

ON A CLASS OF NODAL NONCOMMUTATIVE JORDAN ALGEBRAS

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1. A finite-dimensional power-associative algebra A with unity element 1 over a field F is a nodal algebra [6] if each element x of A may be written $x = \alpha 1 + z$, where α is in F and z is nilpotent, and if A is not of the form $A = F1 + N$ where N is a nil subalgebra of A . The class K , of p^s -dimensional nodal noncommutative Jordan algebras we will consider in this paper is described as follows: let

$$B_n = F[1, x_1, \dots, x_n] \quad \text{with } x_i^p = 0, \quad x_i^0 = 1, \quad \text{for } i = 1, \dots, n,$$

be a commutative associative truncated polynomial algebra over the field F of characteristic $p \neq 2$. An algebra A in K of dimension p^n is taken to be the same vector space as B_n but with multiplication given by the product

$$fg = f \cdot g + \sum_{i,j=1}^n \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_j} \cdot c_{ij},$$

where $f \cdot g$ is the product of $f = f(x_1, \dots, x_n)$ and $g = g(x_1, \dots, x_n)$ in B_n and the $c_{ij} = \frac{1}{2}[x_i, x_j] = \frac{1}{2}(x_i x_j - x_j x_i)$ are arbitrary except for the proviso that at least one of them is nonsingular. That is, there must exist a $c_{ij} = \alpha_{ij} 1 + w_{ij}$ with $\alpha_{ij} \neq 0$. This implies that $n \geq 2$.

The class K was constructed by L. Kokoris who proved [2], [3] that every simple nodal noncommutative Jordan algebra is in K . He also proved that not all the algebras of the class K are simple.

Two papers have appeared which were concerned with studying derivations of some of the algebras in K . The first of these by R. Schafer [7] described derivation algebras for the cases c_{ij} in F and $n=2$ and demonstrated relationships between certain ideals of these algebras and types of simple Lie algebras of characteristic p . He made use of two properties which were later generalized by R. Oehmke, namely that the algebras in K for the cases c_{ij} in F and $n=2$ are Lie-admissible and that, for $n=2$, the generators x_1 and x_2 could be chosen so that $c_{12} = 1 + \alpha x_1^{p-1} \cdot x_2^{p-1}$ for α in F .

In his paper [4], Oehmke determined the derivation algebras of all simple Lie-admissible algebras A of K . He proved that generators could be chosen for these A which satisfy a useful Lie-multiplication table, which is a generalization of

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Schafer’s relation for the Lie product of two generators. We call this multiplication table an Oehmke multiplication table in this paper.

The theorems in our §2 and 3, together with [4] show that existence of generators satisfying Oehmke-type multiplication relations is a necessary and sufficient condition for simplicity and Lie-admissibility of an algebra in K . The result of §3 concerning the Lie-admissibility of algebras of K having an Oehmke multiplication table is applied to the construction of the example of §4. This example is one of a simple algebra in K which is not Lie-admissible and which cannot have an Oehmke multiplication table. Thus, the converse to Theorem 1 is not true and something (Lie-admissibility here) must be added to provide a necessary and sufficient condition for simplicity in this context.

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2. We will say that an algebra $A = F1 + F[x_1, \dots, x_n]$ (vector space direct sum) in K has an Oehmke multiplication table (OMT) if the generators x_1, \dots, x_n can be chosen such that for n even, $n = 2r$, we have:

$$(1) \quad \begin{aligned} [x_i, x_{i+r}] &= 1 + \alpha_i x_i^{p-1} \cdot x_{i+r}^{p-1} && \text{for } i = 1, \dots, r, \alpha_i \text{ in } F, \\ [x_i, x_j] &= 0 && \text{for } j \neq i+r, \end{aligned}$$

and for n odd, $n = 2r + 1$, we have (1) holding and

$$(2) \quad \begin{aligned} [x_{2r+1}, x_j] &= 0 && \text{for } j \neq r, 2r, \\ [x_{2r+1}, x_{2r}] &= \alpha x_r^{p-1} \cdot (1 + \beta x_{2r+1}^{p-1}), \\ [x_{2r+1}, x_r] &= x_{2r}^{p-1} \cdot (1 + \beta x_{2r+1}^{p-1}) && \text{for } \alpha, \beta \text{ in } F. \end{aligned}$$

Here, as in the preceding section, we have used the notation, $[x, y]$, for the commutator $xy - yx$ of x and y . If we denote the mapping $x \rightarrow [x, y]$ by $D(y)$, i.e., $x D(y) = [x, y]$, then the fact that $D(y)$ is a derivation of A^+ for every y in A is a well-known consequence of the flexible identity.

In [4], Oehmke proved that a simple Lie-admissible algebra in K has an OMT.

We further remark here that B is an ideal of A if and only if B is an ideal of both A^+ and A^- .

THEOREM 1. *Let A be an algebra in K . If A has an OMT, then A is simple.*

Proof. Let B be a nonzero ideal of A , for A of dimension p^n . Because of the nature of (1) and (2) we must break up the proof into cases of even and odd numbers of generators.

We first consider an algebra A with an even number, $n = 2r$, of generators for which we can assert the following, due to Oehmke [4, pp. 425–426]. If m is any monomial in B and a generator x_i occurs (to any power) in m , then x_i is in B . We reproduce the proof of this claim here, because of typographical errors in [4].

Suppose $m = x_i^k \cdot q$ where q is independent of x_i . Since B is an ideal of A^- , $[m, x_{i+r}]$ is in B . (If $i > r$, Lie multiplication by x_{i-r} will give us the result.) But

$$\begin{aligned}
 [m, x_{i+r}] &= mD(x_{i+r}) = kx_i^{k-1} \cdot x_i D(x_{i+r}) \cdot q + x_i^k \cdot qD(x_{i+r}) \\
 &= kx_i^{k-1} \cdot (1 + \alpha_i x_i^{p-1} \cdot x_{i+r}^{p-1}) \cdot q = kx_i^{k-1} \cdot q \text{ if } k > 1.
 \end{aligned}$$

Therefore, we might as well suppose that $k = 1$ and that $m = x_i \cdot q$ is in B .

Using an induction argument and the above method, m may be assumed to be just a product of generators, each with exponent = 1.

We may further assume that if the generator x_k appears in m for $k \leq r$, then x_{k+r} does not appear. For, if $m = x_k \cdot x_{k+r} \cdot s$, then $mD(x_k) = -x_k \cdot s$ is in B .

The above considerations justify taking m to be of the form $m = x_i \cdot x_j \cdot \dots \cdot x_k \cdot x_t$, where the subscript of each of the distinct generators is less than or equal to r .

Hence $mD(x_{i+r}) = x_i \cdot x_j \cdot \dots \cdot x_k \cdot x_t D(x_{i+r})$ is in B . But, B an ideal of A^+ and $x_t D(x_{i+r})$ nonsingular imply that $x_i \cdot x_j \cdot \dots \cdot x_k$ is in B . Continuing this process, we find that $x_i \cdot x_j$ is in B . Thus,

$$[(x_i \cdot x_j)D(x_{j+r})] \cdot (x_j D(x_{j+r}))^{-1} = x_i$$

is in B , which proves our claim.

Now we introduce a result of A. A. Albert [1, Lemma 21]: the monomial $m = x_1^{p-1} \cdot \dots \cdot x_{2r}^{p-1}$ is in B . By the claim then, B contains every generator, hence $B = A$. This proves Theorem 1 for $n = 2r$.

We now turn to the proof of Theorem 1 for the case of $n = 2r + 1$ generators. Using the above quoted result of Albert, the maximal degree monomial

$$x_1^{p-1} \cdot \dots \cdot x_{2r+1}^{p-1}$$

is in B . Making use of the multiplication table (1), it is clear that application of the operators $D(x_1), \dots, D(x_{r-1}), D(x_{r+1}), \dots, D(x_{2r-1})$ to this monomial will, just as in the even case above, guarantee that the monomial

$$m = x_1 \cdot \dots \cdot x_{r-1} \cdot x_r^{p-1} \cdot x_{r+1} \cdot \dots \cdot x_{2r-1} \cdot x_{2r}^{p-1} \cdot x_{2r+1}^{p-1}$$

is in B .

Noting that $x_k D(x_{1+r}) = 0$ for $k \neq 1$, we then see that

$$mD(x_{1+r}) \cdot [x_1, x_{1+r}]^{-1} = x_2 \cdot \dots \cdot x_{r-1} \cdot x_r^{p-1} \cdot x_{r+1} \cdot \dots \cdot x_{2r-1} \cdot x_{2r}^{p-1} \cdot x_{2r+1}^{p-1}$$

is in B . The terms $x_i, i = 2, \dots, r-1, r+1, \dots, 2r-1$ are eliminated in the same manner, thus showing that $x_r^{p-1} \cdot x_{2r}^{p-1} \cdot x_{2r+1}^{p-1}$ is in B .

The problem has now been reduced to the case of the ideal B containing a maximal degree monomial in the three generators x_r, x_{2r} , and x_{2r+1} . The technique for completion of the proof will be easier from the standpoint of both understanding and typography if we illustrate it by just showing that an algebra in K with three generators x, y, z satisfying (1) and (2) is simple.

Therefore, we will assume it is clear that the methods to come may be carried over to the general case and we will take $A = F1 + F[x, y, z]$ in K satisfying

$$\begin{aligned}
 (3) \quad & [y, x] = 1 + \gamma x^{p-1} \cdot y^{p-1}, \quad \gamma \text{ in } F, \\
 & [z, y] = \alpha x^{p-1} \cdot (1 + \beta z^{p-1}), \\
 & [z, x] = y^{p-1} \cdot (1 + \beta z^{p-1}),
 \end{aligned}$$

and prove that such an algebra A is simple.

We have $x^{p-1} \cdot y^{p-1} \cdot z^{p-1}$ is in B . Thus there is an integer m = the minimal exponent such that $x^{p-1} \cdot y^{p-1} \cdot z^m$ is in B .

Denote the s th iterate of the mapping $D(x)$ by $D^s(x)$. It may be verified, by virtue of (3), that the following two equations hold:

$$(4) \quad (x^i \cdot y^j \cdot z^k) D^s(x) = P(j, s) x^i \cdot y^{j-s} \cdot z^k$$

and

$$(5) \quad (x^i \cdot y^j \cdot z^k) D^t(y) = \delta P(i, t) x^{i-t} \cdot y^j \cdot z^k,$$

where $i, j > 0, k = 0, 1, \dots, p-1, \delta = \pm 1$, and $P(h, q) = h! / (h-q)!$.

It is clear from (4) and (5) that $x^a \cdot y^b \cdot z^m$ is in B for any a, b such that $1 \leq a, b \leq p-1$. Operating on $x \cdot y \cdot z^m$ by $D(x)$, we have $x \cdot z^m$ in B .

Suppose that $m \geq 1$, and consider the calculation:

$$\begin{aligned}
 (x \cdot z^m) D(x^2) &= 2mx^2 \cdot z^{m-1} \cdot z D(x) \\
 &= 2mx^2 \cdot y^{p-1} \cdot z^{m-1} \quad \text{for } m \geq 2 \\
 &= 2x^2 \cdot y^{p-1} \cdot z^{m-1} + 2\beta x^2 \cdot y^{p-1} \cdot z^{p-1} \quad \text{for } m = 1.
 \end{aligned}$$

Since $x^2 \cdot y^{p-1} \cdot z^{p-1}$ is in B , either of the above two equations implies that $x^2 \cdot y^{p-1} \cdot z^{m-1}$ is in B and hence that $x^{p-1} \cdot y^{p-1} \cdot z^{m-1}$ belongs to B for $m \geq 1$, contrary to the minimality of m . Therefore $m = 0$ and $x^{p-1} \cdot y^{p-1}$ is in B , thus $x \cdot y$ is in B .

Hence the nonsingular element $(x \cdot y) D(x) D(y) = x D(y)$ is in B and $B = A$. Therefore A is simple and the proof of Theorem 1 is complete. The remarks of the introduction yield the following

COROLLARY. *Let A be in K and suppose either all c_{ij} are in F or $n = 2$. A is simple if and only if A has an OMT.*

3. We now examine the Lie-admissibility of algebras A in K having an OMT. The chief tool used in these considerations will be the criteria of R. Schafer [7, p. 318] which we state in the following manner.

An algebra A in K of dimension p^n is Lie-admissible if and only if

$$(6) \quad \sum_{i=1}^n Q_i = 0$$

where

$$(7) \quad Q_t = \partial c_{ij} / \partial x_t \cdot c_{tk} + \partial c_{jk} / \partial x_t \cdot c_{ti} + \partial c_{ki} / \partial x_t \cdot c_{tj}$$

with $2c_{ij} = [x_i, x_j]$ and $1 \leq i < j < k \leq n$.

Suppose A is in K and has an even number, $n=2r$, of generators having an OMT. Since we intend to apply (6) and (7) to this A , we examine Q_t where we now must have

$$1 \leq i < j < k \leq 2r.$$

No generality is lost by assuming $t \leq r$, since a symmetrical result holds for $t \geq r+1$.

The only nonzero c_{iq} occurs when $q=t+r$. Suppose that $c_{tk} = c_{t,t+r}$. (Similar reasoning holds for $i=t+r$ and $j=t+r$.) Therefore $c_{ti} = c_{tj} = 0$. Hence, from (7), $Q_t = \partial c_{ij} / \partial x_t \cdot c_{t,t+r}$.

But $c_{ij} = 0$ unless $j=i+r$ (or $i=j-r$). Thus,

$$Q_t = \frac{\partial c_{i,i+r}}{\partial x_t} \cdot c_{t,t+r} = \frac{1}{2} \alpha_i \frac{\partial (x_i^{p-1} \cdot x_{i+r}^{p-1})}{\partial x_t} \cdot c_{t,t+r}.$$

However, we can assert that $t \neq i$, for otherwise we would have $k=j$, a contradiction, and also we have $t \neq i+r$, for if $t=i+r$, then $k=i+2r$, which is impossible. (Similarly, $t \neq j$ and $t \neq j-r$.)

Therefore the quantity being differentiated is not a function of x_t , which implies $Q_t = 0$. Thus, using (6), A^- is a Lie algebra and we have proved a portion of the following theorem.

THEOREM 2. *Let A be an algebra in K . If A has an OMT, then A is Lie-admissible.*

Proof. To deal with the second portion of this theorem, the case of $2r+1$ generators,

$$1 \leq i < j < k \leq 2r+1$$

must hold where we intend to apply (6) and (7). We see that $Q_t = 0$ for $t \neq r, 2r$, or $2r+1$, since for t different from these values the generators concerned satisfy a multiplication of the form (1). The detailed reasoning involves considerations similar to the even case proof of Theorem 2.

The quantities Q_r, Q_{2r} , and Q_{2r+1} will now be considered separately.

We can exhaust the value possibilities for Q_r by considering the four mutually exclusive cases which arise when $k=2r+1$ and either $i=r$ or $j=r$ or $i \neq r$ and $j \neq r$; or when $k \leq 2r$.

Calculations will show that $Q_r = 0$ in all these cases; we detail some samples of the reasoning for purposes of illustration.

If $k=2r+1$ and $i=r$, then using (7) we find that

$$Q_r = \partial c_{rj} / \partial x_r \cdot c_{r,2r+1} + \partial c_{2r+1,r} / \partial x_r \cdot c_{rj}.$$

Now $c_{2r+1,r}$ is not a function of x_r , hence

$$Q_r = \partial c_{rj} / \partial x_r \cdot c_{r,2r+1}.$$

The only nonzero possibility for Q_r would be for $j=2r$, since we cannot have $j=2r+1=k$. Making the computations for these index values we obtain

$$\begin{aligned} Q_r &= \frac{\partial}{\partial x_r} \left[\frac{1}{2}(1 + \alpha_r x_r^{p-1} \cdot x_{2r}^{p-1}) \right] \cdot \left(-\frac{1}{2} x_{2r}^{p-1} \cdot (1 + \beta x_{2r+1}^{p-1}) \right) \\ &= \frac{1}{4} \alpha_r x_r^{p-2} \cdot x_{2r}^{p-1} \cdot x_{2r}^{p-1} \cdot (1 + \beta x_{2r+1}^{p-1}) = 0. \end{aligned}$$

As a last example, we consider the case in which $k=2r+1$, $i \neq r$, and $j \neq r$. Here we have

$$Q_r = \frac{\partial c_{ij}}{\partial x_r} \cdot c_{r,2r+1} + \frac{\partial c_{j,2r+1}}{\partial x_r} \cdot c_{ri} + \frac{\partial c_{2r+1,i}}{\partial x_r} \cdot c_{rj}.$$

Notice that $Q_r=0$ for $j=2r$, since we cannot have $i=2r=j$ and $i=r$ is disallowed by the case.

For $j < 2r$, the above equation yields

$$Q_r = \partial c_{ij} / \partial x_r \cdot c_{r,2r+1},$$

and we note that c_{ij} cannot be a function of x_r in these circumstances. Thus $Q_r=0$ here.

We can demonstrate that $Q_{2r} = Q_{2r+1} = 0$ in much the same manner. The cases which exhaust the value possibilities for Q_{2r} are $k=2r+1$ and either $i=r$ or $j=r$ or $i \neq r$ and $j \neq r$; or $k \leq 2r$. We need only three mutually exclusive cases to represent all the possibilities for Q_{2r+1} . We could have $k=2r+1$ and either $j=2r$ or $j < 2r$; or have $k \leq 2r$.

The verification that $Q_{2r+1}=0$ is immediate by inspection in all but the first case: $k=2r+1$ and $j=2r$. Here we see

$$Q_{2r+1} = \frac{\partial c_{2r,2r+1}}{\partial x_{2r+1}} \cdot c_{2r+1,i} + \frac{\partial c_{2r+1,i}}{\partial x_{2r+1}} \cdot c_{2r+1,2r}$$

which may admit a nonzero possibility for $i=r$. However, we then find

$$\begin{aligned} 4Q_{2r+1} &= \frac{\partial(-\alpha x_r^{p-1} \cdot (1 + \beta x_{2r+1}^{p-1}))}{\partial x_{2r+1}} \cdot x_{2r}^{p-1} \cdot (1 + \beta x_{2r+1}^{p-1}) \\ &\quad + \frac{\partial(x_{2r}^{p-1} \cdot (1 + \beta x_{2r+1}^{p-1}))}{\partial x_{2r+1}} \cdot \alpha x_r^{p-1} \cdot (1 + \beta x_{2r+1}^{p-1}) \\ &= \alpha \beta x_r^{p-1} \cdot x_{2r+1}^{p-2} \cdot x_{2r}^{p-1} \cdot (1 + \beta x_{2r+1}^{p-1}) \\ &\quad - \alpha \beta x_{2r}^{p-1} \cdot x_{2r+1}^{p-2} \cdot x_r^{p-1} \cdot (1 + \beta x_{2r+1}^{p-1}) \\ &= 0. \end{aligned}$$

Therefore $Q_i=0$ for $1 \leq i \leq 2r+1$ and Theorem 2 is proved.

4. We are now in a position to state the next theorem which makes it clear that simplicity alone is insufficient for an OMT.

THEOREM 3. *There exist simple algebras of K which are not Lie-admissible and which cannot have an OMT.*

Proof. Let x_1, \dots, x_{2R} for $R > 1$, be $2R$ quantities and let F be a field of characteristic $p \neq 2$. Let M be the vector space over F with x_1, \dots, x_{2R} as basis and let $a = a(x, y)$ be a skew-symmetric bilinear form on M with $a(x_i, x_j) = \alpha_{ij} = -\alpha_{ji}$ whose rank is equal to $2R$.

Define A^+ by $A^+ = F1 + N^+$ where $N^+ = F[x_1, \dots, x_{2R}]$, with $x_i^p = 0, i = 1, \dots, 2R$. For convenience write $x_i^0 = 1, i = 1, \dots, 2R$. That is, A^+ is a commutative associative truncated polynomial algebra with nilpotent generators. Now define $A = F1 + N$ to be the same vector space as A^+ with generator products defined by $x_i x_j = \alpha_{ij} 1 + w_{ij}$, with $w_{ij} = 2x_i \cdot x_j - w_{ji}$ in N for $i < j$. Further, define

$$fg = f \cdot g + \frac{1}{2} \sum_{i,j=1}^{2R} \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_j} \cdot [x_i, x_j]$$

for any elements f and g of A .

We note here that Kokoris [3, Theorems 5 and 6] has proved that the above conditions and construction of A are sufficient for A to be a simple nodal noncommutative Jordan algebra in the class K .

Finally, complete the definition of A by setting $w_{23} = w_{32}, w_{13} = w_{31}$, and choosing w_{12} such that $[w_{12}, x_3] \neq 2(\alpha_{13}x_2 + \alpha_{23}x_1)$.

We shall show that A as defined above cannot be Lie-admissible, since the Jacobi identity in A^- is not satisfied. For consider the quantity

$$\begin{aligned} J &= [[x_1, x_2], x_3] + [[x_2, x_3], x_1] + [[x_3, x_1], x_2] \\ &= [w_{12} - w_{21}, x_3] = 2[w_{12} - x_1 \cdot x_2, x_3]. \end{aligned}$$

It has been previously noted that

$$[x_1 \cdot x_2, x_3] = [x_1, x_3] \cdot x_2 + x_1 \cdot [x_2, x_3] = 2(\alpha_{13}x_2 + \alpha_{23}x_1).$$

Thus

$$J = 2[w_{12}, x_3] - 4(\alpha_{13}x_2 + \alpha_{23}x_1) \neq 0$$

and A^- is not a Lie algebra.

Now suppose that there exists a set of generators y_1, \dots, y_{2R} for A satisfying the relations (1). By Theorem 2, we must have A Lie-admissible, which is a contradiction. Theorem 3 is proved.

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