ON HOMOGENEOUS SPACES AND REDUCTIVE SUBALGEBRAS OF SIMPLE LIE ALGEBRAS

BY

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1. Introduction. Let $G$ be a connected Lie group and $H$ a closed subgroup. Then the homogeneous space $M=G/H$ is called reductive if in the Lie algebra $\mathfrak{g}$ of $G$ there exists a subspace $\mathfrak{m}$ such that $\mathfrak{g}=\mathfrak{m}+\mathfrak{h}$ (subspace direct sum) and $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ where $\mathfrak{h}$ is the Lie algebra of $H$ (see [4], [5]). In this case the pair $(\mathfrak{g}, \mathfrak{h})$ is called a reductive pair and the subspace $\mathfrak{m}$ can be made into an anti-commutative algebra as follows. For $X, Y \in \mathfrak{m}$ let $\mathcal{X}(X, Y)=XY+[X, Y]_m$ (resp. $\mathcal{H}(X, Y)=\pi(X, Y)_h$) is the projection of $[X, Y]$ in $\mathfrak{g}$ into $\mathfrak{m}$ (resp. $\mathfrak{h}$). This algebra is related to the canonical $G$-invariant connection $\nabla$ of the first kind on $G/H$ by $\nabla_{\mathcal{X}}(Y)=\pi XY$ where $\pi=H \in M$ (see [5, Theorem 10.1]).

For a fixed decomposition $\mathfrak{g}=\mathfrak{m}+\mathfrak{h}$, the Lie algebra identities of $\mathfrak{g}$ yield the following identities for $\mathfrak{m}$ and $\mathfrak{h}$. For $X, Y, Z \in \mathfrak{m}$ and $U \in \mathfrak{h}$,

(1) $XY=-YX$ (bilinear);
(2) $\mathcal{H}(X, Y)=-\mathcal{H}(Y, X)$ (bilinear);
(3) $[Z, \mathcal{H}(X, Y)] + [X, \mathcal{H}(Y, Z)] + [Y, \mathcal{H}(Z, X)] = \mathcal{J}(X, Y, Z) = (XY)Z + (YZ)X + (ZX)Y$;
(4) $\mathcal{H}(XY, Z) + \mathcal{H}(YZ, X) + \mathcal{H}(ZX, Y) = 0$;
(5) $\mathcal{H}([X, Y], U) = \mathcal{H}([X, U], Y) + \mathcal{H}(X, [Y, U])$;

In particular (6) says the mappings $\text{ad}_{m} U: \mathfrak{m} \to \mathfrak{m}: X \to [U, X]$ are derivations of the algebra $\mathfrak{m}$. Using these identities, there was established in [6] a correspondence between simple algebras $\mathfrak{m}$ and holonomy irreducible simply connected spaces $G/H$ which are not symmetric ($\mathfrak{m} \mathfrak{m} = 0$ if and only if $G/H$ is a symmetric space); for example, if $G/H$ is riemannian, then $G/H$ is holonomy irreducible if and only if $\mathfrak{m}$ is a simple algebra.

In this paper, we consider pairs $(\mathfrak{g}, \mathfrak{h})$ where $\mathfrak{g}$ is a simple Lie algebra over a field $F$ of characteristic zero and $\mathfrak{h}$ is either semisimple, or regular and reductive (see [2]). In each case we show that the associated $\mathfrak{m}$ is either simple or abelian ($\mathfrak{m}^2 = 0$). This together with [6] shows in particular that if $G$ is a simple connected Lie group and $H$ a closed semisimple or regular reductive Lie subgroup of $G$ such that $G/H$ is simply connected, then either $G/H$ is a symmetric space or $G/H$ is holonomy irreducible. This is a reasonable account of the situation since it can be shown that

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if $G/H$ is a holonomy irreducible pseudo-riemannian reductive space with $G$ simple, then $\mathfrak{h}$ is a reductive subalgebra of $\mathfrak{g}$.

2. The regular reductive case.

**Lemma 1.** Let $\mathfrak{a}$ be a nonassociative algebra with derivation algebra $\text{Der} \mathfrak{a}$. Assume that $\mathfrak{a}$ has no proper ideal stable under $\text{Der} \mathfrak{a}$. Then either $\mathfrak{a}$ is simple or $\mathfrak{a}^2 = 0$.

**Proof.** Assume $\mathfrak{a}^2 \neq 0$ and let $\mathfrak{T}(\mathfrak{a})$ denote the associative algebra generated by the left and right multiplications of $\mathfrak{a}$ [3, p. 290]. Let $\mathfrak{R}$ be the radical of $\mathfrak{T}(\mathfrak{a})$. Then $\mathfrak{R}\mathfrak{a}$ is an ideal of $\mathfrak{a}$ since $\mathfrak{T}(\mathfrak{a})/(\mathfrak{R}\mathfrak{a}) \leq \mathfrak{T}(\mathfrak{a})$ since $\text{ad}_{\text{Hom}(\mathfrak{a}, \mathfrak{a})} \mathfrak{D}$ stabilizes the set of right and left multiplications (e.g., $[\mathfrak{D}, L(\mathfrak{A})] = L(D(\mathfrak{A}))$ where $L(\mathfrak{B})$ denotes left multiplication by $\mathfrak{B}$ in $\mathfrak{a}$). Thus $\text{ad}_{\mathfrak{T}(\mathfrak{a})} \mathfrak{D}$ is a derivation of $\mathfrak{T}(\mathfrak{a})$ and it follows that $[\mathfrak{D}, \mathfrak{R}] \leq [\mathfrak{D}, \mathfrak{R}][3, \text{ p. 30, exercise 22}]$. Thus $[\mathfrak{D}, \mathfrak{R}] \leq [\mathfrak{D}, \mathfrak{R}] + [\mathfrak{R}(\mathfrak{A}), \mathfrak{R}]$. Thus $\mathfrak{R}$ is a $\text{Der} \mathfrak{a}$-stable ideal of $\mathfrak{a}$. By assumption, we must have $\mathfrak{R}=\mathfrak{a}$ or $\mathfrak{R}=0$. If $\mathfrak{R}=\mathfrak{a}$, then for some $i$, $0=\mathfrak{R}^i=\mathfrak{R}^{i-1}a=\cdots=\mathfrak{R}^i=\mathfrak{a}$ and $a=0$. Thus we may assume that $\mathfrak{R}=0$. Then $\mathfrak{R}=0$ and $\mathfrak{T}(\mathfrak{a})$ is completely reducible on $\mathfrak{a}$. $\mathfrak{a}^2$ is clearly $\text{Der} \mathfrak{a}$-stable. Assuming that $\mathfrak{a}^2 \neq 0$, we must have $\mathfrak{a}^2 = \mathfrak{a}$ by hypothesis. We claim that $\mathfrak{a}^2 = \mathfrak{a}$ implies that $\mathfrak{a}$ is simple. For if $\mathfrak{b}$ were a proper ideal of $\mathfrak{a}$, then $\mathfrak{b}$ would be $\mathfrak{T}(\mathfrak{a})$-stable and hence $a = b \oplus b'$ for some $\mathfrak{T}(\mathfrak{a})$-stable $b'$. This $b'$ would be an ideal and $\mathfrak{a} = \mathfrak{a}^2 = b^2 + (b')^2$ shows that $b^2 = b$. But then $b = b^2$ would be $\text{Der} \mathfrak{a}$-stable since for $B_1, B_2$ in $\mathfrak{b}$, $D(B_1B_2) = (D(B_1))B_2 + B_1(D(B_2)) \in \mathfrak{b}$. Thus $\mathfrak{a}$ is simple.

We now consider reductive pairs $(\mathfrak{g}, \mathfrak{h})$. Thus let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h}$ a Lie subalgebra of $\mathfrak{g}$, $\mathfrak{m}$ a complementary subspace of $\mathfrak{h}$ in $\mathfrak{g}$ such that $[\mathfrak{m}, \mathfrak{h}] \leq \mathfrak{m}$. For $\mathfrak{X}, \mathfrak{Y} \in \mathfrak{m}$ we define $XY$ in $\mathfrak{m}$ and $\mathfrak{h}(\mathfrak{X}, \mathfrak{Y})$ in $\mathfrak{h}$ by requiring that $[XY] = XY + \mathfrak{h}(\mathfrak{X}, \mathfrak{Y})$. We regard $\mathfrak{m}$ as a nonassociative algebra with respect to the product $XY$. Then $\mathfrak{m}$ is clearly anti-commutative and $\text{ad}_\mathfrak{m} \mathfrak{U}$ is a derivation of $\mathfrak{m}$ for $\mathfrak{U} \in \mathfrak{h}$ (by (6)).

**Lemma 2.** Let $\mathfrak{n}$ be an $\mathfrak{ad} \mathfrak{h}$-stable ideal of $\mathfrak{m}$. Let $\mathfrak{q} = \mathfrak{n} + \mathfrak{h}(\mathfrak{n}, \mathfrak{n})$. If $[\mathfrak{n}, \mathfrak{n}'] \leq \mathfrak{q}$ for some complementary subspace $\mathfrak{n}'$ of $\mathfrak{n}$ in $\mathfrak{m}$, then $\mathfrak{q}$ is an ideal of $\mathfrak{g}$.

**Proof.** $[\mathfrak{q}, \mathfrak{n}] \leq [\mathfrak{n}, \mathfrak{n}] + [\mathfrak{h}(\mathfrak{n}, \mathfrak{n}), \mathfrak{n}] \leq \mathfrak{nn} + \mathfrak{h}(\mathfrak{n}, \mathfrak{n}) + \mathfrak{n}$ by (3) since $\mathfrak{n}$ is ad $\mathfrak{h}$-stable. Thus $[\mathfrak{q}, \mathfrak{n}] \leq \mathfrak{q}$. And $[\mathfrak{q}, \mathfrak{h}] \leq \mathfrak{q}$ since $\mathfrak{q}$ is an ad $\mathfrak{h}$-stable and $\mathfrak{q} = \mathfrak{n} + [\mathfrak{n}, \mathfrak{n}]$. It remains to show that $[\mathfrak{q}, \mathfrak{n}'] \leq \mathfrak{q}$. But we have

$$[\mathfrak{q}, \mathfrak{n}'] \leq \mathfrak{nn}' + \mathfrak{h}(\mathfrak{n}, \mathfrak{n}') + [\mathfrak{h}(\mathfrak{n}, \mathfrak{n}), \mathfrak{n}'],$$

$$[\mathfrak{h}(\mathfrak{n}, \mathfrak{n}), \mathfrak{n}'] \leq [\mathfrak{nn}, \mathfrak{n}'] + [\mathfrak{n}, \mathfrak{n}'] \leq [\mathfrak{n}, \mathfrak{n}'] + [\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}']].$$

$$\mathfrak{h}(\mathfrak{n}, \mathfrak{n}') \leq \mathfrak{nn}' + [\mathfrak{n}, \mathfrak{n}'].$$

But since $[\mathfrak{n}, \mathfrak{n}'] \leq \mathfrak{q}$ by hypothesis, $\mathfrak{q}$ contains $[\mathfrak{h}(\mathfrak{n}, \mathfrak{n}), \mathfrak{n}']$ (using (3)) and $\mathfrak{h}(\mathfrak{n}, \mathfrak{n}')$. Since $\mathfrak{nn}' \leq \mathfrak{n}$ (n is an ideal of $\mathfrak{m}$), $[\mathfrak{q}, \mathfrak{n}'] \leq \mathfrak{q}$. Thus $\mathfrak{q}$ is an ideal of $\mathfrak{g}$.

**Lemma 3.** Suppose that the Killing form $B(\cdot, \cdot)$ of $\mathfrak{g}$ is nondegenerate and that $B(\cdot, \mathfrak{h}) = 0$. Then $B(\cdot, \mathfrak{m})$ is nondegenerate and invariant, i.e., $B(\mathfrak{XY}, \mathfrak{Z}) = B(\mathfrak{X}, \mathfrak{YZ})$. Moreover every ad $\mathfrak{h}$-stable ideal $\mathfrak{n}$ of $\mathfrak{m}$ satisfies $[\mathfrak{n}, \mathfrak{n}'] = 0$ where $\mathfrak{n}' = \{\mathfrak{X} \in \mathfrak{m} | B(\mathfrak{X}, \mathfrak{n}) = 0\}$. 

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Proof. For $X, Y, Z \in m$ we have:

$$B(X, Y, Z) = B([X, Y], Z) = B(X, [Y, Z]) = B(X, YZ).$$

Now $B(n^i, n^j) = 0$ implies that $0 = B(n^i, n^j) = B(n^i, n^j)$. And $B(n, n^j) = 0$ implies that $B(n^i, n^j) = 0$. Thus $B(n^i, n^j) = B(n^i, n^j) = B(n^i, n^j)$.

Theorem 1. Let $g$ be a split simple Lie algebra. Let $\mathfrak{h}$ be a reductive subalgebra of $g$ which is normalized by a split Cartan subalgebra $c$ of $g$ (i.e., $\mathfrak{h}$ is reductive and regular [2]). Then $\mathfrak{h}$ has an $\text{ad}(c + \mathfrak{h})$-stable complement $m$. Such an $m$ is either simple or abelian ($m^2 = 0$).

Proof. We first show that $c + \mathfrak{h}$ is reductive. Letting $g = g_0 + \sum g_a$ be the root space decomposition of $g$, it suffices to show that for $a \neq 0$, $g_a \subseteq c + \mathfrak{h}$ implies $g_{-a} \subseteq c + \mathfrak{h}$ [7, p. 669]. Since $[c, \mathfrak{h}] \subseteq \mathfrak{h}$ we have $[c, b] \subseteq b$ where $b$ is the center of $\mathfrak{h}$. Thus $c + b$ is solvable. Thus $\text{ad}(c + b)$ is triangularizable and $0 = \text{ad}(c, b) = \text{ad}(c, b)$ since $\text{ad}(c, b) \subseteq \text{ad}(b)$ and $\text{ad}(b)$ consists of semisimple transformations. Thus $[c, b] = 0$ and $b \subseteq c = g_a$. Now $\mathfrak{h} = b \oplus \mathfrak{h}^{(1)}$ with $\mathfrak{h}^{(1)}$ semisimple, since $\mathfrak{h}$ is reductive. Let $\alpha$ be a nonzero root such that $g_{-\alpha} \subseteq c + \mathfrak{h}$. Then since $\mathfrak{h}^{(1)}$ is ad-$c$-stable and $c + \mathfrak{h} = g_0 + b + \mathfrak{h}^{(1)} = g_0 + \mathfrak{h}^{(1)}$, we have $g_{-\alpha} \subseteq \mathfrak{h}^{(1)}$. Now the restriction of the Killing form $B(\ , \ )$ of $g$ to $\mathfrak{h}^{(1)}$ is nondegenerate since it is the trace form of a faithful representation of the semisimple Lie algebra $\mathfrak{h}^{(1)}$ (see [3, p. 69]). Thus $B(g_{-\alpha}, \mathfrak{h}^{(1)}) = 0$. Since $B(g_{-\alpha}, g_0) = 0$ for $\alpha + \beta \neq 0$, it follows $g_{-\alpha} \subseteq \mathfrak{h}^{(1)}$. Thus $g_{-\alpha} \subseteq c + \mathfrak{h}$ implies $g_{-\alpha} \subseteq c + \mathfrak{h}$ and $c + \mathfrak{h}$ is reductive.

It follows that $\mathfrak{h}$ has a complement $m$ stable under $\text{ad}(c + \mathfrak{h})$. Any complement $m$ is the sum of $m \cap g_0$ and those root spaces $g_a$ not occurring in $\mathfrak{h}$. In particular, $g_a \subseteq m$ implies $g_{-a} \subseteq m$.

We now show that such an $m$ is either simple or abelian. Assume that $m^2 \neq 0$ and $m$ not simple. Then by Lemma 1, $m$ has a proper Der $m$-stable ideal. Since $m$ is $\text{ad}(c + \mathfrak{h})$-stable, $\text{ad}(c + \mathfrak{h})$ consists of derivations of $m$. Thus $m$ has a proper ideal $n$ stable under $\text{ad}(c + \mathfrak{h})$.

Let $\sigma$ be an automorphism of $g$ such that $\sigma|c = -id_c$ and $g_{-\alpha} = g_{-\alpha}$ for all $\alpha$ (see [3, p. 127]). Then the above discussion shows that $m$ and $\mathfrak{h}$ are $\sigma$-stable. It follows that $(XY)^\sigma = X^\sigma Y^\sigma$ and $(\mathfrak{h}(X, Y))^\sigma = \mathfrak{h}(X^\sigma, Y^\sigma)$. Thus $\sigma|m$ is an automorphism of $m$ and $n^\sigma$ is an ideal of $m$. Since $[n^\sigma, c + \mathfrak{h}] = [n^\sigma, (c + h)^\sigma] = [n, c + \mathfrak{h}]^\sigma \subseteq n^\sigma$, $n^\sigma$ is also $\text{ad}(c + \mathfrak{h})$-stable.

Suppose that one of the ideals $m \cap n^\sigma, n + n^\sigma$ is proper in $m$. Call it $\nu$. Then $\nu$ is the sum of $\nu \cap g_0$ and root spaces $g_a$. Moreover $g_{-\alpha} \subseteq \nu$ implies $g_{-\alpha} \subseteq \nu$. It follows that $m = m \cap g_0 + \nu + \nu^\perp$ where $\nu^\perp = \{X \in m \mid B(X, \nu) = 0\}$ (thus $g_a \subseteq m - g_0$ and $g_{-\alpha} \subseteq \nu$ implies $g_{-\alpha} \subseteq \nu$ which implies $B(g_a, \nu) = 0$). We use this to show that $m = m + \mathfrak{h}(\nu, \nu)$ is an ideal of $g$. By Lemma 2 it suffices to show that $[\nu, \nu^\perp] \subseteq \nu$ where $\nu^\perp = \nu^\perp + m \subseteq g_0$. But $[\nu, m \cap g_0] \subseteq [\nu, c] \subseteq \nu$. Thus it suffices to show that $[\nu, \nu^\perp]$
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Thus \( B(\{p, p^1\}, c + h) = B(p^1, \{p, c + h\}) = B(p^1, p) = 0 \) and \( [p, p^1] \subseteq (c + h)^1 \subseteq m. \)

Thus \( h(p, p^1) = [p, p^1] + p^1 \subseteq m \) and \( h(p, p^1) = 0. \) Thus \( [p, p^1] = p^1 \subseteq p \subseteq q \) and \( q \)

is an ideal of \( g. \) Thus \( q = g \) and \( n \) cannot be proper in \( m, \) a contradiction.

Thus we have \( n \cap n^2 = 0 \) and \( n + n^2 = m. \) Thus \( n \cap g_0 = (n \cap g_0)^2 = 0 \) (since

\( \sigma|g_0 = -id_{g_0}. \) Thus \( m \cap g_0 = n \cap g_0 + (n \cap g_0)^2 = 0. \) It follows that \( B(m, h) = 0 \)

(e.g., \( m = \sum_{\alpha \in S} g_{\alpha} \) for some set \( S \) of nonzero roots, and \( \alpha \in S \) implies \( -\alpha \in S \) which

implies \( g_{-\alpha} \notin h \) and therefore \( B(g_{\alpha}, h) = 0). \) Also \( B(n, n) = 0 \) (e.g., \( n = \sum_{\alpha \in T} g_{\alpha} \)

for some set \( T \) of nonzero roots, and \( \alpha \in T \) implies \( -\alpha \notin T \) which implies \( B(g_{\alpha}, n) = 0). \)

It follows from Lemma 3 that \( [n, n] = m = 0. \) Thus \( n^2n^2 = 0. \) Finally

\( m^2 = (n + n^2)^2 = n^2 + n(n^2) = 0. \) a contradiction.

3. The semisimple case. We now consider the reductive pair \((g, h)\) where \( g \) is a

simple Lie algebra and \( h \) is a semisimple Lie subalgebra. We note that the Killing

form \( B(\ , \ ) \) of \( g \) restricted to \( h \) is nondegenerate. For if \( U, V \in h, \) then \( B(U, V) = \text{tr} \ ad_h \ U \ ad_h \ V \)

is the trace form of the representation \( ad_h \) in \( g, \) and is non-

degenerate by Cartan's criterion [3, p. 69]. (Note that \( ad_h \ U = 0 \) implies \( UF \)

is a one-dimensional ideal in the simple algebra \( g \) so that \( U = 0. \)) Thus if \( h^1 = \{X \in g \mid B(X, h) = 0\}, \)

then \( h \cap h^1 = 0 \) and therefore \( g = h^1 + h. \) And \( B([h^1, h], h) = B(h^1, h) = 0. \) So that for \( m = h^1, \)

\((g, h)\) is a reductive pair with \((fixed) \) decom-

position \( g = m + h. \) Note that since \( m = h^1, \) the Killing form \( B, \)

restricted to \( m, \) is a

nondegenerate invariant form, i.e., \( B(XY, Z) = B(X, YZ). \)

Theorem 2. Let \( g \) be a simple Lie algebra and \( h \) a semisimple subalgebra. Then

\((g, h)\) is a reductive pair with \( m = h^1. \) Furthermore \( m^2 = 0 \) or \( m \) is simple.

Proof. Assume \( m^2 \neq 0. \) Then we have from Lemma 1 that \( m \) has a minimal

proper ad \( h\)-stable ideal \( n. \) Then since \( B \) is a nondegenerate invariant form on \( m \)

and \( B([XU], Y) = B(X, [UY]) \) for \( X, Y \in m, U \in h, \) we have \( n^1 = \{X \in m \mid B(X, n) \)

= 0) is an ad \( h\)-stable ideal of \( m. \) Thus \( n \cap n^1 \) is an ad \( h\)-stable ideal of \( m; \) and

since \( n \) is minimal, either \( n \cap n^1 = 0 \) or \( n \cap n^1 = n. \)

In case \( n \cap n^1 = 0 \) we have \( m = n \oplus n^1. \) And we know from Lemma 3 that

\( [n, n^1] = 0. \) Thus \( q = n + h(n, n) \) is a proper ideal of \( g \) by Lemma 2. This contradiction

shows we must have \( n \cap n^1 = n. \)

In the case \( n \cap n^1 = n \) we can find an ad \( h\)-stable complement, \( n' \) (since ad \( h\) is

semisimple and therefore completely reducible); and we write \( m = n + n'. \) Thus since

\( B(n, n) = 0, \) to show that \( n = 0 \) it suffices to show \( B(n, n') = 0. \)

To find a formula for \( B(X, Y) \) with \( X, Y \in m, \) define \( \epsilon(X) \) and \( \delta(X) \) by

\[ \epsilon(X): m \to h: \quad Y \mapsto h(X, Y) = \epsilon(X)(Y), \]

\[ \delta(X): h \to m: \quad U \mapsto \{X, U\} = \delta(X)(U), \]

where \( U \in h. \) Using these maps we have for any \( Z, X \in m, U \in h \) that

\( (ad_h Z)(X) = [Z, X] = ZX + h(Z, X) \)

\( = (L(Z) + \epsilon(Z))(X) \)

\( (ad_h Z)(U) = [Z, U] = \delta(Z)(U) \)
and therefore
\[ \text{ad}_g Z = \begin{pmatrix} L(Z) & e(Z) \\ \delta(Z) & 0 \end{pmatrix}. \]

From this, note that since \( g \) is simple \( 0 = \text{tr} \text{ad}_g Z = \text{tr} L(Z) \). Also since \( \mathfrak{h} = [\mathfrak{h}, \mathfrak{h}] \) is semisimple, and since \( \mathfrak{h} \rightarrow \text{ad}_m \mathfrak{h} : U \rightarrow \text{ad}_m U \) and \( \mathfrak{h} \rightarrow \text{ad}_g \mathfrak{h} : U \rightarrow \text{ad}_g U \) are representations of \( \mathfrak{h} \), we have \( \text{tr} \text{ad}_m U = \text{tr} \text{ad}_g U = 0 \) for all \( U \in \mathfrak{h} \).

Next for \( X, Y \in \mathfrak{m} \) define the linear transformation \( \sigma(X, Y) : \mathfrak{m} \rightarrow \mathfrak{m} \) by \( \sigma(X, Y)Z = [X, \mathfrak{h}(Y, Z)] \). From (3) we have the identity
\[ \text{ad}_m \mathfrak{h}(X, Y) - \sigma(X, Y) + \sigma(Y, X) = [L(X), L(Y)] - L(XY) \]
and therefore \( \text{tr} \sigma(X, Y) = \text{tr} \sigma(Y, X) \). From this and the matrix for \( \text{ad}_g Z \) we obtain for \( X, Y \in \mathfrak{m} \) that
\[ B(X, Y) = \text{tr} \text{ad}_g X \text{ad}_g Y \]
\[ = \text{tr} L(X)L(Y) + \text{tr} e(X)e(Y) + \text{tr} \delta(X)
\]
\[ = \text{tr} L(X)L(Y) + \text{tr} \delta(Y)e(X) + \text{tr} \delta(X)e(Y) \]
\[ = \text{tr} L(X)L(Y) + \text{tr} \sigma(Y, X) + \text{tr} \sigma(X, Y) \]
\[ = \text{tr} L(X)L(Y) + 2 \text{tr} \sigma(X, Y), \]
using for the third equality that if \( S \in \text{Hom}(V, W) \) and \( T \in \text{Hom}(W, V) \) for vector spaces \( V \) and \( W \), then \( \text{tr} ST = \text{tr} TS \).

Now recall that in the decomposition \( \mathfrak{m} = \mathfrak{n} + \mathfrak{n}' \) we must show \( B(n, n') = 0 \). Thus for \( X \in \mathfrak{n}, Y \in \mathfrak{n}' \) we have (from the fact that \( \mathfrak{n} \) is an ideal and \( \mathfrak{n}\mathfrak{n} = 0 \)) the matrices
\[ L(X) = \begin{pmatrix} 0 & 0 \\ X_{11} & 0 \end{pmatrix} \] and \( L(Y) = \begin{pmatrix} Y_{11} & 0 \\ Y_{21} & Y_{22} \end{pmatrix} \)
and therefore \( \text{tr} L(X)L(Y) = 0 \) and \( B(X, Y) = 2 \text{tr} \sigma(X, Y) \).

To find the matrix for \( \sigma(X, Y) \) (with \( X \in \mathfrak{n}, Y \in \mathfrak{n}' \) let \( Z \in \mathfrak{n}, Z' \in \mathfrak{n}' \). Then
\[ \sigma(X, Y)Z = [\mathfrak{h}(Z, Y), X] \in \mathfrak{n}, \]
\[ \sigma(X, Y)Z' = [\mathfrak{h}(Z', X), X] \in \mathfrak{n}. \]

Therefore
\[ \sigma(X, Y) = \begin{pmatrix} \sigma_{11} & 0 \\ \sigma_{21} & 0 \end{pmatrix} \]
and \( \text{tr} \sigma(X, Y) = \text{tr} \sigma_{11} = \text{tr}_n \sigma(X, Y) \). To find the action of \( \sigma(X, Y) \) on \( \mathfrak{n} \) again let \( Z \in \mathfrak{n} \). Then since \( \mathfrak{n} \) is an ideal, \( \mathfrak{n}\mathfrak{n} = 0 \) and \( \mathfrak{h}(n, n) = 0 \), we have from (3) that
\[ 0 = J(Z, X, Y) = [Z, \mathfrak{h}(X, Y)] + [X, \mathfrak{h}(Y, Z)] \]
\[ = [- \text{ad}_n \mathfrak{h}(X, Y) + \sigma(X, Y)]Z. \]

Therefore on \( \mathfrak{n} \) we have \( \sigma(X, Y) = \text{ad}_n \mathfrak{h}(X, Y) \) and since \( U \rightarrow \text{ad}_n U \) is a representation of the semisimple Lie algebra \( \mathfrak{h} \), \( 0 = \text{tr} \text{ad}_n \mathfrak{h}(X, Y) = \text{tr}_n \sigma(X, Y) \). Thus \( B(n, n') = 0 \) and \( \mathfrak{m} \) is simple, a contradiction. Thus either \( \mathfrak{m}\mathfrak{m} = 0 \) or \( \mathfrak{m} \) is simple.
4. **Remarks.** (i) The above discussion for $\mathfrak{h}$ semisimple holds for $\mathfrak{h}$ reductive in $\mathfrak{g}$ except for the assertion that $\text{tr \ ad}_\mathfrak{n} \mathfrak{h}(X, Y) = 0$ and its consequences. The authors do not know whether the theorem holds for all reductive $\mathfrak{h}$.

(ii) If $\mathfrak{h}$ is the zero-space of a derivation of $\mathfrak{g}$ or the one-space of an automorphism of $\mathfrak{g}$, then $\mathfrak{h}$ is reductive and contains a regular element of $\mathfrak{g}$ [1]. Thus if $\mathfrak{g}$ is simple and the underlying field algebraically closed, the associated $m$ is simple or abelian by Theorem 1.

(iii) It would be of value to determine all pairs $(\mathfrak{g}, \mathfrak{h})$ with $\mathfrak{g}$ semisimple for which an associated $m$ is simple. We now give an example of one nontrivial such pair $(\mathfrak{g}, \mathfrak{h})$ where $\mathfrak{g}$ is not simple. Thus let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ (direct) where the $\mathfrak{g}_i$ ($i = 1, 2$) are real compact simple Lie algebras. Suppose that $\mathfrak{b}$ is a simple subalgebra of $\mathfrak{g}_1$, $\mathfrak{b}'$ a simple subalgebra of $\mathfrak{g}_2$, $B \rightarrow B'$ an isomorphism from $\mathfrak{b}$ onto $\mathfrak{b}'$. Let $\mathfrak{h} = \{B + B' \mid B \in \mathfrak{b}\}$ and $m = \mathfrak{h}^\perp$. Then $\mathfrak{g}_1$, $\mathfrak{g}_2$, $\mathfrak{b}$, and $\mathfrak{b}'$ can easily be chosen such that $m^2 \neq 0$. We claim that for any such choice, $m$ is simple. By Lemma 1, it suffices to show that $m$ has no proper $\text{ad} \mathfrak{h}$-stable ideal. If $n$ were such an ideal, then since the Killing form is negative definite on $\mathfrak{g}$, $m = n \oplus n^\perp$. It is now clear that $n + \mathfrak{h}(n, n)$ is an ideal of $\mathfrak{g}$ by Lemma 2, since $[n, n^\perp] = 0$ by Lemma 3. But then $n + \mathfrak{h}(n, n) = \mathfrak{g}_1$ or $\mathfrak{g}_2$. But by construction, $\mathfrak{h} \cap \mathfrak{g}_1 = \mathfrak{h} \cap \mathfrak{g}_2 = 0$. Thus $n = \mathfrak{g}_1$ or $\mathfrak{g}_2$. This is impossible since $\mathfrak{B}(n, \mathfrak{h}) = 0$ whereas $\mathfrak{B}(\mathfrak{g}_i, \mathfrak{h}) \neq 0$ for $i = 1, 2$.

**Bibliography**