - M \times (0, \frac{1}{2})$ a homotopy equivalence with $M_h - M \times (0, \frac{1}{2}) = M \times [\frac{1}{2}, 1]$. It follows that $W - M \times (0, \frac{1}{2})$ is a h-cobordism between $M \times 0$ and $M \times \frac{1}{2}$ and so by the h-cobordism theorem [15], $W - M \times (0, \frac{1}{2})$ is diffeomorphic to $M \times [0, 1]$, with the diffeomorphism being the identity on $M \times 0$. It follows that the diffeomorphism of $M \times \frac{1}{2}$ with $M \times 1$ is homotopic to h and W is the mapping torus

(2) In this paper we always restrict our attention to orientation preserving maps, just as in [1] we considered only oriented manifolds. However, Dennis Sullivan has pointed out to me that all the arguments of [1] carry over to nonorientable manifolds with $\pi_1 = Z$, if one changes the definition of a splitting satisfying Poincaré duality to be (in the notation of [1]):

A splitting $A \times I \cup_f Y$ satisfies Poincaré duality if the pair $(Y, A \times 0 \cup A \times 1)$ satisfies relative Poincaré duality (thinking of f as an inclusion, so that $A \times 0 \cup A \times 1 \subset Y$). Hence, the results of this paper may be carried over to orientation reversing diffeomorphisms, using this strengthening of the results of [1].

of the diffeomorphism. The remaining statements follow from the analogous ones in the statement of Theorem (1.1) of [1].

Considering the case of manifolds with boundary, we have

THEOREM 2. *Let $(W^n, \partial W)$ be a 1-connected n-dimensional oriented smooth manifold with 1-connected boundary, $n \geqq 6$. An orientation preserving homotopy equivalence of pairs $h: (W, \partial W) \to (W, \partial W)$ is homotopic to a diffeomorphism of W if and only if there is an oriented k-plane bundle ξ^k over W_h such that $\xi | W = \nu^k$, the normal bundle to W for an embedding in D^{n+k} (with $\partial W \subset \partial D^{n+k}$) and an element $\alpha \in \pi_{n+k+1}(T(\xi), T(\xi|(\partial W)_h))$ such that $q_*(\alpha) \in \pi_{n+k+1}(T(\nu \oplus \varepsilon^1), T(\nu \oplus \varepsilon^1 | \partial W))$ is a normal invariant for W (where q is the natural collapsing as in §0). Further ξ will be the normal bundle of the mapping torus of the diffeomorphism and α will be its normal invariant.*

This follows from Theorem 2.1 of [1], in a way similar to Theorem 1.

A bundle ξ over M_h (or W_h) such that $\xi | M$ (or $\xi | W$) is ν, is given by a bundle equivalence $b: \nu \to \nu$ which covers h. Two such bundles ξ_1, ξ_2 over M_h (or W_h) are isomorphic if and only if the bundle maps b_1 and b_2 are homotopic (as bundle maps). Thus the first condition for h to be conjugate to a diffeomorphism is simply the obvious condition that h is covered by an equivalence of stable normal bundles.

Similarly, the second condition on h reduces to the statement that the equivalence of stable tangent bundles carries a normal invariant for M (or W) into itself.

2. **Pseudo-isotopy of diffeomorphisms.** Two diffeomorphisms f_0 and f_1 of M are called pseudo-isotopic if there is a diffeomorphism $F: M \times [0, 1] \to M \times [0, 1]$ such that $F(x, 0) = (f_0(x), 0)$, $F(x, 1) = (f_1(x), 1)$, for all $x \in M$. We will say f_0 is conjugate to f_1 if f_0 is pseudo-isotopic to $g f_1 g^{-1}$ for some diffeomorphism g. It follows that for $n \geqq 5$, M^n closed and 1-connected, f_0 is conjugate to f_1 if and only if there is an h-cobordism of M with itself and a diffeomorphism $F: U \to U$ which agrees with f_0 on one end and f_1 at the other. Note that f conjugate to the identity implies f is pseudo-isotopic to the identity. The set of pseudo-isotopy classes of orientation preserving diffeomorphisms of M forms a group $\mathscr{D}(M)$.

If f is an orientation preserving diffeomorphism of a smooth n-manifold M, then we may change f by an isotopy so that $f | D^n =$ identity for an embedded closed disk $D^n \subset M^n$, (see [14]). If g is an orientation preserving diffeomorphism of S^n, then we may define a diffeomorphism ψg of M as follows: By an isotopy, make $g | D^n_- =$ identity, and define $\psi g(x) = x$ if $x \in M - D^n$ and $\psi g | D^n = g | D^n_+$. The pseudo-isotopy class of ψg depends only on the pseudo-isotopy class of g, and it is easy to see that ψ defines a homomorphism $\psi: \mathscr{D}(S^n) \to \mathscr{D}(M^n)$, and that image $\psi \subset$ center of $\mathscr{D}(M^n)$. Now the group $\mathscr{D}(S^n)$ is isomorphic to Γ^{n+1}, the group of differential structures on S^{n+1}, which, if $n \geqq 4$, is the same as θ^{n+1}, the group of homotopy $(n+1)$-spheres (see [10]).

LEMMA 1. *Let f be a diffeomorphism of a smooth closed manifold M. If $\Sigma \in \Gamma^{n+1}$, then $M_f \# \Sigma$ is diffeomorphic to $M_{(f\cdot\psi g)}$, where $g \in \mathscr{D}(S^n)$ is the element corresponding to Σ under the isomorphism of $\mathscr{D}(S^n)$ with Γ^{n+1}.*

Proof. $M_f \# \Sigma = (M_f - {}^\circ D_a^{n+1}) \cup (\Sigma - {}^\circ D_-^{n+1})$ where D^{n+1} is an $(n+1)$-cell, ${}^\circ D^{n+1}$ its interior. Now $\Sigma = D_+^{n+1} \cup_g D_-^{n+1}$, where $g \in \mathscr{D}(S^n)$. Hence $M_f \# \Sigma = (M_f - {}^\circ D_a^{n+1}) \cup_g D_+^{n+1}$. Now g may be assumed to leave a disk $D_-^n \subset S^n$ fixed and we may assume that $D_a^{n+1} \cap M^n \times t = D_b^n$ where $f | D_b^n =$ identity. Hence $M_f \# \Sigma = (M_f - {}^\circ D_a^{n+1}) \cup_g D_+^{n+1} = M_{f\cdot\psi g}$.

THEOREM 3. *Let M be a closed oriented 1-connected n-manifold, f_i: $M \to M$, $i = 1, 2$, two orientation preserving diffeomorphisms, ν_i their normal bundles, so ν_i is a bundle over M_{f_i} such that $\nu_i | M = \nu$ the normal bundle of M. Suppose f_1 is homotopic to f_2 so that M_{f_1} is homotopy equivalent to M_{f_2}, and the homotopy equivalence is covered by a bundle map b: $\nu_1 \to \nu_2$ such that $T(b)_*(\alpha_1) = \alpha_2 \in \pi_{n+k+1}(T(\nu_2))$, where α_i is some normal invariant for M_{f_i}. If n is even, $n \geqq 6$, then f_1 is conjugate to $f_2 \cdot \psi g$ for some $g \in \mathscr{D}(S^n)$ corresponding to some element $\Sigma \in \theta^{n+1}(\partial\pi) \subset \theta^{n+1} = \Gamma^{n+1}$.*

This follows from Lemma 1 and [1, (1.3)]. The case of n odd is slightly more complicated.

If f: $M^n \to M^n$ is a diffeomorphism fixed on a cell $D^n \subset M^n$, then f defines in an obvious way a diffeomorphism f': $M \# \Sigma \to M \# \Sigma$ for any $\Sigma \in \theta^n$. If Σ^n is diffeomorphic to S^n then $f' = f$. Any orientation preserving diffeomorphism of M^n is isotopic to one which is fixed on a disk [14]. On the other hand, if g: $M \to M \# \Sigma$ is a diffeomorphism, then any diffeomorphism d of $M \# \Sigma$ induces a diffeomorphism, namely $g^{-1}\, dg$, on M. Hence given a diffeomorphism f on M, we may change it by an isotopy to $\bar{f}$, which is fixed on a disk, and if g: $M \to M \# \Sigma$ is a diffeomorphism, we shall say that f and $g^{-1}\bar{f}'g^{-1}$ *are weakly conjugate*, mod Σ. Note that if $\Sigma^n = S^n$ (or if we are in the p.l. category) weakly conjugate is the same as conjugate. *Also, f weakly conjugate to the identity implies f is pseudo-isotopic to the identity.*

THEOREM 4. *Let f_i: $M^n \to M^n$, $i = 1, 2$, be diffeomorphisms satisfying all the conditions of Theorem 3, but with n odd, $n \geqq 5$. Then f_1 is weakly conjugate to f_2,* mod Σ, *for some $\Sigma \in \theta^n(\partial\pi)$.*

(In particular, if M has the property:

$$M \# \Sigma \text{ diffeomorphic to } M, \quad \Sigma \in \theta^n(\partial\pi) \text{ implies } \Sigma = S^n,$$

(cf. [2]), then in Theorem 4, f_1 is conjugate to f_2.)

Proof of Theorem 4. From [1, Theorem 1.4] we deduce that M_{f_1} is diffeomorphic to $M_{f_2} \#_\beta \Sigma^n$, for some $\Sigma^n \in \theta^n(\partial\pi)$. That is, there is an embedding β: $S^1 \times D^n \subset M_{f_2}$ such that $\beta(S \times 0)$ represents a generator of π_1 such that M_{f_1} is diffeomorphic to $(M_{f_2} - \beta(S^1 \times \operatorname{int} D^n)) \cup (S^1 \times \Sigma^n - S^1 \times \operatorname{int} D^n)$ with the obvious identification along $S^1 \times S^{n-1}$, where $\Sigma^n \in \theta^n(\partial\pi)$, $\Sigma^n = D^n \cup D^{n'}$. By changing f_2 by an isotopy,

we may assume that f_2 is the identity on $D^n \subset M^n$, so that $M_{f_2} \#_\beta \Sigma$ is obviously the mapping torus of the diffeomorphism f_2' of $M \# \Sigma$. Now the result follows from:

LEMMA 2. *Let M_f and M_g' be differentiable fiber bundles over S^1 with projections p and p', and with fibers M^n and M'^n which are 1-connected closed manifolds of dimension $n \geqq 5$. If h: $M_f \to M_g'$ is a diffeomorphism such that $p'h$ is homotopic to p, then h is pseudo-isotopic to h' such that $p'h' = p$, so that h' restricts to a diffeomorphism of M with M'.*

Proof of Lemma 2. We consider the fibration of $M_f \times \{0, 1\}$ given by p on $M_f \times 0$ and $p'h$ on $M_f \times 1$. By the result of [4], this fibration extends to a fibration of all of $M_f \times [0, 1]$, so that the fiber is an h-cobordism between $M \times 0$ and $h^{-1}(M') \times 1$. Using the h-cobordism theorem of Smale on this h-cobordism we see that we have an embedding of $M \times I$ in $M_f \times I$ with $M \times 1 = h^{-1}(M') \times 1$, so that it follows using the h-cobordism theorem again that there is a pseudo-isotopy of the identity of M_f carrying M into $h^{-1}(M')$. Then the composite of this and h is a pseudo-isotopy of h carrying M into M'.

In the bounded case we have the following

THEOREM 5. *Let W^n be a 1-connected n-manifold with nonempty 1-connected boundary, ∂W, $n \geqq 6$. Let f_i: $W \to W$, $i = 1, 2$ be two orientation preserving diffeomorphisms which are homotopic, such that the induced homotopy equivalence of M_{f_1} with M_{f_2} is covered by a bundle equivalence b: $\nu_1 \to \nu_2$, and such that $T(b)_*(\alpha_1) = \alpha_2$ where $\alpha_1 \in \pi_{n+k+1}(T(\nu_1))$ is a normal invariant for f_1 (for M_{f_1}) and α_2 is a normal invariant for f_2 (for M_{f_2}), ($T(b)$ is the induced map of Thom complexes). Then f_1 is conjugate to f_2.*

3. **Examples of diffeomorphisms.** Now we apply Theorem 1 to create some examples of diffeomorphisms.

THEOREM 6. *Let M^n be a 1-connected manifold, closed or with 1-connected boundary, and suppose $H^{4k-1}(M; \boldsymbol{Q}) \neq 0$ for some k, and $p_k(M) = 0$. If M is closed, suppose n is even $\geqq 6$ or $n = 5$, 13. Then there is a diffeomorphism f: $M \to M$ which is homotopic to the identity, but such that f^q is not pseudo-isotopic to the identity for any $q > 0$.*

In fact we shall construct an f using Theorem 1, such that $d(f^q)$ induces a tangential equivalence different from the identity for all $q > 0$.

Since $H^{4k-1}(M; \boldsymbol{Q}) = H^{4k}(\Sigma M^+; \boldsymbol{Q}) \neq 0$, it follows (using the Pontrjagin character) that there is a bundle η over ΣM^+ with nonzero $p_k(\eta)$. Now the group of homotopy classes $[\Sigma M^+, B_F]$ is finite since ΣM^+ is a finite polyhedron and $\pi_i(B_F) = \pi_{i+n-1}(S^n)$ (for large n) is finite for all i. Hence some multiple $m\eta$ is fiber homotopy trivial (see [5]). By the Whitney sum formula $p_k(m\eta) = mp_k(\eta)$, since cup products are zero in ΣM^+.

Now we apply Theorem 1 or 2 using $h=$identity so that $M_h=M\times S^1$, and $\xi^s=(\nu\times\varepsilon^l)+c^*(m\eta)$, where $c\colon M\times S^1\to M\times S^1/M\times t=\Sigma M^+$. Since $m\eta$ is fiber homotopy trivial, $(\nu\times\varepsilon^l)+c^*(m\eta)$ is fiber homotopy equivalent by a map g to $\nu\times\varepsilon^r$ for some r, i.e., to the normal bundle of the identity diffeomorphism. Then a normal invariant ι for the identity map is sent by the induced map of Thom complexes $T(g)$ to an element $T(g)_*(\iota)\in\pi_{n+s+1}(T(\xi))$ which satisfies the hypothesis of Theorem 1 or 2. Hence there is a diffeomorphism f homotopic to the identity with normal bundle ξ.

Now we define an invariant $p'_k\in H^{4k}(\Sigma M^+;\boldsymbol{Q})$ for diffeomorphisms of M homotopic to the identity as follows:

Consider the exact sequence

$$0\longrightarrow H^{4k}(\Sigma M^+;\boldsymbol{Q})\xrightarrow{j^*}H^{4k}(M_d;\boldsymbol{Q})\xrightarrow{i^*}H^{4k}(M;\boldsymbol{Q})\longrightarrow 0$$

(recall that $M_d\cong M\times S^1$). Then $i^*p_k(M_d)=p_k(M)=0$ so $p_k(M_d)=j^*(x)$ and we define $p'_k(d)=x$. Then $p'_k(d)$ is uniquely defined and is nonzero for $d=f^q$ for $q>0$, since $p_k(M_d)\neq 0$ for $d=f^q$.

This proves Theorem 5. Note that p'_k is a p.l. and topological invariant (see [16] and [13]) so that f^q is not p.l. or topologically pseudo-isotopic to the identity.

If $n=4(k+l)-1$, and M is closed, then an additional condition on $L_l(\tau_M)$ will yield a similar result, namely we must assume also that the annihilator of $L_l(\tau_M)$ in $H^{4k-1}(M;\boldsymbol{Q})$ is nonzero.

As an example let us consider $M^n=S^2\times S^3$. Then by Theorem 6 there exist diffeomorphisms homotopic to the identity, which are not pseudo-isotopic to the identity (in the differential, p.l., or topological sense). W.-C. Hsiang has pointed out to me that this also follows from results of Haefliger [7] and Montgomery and Yang [11]. Similar examples of diffeomorphisms of $S^p\times S^q$ have been found by J. Hodgson [9].

In the case of $S^3\times S^3$ such a diffeomorphism f may be constructed as follows: Let $\alpha\colon S^3\to SO(4)$ be such that the induced bundle over S^4 has a nonzero Pontrjagin class p_1 but is fiber homotopically trivial. Then define $f(x,y)=(x,\alpha(x)y)$.

4. Group properties. Denote by $\mathscr{T}(M)$ the group of tangential equivalences of M, i.e., bundle maps of the stable normal bundle of M, up to homotopy as bundle maps.

LEMMA 3. *The elements of $\mathscr{T}(M)$ which cover a fixed map of M to M are isomorphic to the group $[M,SO]=KSO^{-1}(M)$.*

Proof. (Cf. Hirsch-Mazur [8].) Let End(ν) denote the group of bundle maps covering the identity. If ν is the bundle and if ξ is its inverse, define a map

$$i\colon \mathrm{End}(\nu)\to\mathrm{End}(\nu+\xi)\text{ by }i(f)=f+id.$$

Then the composite End(ν) $\to$ End($\nu+\xi$) $\to$ End($\nu+\xi+\nu$) is the same as j: End(ν) $\to$ End($\nu+$trivial bundle) defined by $j(f)=f+id$. If dimension ν is much larger than

dimension M, it follows that j is an isomorphism. On the other hand $\mathrm{End}(\nu+\xi) = \mathrm{End}(\text{trivial bundle}) = [M, SO]$, so that $\mathrm{End}(\nu)$ is a direct summand of $[M, SO]$. A similar argument starting from End(trivial) shows that $\mathrm{End}(\nu) \cong [M, SO]$.

If a, b are two bundle maps covering the same map of base spaces, then ab^{-1} covers the identity, so that Lemma 3 follows.

If $\mathscr{H}(M)$ denotes the group of homotopy equivalences of M, Lemma 2 gives us an exact sequence

$$0 \to [M, SO] \to \mathscr{T}(M) \to \mathscr{H}(M).$$

The image of $\mathscr{T}(M)$ consists of those $h \in \mathscr{H}(M)$ such that $h^*(\nu) = \nu$, or in other words those h which commute with the classifying map of ν.

Now suppose M and ∂M are 1-connected, $\dim M \geqq 5$, if $\partial M = \varnothing$, $\dim M > 5$, if $\partial M \neq \varnothing$.

Theorems 1 and 2 characterize the image of $\mathscr{D}(M)$ in $\mathscr{T}(M)$, provided $n \not\equiv 1 \bmod 4$ or $n = 5$, 13 when $\partial M = \varnothing$.

If $f \in \mathscr{D}(M)$ is in the kernel, i.e., if f induces the identity on the stable normal k-plane bundle ν^k of M, then $M_f \cong M \times S^1$ and $\nu_f = \nu \times (0) = p_1^*(\nu)$. Since ν_f is induced by the projection map p_1 from ν over M, it follows that $T(\nu_f)$ has $T(\nu)$ as a retract. Hence we get the exact sequence

$$0 \longrightarrow \pi_{n+k+1}(T(\nu^k)) \longrightarrow \pi_{n+k+1}(T(\nu_f)) \xrightarrow{q_*} \pi_{n+k+1}(T(\nu^k+\varepsilon^1)) \longrightarrow 0$$

(we are in stable homotopy so the sequence is exact). It follows that the different possible normal invariants for diffeomorphisms homotopic to f are in 1-1 correspondence with kernel $q_* = \pi_{n+k+1}(T(\nu^k))$. It follows from standard homotopy theory that this group is finite, since $H_{n+k+1}(T(\nu^k)) = 0$. Then it follows from Theorems 3, 4, and 5 that each of these elements of $\pi_{n+k+1}(T(\nu))$ corresponds to only a finite number of conjugacy classes of diffeomorphisms, so that the kernel $\mathscr{D}(M) \to \mathscr{T}(M)$ is finite up to conjugacy.

5. **Automorphisms of $S^n \times S^1$.** In this section we point out some easy consequences of Lemma 2 in §2. By automorphism we will mean diffeomorphism or p.l. equivalence, depending on the context.

Proof. By Lemma 2 if h: $M_f \to M_f$ is an automorphism (Lemma 2 holds in both the smooth and p.l. categories), then h is pseudo-isotopic to h' such that $h'(M) = M$. Then clearly $h' \mid M \in \mathscr{C}$ and h is pseudo-isotopic to $\eta(h' \mid M)$.

COROLLARY 1. *If $\mathscr{D}(S^n \times S^1)$ is the group of pseudo-isotopy classes of diffeomorphisms of $S^n \times S^1$, then the sequence*

$$\Gamma^{n+2} + \Gamma^{n+1} \longrightarrow \mathscr{D}(S^n \times S^1) \xrightarrow{\gamma} \boldsymbol{Z}_2 + \boldsymbol{Z}_2 + \boldsymbol{Z}_2 \longrightarrow 0$$

is exact for $n \geqq 5$.

Here $\Gamma^{n+1}=\mathscr{D}(S^n)$ pseudo-isotopy classes of diffeomorphisms of S^n of degree $+1$, and acts by operating only on the fibers $S^n\times t$ in $S^n\times S^1$, and Γ^{n+2} acts by leaving the complement of an $(n+1)$-cell in $S^n\times S^1$ fixed (cf. §2). (Recall that Γ^{n+2} goes into the center of $\mathscr{D}(M)$ for any $(n+1)$-manifold M.)

The first $\boldsymbol{Z}_2+\boldsymbol{Z}_2$ on the right is simply the automorphism group of $H_*(S^n\times S^1)$ if $n>1$, and the map is simply $f\to f_*$. The map into the third $\boldsymbol{Z}_2$ is obtained by taking $f\to w_2(M_f)$. There is a subgroup $\boldsymbol{Z}_2+\boldsymbol{Z}_2+\boldsymbol{Z}_2\subset\mathscr{D}(S^n\times S^1)$ which maps isomorphically onto the group on the right, which is described as follows:

The first and second $\boldsymbol{Z}_2$'s are generated by the orientation reversing diffeomorphisms of S^1 and S^n, while the third $\boldsymbol{Z}_2$ is generated by the map $f(x, y)=(\alpha(y)x, y)$ where the map α: $S^1\to SO(n+1)$ represents a generator of $\pi_1(SO(n+1))=\boldsymbol{Z}_2$. It follows that by composing with these automorphisms we may get an automorphism h: $S^n\times S^1\to S^n\times S^1$ which is the identity in homology and leaves fixed a tubular neighborhood of an embedded $S^1\subset S^n\times S^1$. (This procedure generalizes Gluck [6].)

Lemma 2 of §2 implies that if f_* is the identity on $H_1(S^n\times S^1)$, f is pseudo-isotopic to an automorphism sending $S^n\times t$ into itself, (for some $t\in S^1$).

Corollary 1 now follows by showing that if $\gamma(f)=0$ then we may compose with an automorphism of S^n along all the fibers $S^n\times t$ to get f' which is fixed on a neighborhood of $S^n\times t_0\cup x_0\times S^1$, such that the complement of this neighborhood is an $(n+1)$-cell, so that the resultant diffeomorphism comes from Γ^{n+2}.

Applying the above argument in the p.l. case, where by the theorem of Alexander the p.l. analogues of Γ^{n+1} and Γ^{n+2} are zero we get:

COROLLARY 2. *The group of pseudo-isotopy classes of p.l. automorphisms of $S^n\times S^1$ for $n\geqq 5$, is isomorphic to $\boldsymbol{Z}_2+\boldsymbol{Z}_2+\boldsymbol{Z}_2$.*

This is a higher dimensional analogue of the theorem of Gluck [6] for $S^2\times S^1$. Following Gluck, we get:

COROLLARY 3. *There are at most two different smooth or p.l. knots (S^{n+2}, S^n) $n\geqq 5$, which have equivalent complements.*

Here p.l. knot is taken to mean that S^n has a product neighborhood $S^n\times D^2\subset S^{n+2}$.

The different knots with equivalent complements correspond to the different ways of attaching $S^n\times D^2$ to the complement along $S^n\times S^1$, so that these knots correspond to equivalences of $S^n\times S^1$. But it follows easily that the image of Γ^{n+1} in $\mathscr{D}(S^n\times S^1)$ is extendable to $S^n\times D^2$ so that these yield the same knot. The image of Γ^{n+2} in $\mathscr{D}(S^n\times S^1)$ changes a knot (S^{n+2}, S^n) to a knot (Σ^{n+2}, S^n) where Σ^{n+2} is the corresponding homotopy $(n+2)$-sphere. Hence two knots in S^{n+2} cannot differ by such a diffeomorphism. Finally in the quotient $\boldsymbol{Z}_2+\boldsymbol{Z}_2+\boldsymbol{Z}_2$ (which is the whole thing in the p.l. case), the only element which does not extend to $S^n\times D^2$ is the element arising from $\pi_1(SO(n+1))$, and the corollary follows. (I am indebted to the referee for pointing out the smooth part of this corollary.)

BIBLIOGRAPHY

1. W. Browder, *Manifolds with* $\pi_1 = Z$, Bull. Amer. Math. Soc. **72** (1966), 238–245.

2. ———, *On the action of* $\theta^n(\partial\pi)$, Differential and Combinatorial Topology, a symposium in honor of Marston Morse, Princeton Univ. Press, Princeton, N. J., 1965.

3. W. Browder and M. W. Hirsch, *Surgery on piecewise linear manifolds and applications*, Bull. Amer. Math. Soc. **72** (1966), 959–964.

4. W. Browder and J. Levine, *Fibring manifolds over* S^1, Comment. Math. Helv. **40** (1966), 153–160.

5. A. Dold and R. Lashof, *Principal quasi-fibrations and fibre homotopy equivalence of bundles*, Illinois J. Math. **3** (1959), 285–305.

6. H. Gluck, *The embedding of two-spheres in the four-sphere*, Bull. Amer. Math. Soc. **67** (1961), 586–589.

7. A. Haefliger, *Knotted* $(4k-1)$*-spheres in* $6k$*-space*, Ann. of Math. (2) **75** (1962), 452–466.

8. M. Hirsch and B. Mazur, *Smoothings of piecewise linear manifolds*, Mimeographed, Cambridge Univ., Cambridge, 1964.

9. J. Hodgson, Thesis, Cambridge Univ., Cambridge, 1966.

10. M. Kervaire and J. Milnor, *Groups of homotopy spheres*. I, Ann. of Math. (2) **77** (1963), 504–537.

11. D. Montgomery and C. T. Yang, *Differentiable actions on homotopy seven spheres*, Trans. Amer. Math. Soc. **122** (1966), 480–498.

12. S. P. Novikov, *Homotopy equivalent smooth manifolds*. I, Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 365–474; Amer. Math. Soc. Transl. (2) **48** (1965), 271–396.

13. ———, *On manifolds with a free abelian fundamental group and their applications (Pontrjagin classes, smoothings, multidimensional knots)*, Izv. Akad. Nauk SSSR **30** (1966), 207–246. (Russian)

14. R. Palais, *Extending diffeomorphisms*, Proc. Amer. Math. Soc. **11** (1960), 274–277.

15. S. Smale, *On the structure of manifolds*, Amer. J. Math. **84** (1962), 387–399.

16. R. Thom, *Classes de Pontrjagin des variétés triangulées*, Colloq. Internat. Topological Algebra, Mexico, 1958.

PRINCETON UNIVERSITY,
PRINCETON, NEW JERSEY