ON A CONSTRUCTIVE DEFINITION OF THE
RESTRICTED DENJOY INTEGRAL

BY
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1. Introduction. Let \( I = [a, b] \) and \( F \) be a function of the closed subintervals of \( I \). One says that \( F \) has Burkill integral \( \int_J F \) over the closed interval \( J \subseteq I \) if

\[
\int_J F = \lim_{P \to 0} \sum_{I' \in \mathcal{P}} F(I'),
\]

where \( \mathcal{P} \) is the set of partitions of \( J \) and \( |P| \) is the norm of \( P \). The Burkill integral has been employed in a more general setting [1] to give a descriptive definition of the restricted Denjoy integral of point functions \( f \). In this paper we show how this integral can be used to give a constructive definition of the restricted Denjoy integral and compare the classical construction with ours. We adopt the convention that \( I \) and \( J \), with or without subscripts or superscripts, always denote a closed interval.

Before we begin our discussion, let us recall the classical constructive definition ([2], 255-259) of the restricted Denjoy integral.

Let \( T \) be a real-valued function whose domain, \( \text{dom } T \), is a set of ordered pairs \( \{(f, J)\} \), where \( f \) is a real-valued point function defined on \( J \). The set

\[
\{f: (f, J) \in \text{dom } T\}
\]

will be denoted by \( \text{dom}_T T \).

\( T \) is called an integral if

(i) \( f \in \text{dom}_T T \) implies \( f \in \text{dom}_T T \) for all closed subintervals \( J' \subseteq J \), and \( T(f, J') \) is an additive, continuous function of \( f \);

(ii) if \( f \in \text{dom}_T T \), \( i = 1, 2 \), where \( I_1 \) and \( I_2 \) are abutting, then \( f \in \text{dom}_{T_1 \cup T_2} T \);

(iii) if \( f \equiv 0 \) on \( I \), then \( f \in \text{dom}_T T \) and \( T(f, I) = 0 \).

One says that \( f \) is \( T \)-integrable on \( I \) if \( (f, I) \in \text{dom } T \).

Two integrals \( T_1 \) and \( T_2 \) are compatible if \( T_1(f, I') = T_2(f, I') \) whenever they both exist. We say \( T_1 \subseteq T_2 \) if \( T_1 \) and \( T_2 \) are compatible and \( \text{dom } T_1 \subseteq \text{dom } T_2 \).

Given a function \( f \) defined on \( I' \subseteq I \) and an integral \( T \), one says that a point \( x \in I' \) is a \( T \)-singular point of \( f \) in \( I' \) if there exists \( \{I_n\} \), with \( I_n \subseteq I' \), \( |I_n| \to 0 \), \( x \in I_n \), such that \( (f, I_n) \notin \text{dom } T \).

If \( \mathcal{S} \) is the set of \( T \)-singular points in \( I \), clearly \( \mathcal{S} \) is closed and \( (f, I') \in \text{dom } T \) for all \( I' \subseteq I \) such that \( I' \cap \mathcal{S} = \emptyset \).

Received by the editors October 4, 1966.

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Let $T$ be an integral. One defines $\text{dom}_T T^c$ by the following conditions:

- $\mathcal{S} \cap I$ is finite or void;
- there exists a continuous, additive $F$ such that $F(I') = T(f, I')$ whenever $(f, I') \in \text{dom}_T I'$.

Define $T^c(f, I) = F(I)$. It is clear that $T^c$ is an integral.

Let $E \subseteq I$. Let $f_E = f_{X_E}$, where $X_E$ is the characteristic function of $E$. One says $f$ is $T$-integrable on $E$ if $f_E$ is $T$-integrable on $I$.

One defines $\text{dom}_T T^h$ by the following conditions:

- $f$ is $T$-integrable on $\mathcal{S}$ and on each of the intervals $I_k$ contiguous to $\mathcal{S} \cup \{a, b\}$.
- $\sum O(T; f; I_k) < \infty$, where $O(T; f; J) = \sup_{J \subseteq J} |T(f, J')|$. Define $T^h(f, I) = T(f, \mathcal{S}) + \sum T(f, I_k)$. Clearly $T^h$ is also an integral.

Let $\{T^a\}$ be a sequence of integrals, in general transfinite, such that $T^a \subseteq T^b$ whenever $a < b$. Define $\text{dom}_T \sum_{b < a} T^b = \bigcup_{b < a} \text{dom}_T T^b$ and if $(f, I') \in \text{dom}_T \sum_{b < a} T^b$, define $(\sum_{b < a} T^b)(f, I') = T^0(f, I')$, where $b_0$ is the least ordinal such that $(f, I') \in \text{dom}_T T^{b_0}$.

Write $T^{CH} = (T^c)^H$. We define a transfinite sequence $\{D^a\}$ of integrals as follows: let $\mathcal{L}$ be the Lebesgue integral,

$$D^0 = \mathcal{L}$$

$$D^a = \left( \sum_{b < a} D^b \right)^{CH}$$

if $a > 0$. Let $\Omega$ be the first uncountable ordinal. Then it is well known ([2], 258) that if $D_\infty$ is the restricted Denjoy integral,

$$D_\infty = \sum_{a < \infty} D^a.$$

2. A constructive definition using the Burkill integral. We define $\text{dom}_T T^{H^*}$ by the following conditions:

- there exists a closed set $W \subseteq I$ such that $f$ is $T$-integrable on $W$ and on each $I' \subseteq I$ with $I' \cap W = \emptyset$;
- if we define

$$\psi(I') = T(f, I') \text{ if } I' \cap W = \emptyset,$$

$$= 0 \text{ if } I' \cap W \neq \emptyset,$$

then $\int_I \psi$ exists and $\int_I \psi$ is continuous (note that if $\int_I F$ exists, then $\int_I F$ exists for all $I' \subseteq I$ and $\int_I F$ is additive).

Define $T^{H^*}(f, I) = T(f_w, I) + \int_I \psi$. We note that $T^{H^*}$ is an integral. For suppose that $f \in \text{dom}_T T^{H^*}$. Let $J' \subseteq J$. If $J' \cap W = \emptyset$, then $f \in \text{dom}_T T$ so that, taking $W' = J'$ in the definition of $\text{dom}_T T^{H^*}$, we see that $f \in \text{dom}_T T^{H^*}$. If $J' \cap W \neq \emptyset$, then since $f_w \in \text{dom}_T T$ and $T$ is an integral, $f_w \in \text{dom}_T T$. Now, since $\int_I \psi$ exists, so does $\int_I \psi$ ([1], p. 70). Recall that continuity of $\int_I \psi$ is assumed in the definition.
of $\text{dom}_T T^{**}$. Therefore, in either case $f \in \text{dom}_I T^{**}$. Since $T$ is an integral, $T(f_w, J')$ is an additive, continuous function of the $J' \subseteq J$; it is known ([1], p. 70) that $\int \psi$ is additive. Therefore $T^{**}(f, J')$ is an additive, continuous function of the $J' \subseteq J$, and condition (i) is satisfied. Now suppose $f \in \text{dom}_I T^{**}, \ i=1, 2$, where $I_1$ and $I_2$ are abutting. With no loss in generality we may assume $I_1$ is to the left of $I_2$. Choose $W_i \subseteq I_i, \ i=1, 2$, closed sets satisfying the requirements of the definition of $f \in \text{dom}_I T^{**}$, and let $\psi_i$ be the functions corresponding to $T^{**}, f$ and $W_i$, $i=1, 2$, in this definition. Set $a_i = \sup W_i, \ b_i = \inf W_i$. If $a_1 = b_1$, choose

$$W = W_1 \cup W_2.$$ 

Then $W$ is closed and, since $T$ is an integral, by condition (ii), $f_w \in \text{dom}_{I_1 \cup I_2} T$. Let $\varepsilon > 0$ and choose $\eta_i = \eta_i(\varepsilon) > 0, \ i=1, 2$, so that if $P_i$ is a partition of $I_i, \ i=1, 2$, with $|P_i| < \eta_i$, then $|\sum_{r \in P_i} \psi_i(I') - \int_{I_i} \psi_i| < \varepsilon/8$, and also so that if $\{J_1, \ldots, J_3\}$ is any finite sequence of nonoverlapping subintervals of $I_i$ with $\text{Max} |J_k| < \eta_i$, then $\sum_i |\psi_i(J_k) - \int_{J_k} \psi_i| < \varepsilon/8$ (see [1], p. 70). Also, since $\int \psi_i$ is additive and continuous on $I_i, \ i=1, 2$, we may choose $\eta_i, 0 < \eta_i < \min(\eta_1, \eta_2)$, so that $J \subseteq I_i, \ |J| < \eta_i$ imply that $|\int_{J} \psi_i| < \varepsilon/8$. Now let $P$ be any partition of $I_1 \cup I_2$ with $|P| < \eta$. Then

$$\left| \sum_{r \in P} \psi(I') - \left( \int_{I_1} \psi_1 + \int_{I_2} \psi_2 \right) \right| < \varepsilon.$$ 

It therefore follows that $\int_{I_1 \cup I_2} \psi = \int_{I_1} \psi_1 + \int_{I_2} \psi_2$. It is immediate that $\int \psi$ is continuous on $I_1 \cup I_2$. If $a_1 < b_1$, choose $W = W_1 \cup W_2 \cup [a_1, b_1]$. A similar argument shows that, again, condition (ii) is satisfied. Finally, if $f \equiv 0$ on $I$, then since $T$ is an integral, $f \in \text{dom}_I T$ and $T(f, I) = 0$. Choosing $W = I$, we see that $f \in \text{dom}_T T^{**}$ and $T^{**}(f, I) = 0$. Therefore $T^{**}$ is an integral. By a known result ([1], 90), if $T \subseteq D_\ast$, then $T^{**} \subseteq D_\ast$.

Romanovski ([1], 91) has defined the following sequence $\{K_\alpha\}$ of classes of $D_\ast$-integrable functions defined on $I$. Put $f \in K_0$ if $f$ is Lebesgue integrable on $I$. Let $\alpha > 0$. Then we put $f \in K_\alpha$ if one of the following four conditions is satisfied:

1. there is a partition $P$ such that $f$ is in a class $K_\beta$, with $\beta < \alpha$, on each member of $P$;
2. $f$ can be extended to a function which belongs to a class $K_\beta$, with $\beta < \alpha$, on some $I' \supseteq I$;
3. $f$ belongs to a class $K_\beta$, with $\beta < \alpha$, on each closed sub-interval of $I^0 = (a, b)$;
4. there is a closed set $W$ such that $f_w$ is Lebesgue integrable on $I$, and such that, on each $I' \subseteq I$ with $I' \cap W = \emptyset$, $f$ belongs to a class $K_\beta$, with $\beta < \alpha$; in addition, if

$$\psi(I') = (D_\ast) \int_{I' \cap W} f \ dx \quad \text{if} \quad I' \cap W = \emptyset,$$

$$= 0 \quad \text{if} \quad I' \cap W \neq \emptyset,$$

then $\int \psi$ exists and $\int \psi$ is continuous. It is known ([1], 91) that if $f$ is $D_\ast$-integrable, then $f$ belongs to some class $K_\alpha$. 

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We define a transfinite sequence \( \{D_\beta^\alpha\} \) of integrals as follows:

\[
D_0^\alpha = \mathcal{L},
\]

\[
D_\beta^\alpha = \left( \sum_{\beta < \alpha} D_\beta^\alpha \right)^{H^*}
\]

if \( \alpha > 0 \).

**Theorem 1.** \( K_\alpha \subseteq \text{dom}_f D_\beta^\alpha + 1 \).

**Proof.** Clearly \( \text{dom}_f D_0^\alpha = K_\alpha \). Suppose the theorem is true for all \( \beta < \alpha \), for some \( \alpha > 0 \) and that \( f \in K_\alpha \). If condition (1) of the definition of \( K_\alpha \) is satisfied, then there is a partition \( P \) of \( I \) such that \( f \in \text{dom}_f D_\beta^\alpha + 1 \) for all \( I' \in P \), where \( \beta < \alpha \). By condition (ii) for an integral, \( f \in \text{dom}_f D_\beta^\alpha + 1 \). Since \( \beta + 1 < \alpha + 1 \), \( f \in \text{dom}_f D_\beta^\alpha + 1 \). If condition (2) of the definition of \( K_\alpha \) is satisfied, then there exists \( I' \supseteq I \) such that \( f \in K_\beta \) on \( I' \), \( \beta < \alpha \). But then by the induction hypothesis, \( f \in \text{dom}_f D_\beta^\alpha + 1 \) and by condition (i) for integrals, \( f \in \text{dom}_f D_\beta^\alpha + 1 \). Since \( \beta + 1 < \alpha + 1 \), again \( f \in \text{dom}_f D_\beta^\alpha + 1 \). Now suppose condition (4) of the definition of \( K_\alpha \) is satisfied. Then there is a closed set \( W \subseteq I \) such that \( f_w \in \text{dom}_f \mathcal{L} \), and such that, on each \( I' \subseteq I \) with \( I' \cap W = \emptyset \), \( f \in \text{dom}_f D_\beta^\alpha + 1 \), where \( \beta < \alpha \); in addition, \( \int I \psi \) exists and \( \int \psi \) is continuous. It follows that \( f \in \text{dom}_f D_\beta^\alpha + 1 \).

We finally show that if condition (3) of the definition is satisfied, then \( f \) is \( D_\beta^\alpha + 1 \)-integrable. Now \( f \in \text{dom}_f D_\beta^\alpha \), with \( \beta < \alpha + 1 \), for all \( I' \subseteq (a, b) \). Let

\[
F = (D_\beta^\alpha) \int f \, dx.
\]

Then \( F \) is continuous on \( I \). In \((h^*_\beta)\) and \((h^*_\beta)\) choose \( T = D_\beta^\alpha \). Note that \( F(I') = T(f, I') \) for all \( I' \subseteq I^\beta \). Let \( W = (a, b) \) and \( \epsilon > 0 \). Choose \( \delta > 0 \) such that \( I' \subseteq I \), \( |I' - I^\beta| < \delta \) imply \( |F(I) - F(I')| < \epsilon \). Let \( P \) be any partition of \( I \) with \( |P| < \delta/2 \). Then

\[
\left| \sum_{I \in P'} \psi(I') - F(I) \right| = \left| \sum_{I \in P'} F(I') - F(I) \right| < \epsilon,
\]

where \( P' \) is the set of those members of \( P \) which contain neither \( a \) nor \( b \). It follows that \( \int I \psi = F(I) \). Clearly \( \int \psi \) is continuous. Therefore

\[
K_\alpha \subseteq \text{dom}_f D_\beta^\alpha + 1.
\]

**Theorem 2.** \( D_\beta = \sum_{\beta < \alpha} D_\beta^\alpha \).

**Proof.** Let \( f \) be \( D_\beta^\alpha \)-integrable on \( I \). Define \( f = 0 \) outside of \( I \). Let \( S^\alpha \) denote the set of \( D_\beta^\alpha \)-singular points of \( f \) in \( I \). Then \( \{S^\alpha\} \) is decreasing and, thus, stationary (i.e., there is an \( \alpha_0 < \Omega \) such that \( S^\alpha = S^\alpha_0 \) for all \( \alpha > \alpha_0 \); see, e.g., [2], 258). However, \( f \in K_\beta \) for some \( \beta \) and, therefore, \( f \in \text{dom}_f D_\beta^\alpha + 1 \). Thus \( S^{\beta + 1} = \emptyset \), which implies \( S^{\alpha_0} = \emptyset \) and \( f \in \text{dom}_f D_\beta^\alpha_0 \). We have observed above that \( T \subseteq D_\beta^\alpha \) implies \( T^{\alpha_0} \subseteq D_\beta^\alpha_0 \). To show that \( D_\beta = \sum_{\beta < \alpha} D_\beta^\alpha \), it is easy to see that if \( D_\beta^\alpha \subseteq D_\beta \) for all \( \beta < \alpha \), then \( \sum_{\beta < \alpha} D_\beta^\alpha \subseteq D_\beta \), and that therefore \( (\sum_{\beta < \alpha} D_\beta^\alpha)^{H^*} \subseteq D_\beta \). Since \( D_\beta^\alpha = \mathcal{L} \subseteq D_\beta \), we have the desired containment relation by applying transfinite induction.
3. Comparison of our construction with the classical construction. In the following theorem we restrict our attention to those integrals $T$ which have the property that $f \in \text{dom}_r T$ whenever $I \cap \{f \neq 0\}$ is finite.

**Theorem 3.** $T_0 \subseteq T_r$.

**Proof.** Let $f$ be $T_0$-integrable on $I$. Let $\mathcal{S}$ be the set of $T$-singular points in $I$ and \{I_1, \ldots, I_k\} be the set of intervals contiguous to $\mathcal{S} \cup \{a, b\} = E$. Let $F(I') = T_0(f, I')$ for all $I' \subseteq I$. Let $\varepsilon > 0$. Choose $\delta > 0$ such that for each $i$, $I' \subseteq I_i$ and $|I_i - I'| < \delta$ imply $|F(I) - F(I')| < \varepsilon/k$. Let $S = \{J_1, \ldots, J_m\}$ be any partition of $I$ with $|S| < \delta/2$.

Define

$$\psi(I') = F(I') \quad \text{if} \quad I' \cap E = \emptyset,$$

$$= 0 \quad \text{if} \quad I' \cap E \neq \emptyset,$$

for all $I' \subseteq I$. If $S \subseteq S_i$ is the subset of $S_i$ consisting of those $J_i$ such that $J_i \cap E = \emptyset$, we see that, since $F$ is additive,

$$\left| \sum_{i=1}^{m} \psi(J_i) - F(I) \right| = \left| \sum_{J \in S} \psi(J_i) - \sum F(I_i) \right| < \varepsilon.$$

It follows that $\int_I \psi = F(I)$.

We shall show below that the containment guaranteed by Theorem 3 is proper in general, i.e., there are $T_r$-integrable functions which are not $T_0$-integrable.

We note that (h) and (h') do not require that $f$ be $T$-integrable on the intervals $I_i$ contiguous to $W$. Since $\int_I \psi$ exists, $\int \psi$ is continuous, and $T(f, I')$ is an additive function of the closed subintervals of $I_j$ for all $j$, it follows that $\int_I \psi = T(f, I')$ for all such $I'$, and therefore $f$ is $T_0$-integrable on $I_i$ for all $i$. Moreover,

$$\int_{I_i} \psi = \lim_{|I_i| - I'_i \to 0} T(f, I').$$

The following theorem will immediately establish the inclusion relationships between $T_r$, $T_0$, and $T_{CH}$.

**Theorem 4.** Suppose $E \subseteq I$ is closed. Let $\{I_k\}$ be the sequence of intervals contiguous to $E \cup \{a, b\}$. Consider the following two conditions:

1. For every $k$, $f \in \text{dom}_r T$ for all $I' \subseteq I_k^c$ and $f \in \text{dom}_{t_k} T_0$; moreover, $\sum_0^k O(T_0; f; I_k) < \infty$;

2. $f$ is $T$-integrable on all $I' \subseteq I$ such that $I' \cap E = \emptyset$; moreover, if

$$\psi(I') = T(f, I') \quad \text{if} \quad I' \cap E = \emptyset,$$

$$= 0 \quad \text{if} \quad I' \cap E \neq \emptyset,$$

then $\int_I \psi$ exists and $\int \psi$ is continuous. Then (1) and (2) are equivalent.

**Corollary 5.** $T_r \subseteq T_{0r}$.

**Corollary 6.** $T_{0r} \subseteq T_{CH}$. 

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These corollaries follow from a comparison of the conditions in the definitions of $T^H$, $T^H^*$, and $T^{C^H}$ with conditions (1) and (2) of Theorem 4.

**Proof of Theorem 4.** We show first that (1) implies (2). Let $\varepsilon > 0$. Choose $N = N(\varepsilon)$ such that

$$\sum_{n=1}^\infty O(T^C; f; I_n) < \varepsilon/3.$$ 

Let $I_k = [a_k, b_k]$ for all $k$. Consider $I_j$ with $j \leq N$. By continuity of $T^C(f, I')$ there is a $\delta_j = \delta_j(\varepsilon) > 0$ such that for $I' \subseteq I_j$, $|T^C(f, I_j) - T^C(f, I')| < \varepsilon/6N$ whenever $|I_j - I'| < \delta_j$. In particular, then, if $J' = [a_j, b']$ and $J'' = [a', b_j]$ are contained in $J_j$, and if $b' - a_j < \delta_j$ and $b_j - a' < \delta_j$, then

$$|T^C(f, J')| + |T^C(f, J'')| < \varepsilon/3N.$$

Note, also, that since $T^C(f, I)$ is additive, if $J' \subseteq I_j$ and $\{J_1', \ldots, J_p\}$ is a partition of $J'$, then $\sum T^C(f, J_i') = T^C(f, J')$.

Now choose $\delta = \min_{j=1,\ldots,N} \delta_j$. Suppose $S = \{J_1, \ldots, J_p\}$ is a partition of $I$ with $|S| < \delta$. The only members $J_i$ of $S$ for which it is possible that $\psi(J_i) \neq 0$ are those for which $J_i \cap E = \emptyset$. Consider $I_k$ with $k \leq N$. Let $\{J_{m_k}^1, \ldots, J_{m_k}^{n_k}\}$ be those $J_i$ which are contained in $I_k$, such that $J_{m_k}^1 \cap E = \emptyset$. Then by the above discussion we see that

$$\left| \sum_{m=1}^{n_k} \psi(J_{m_k}^1) - T^C(f, I_k) \right| < \varepsilon/3N.$$ 

For $k > N$, clearly $\left| \sum_{m=1}^{n_k} \psi(J_{m_k}^1) \right| \leq O(T^C; f; I_k)$. Thus,

$$\left| \sum_{k=1}^p \psi(J_k) - \sum_{k=1}^\infty T^C(f, I_k) \right| = \left| \sum_{k=1}^\infty \sum_{m=1}^{n_k} \psi(J_{m_k}^1) - \sum_{k=1}^N T^C(f, I_k) \sum_{k=N+1}^\infty T^C(f, I_k) \right| \leq \left| \sum_{k=1}^N \left| \sum_{m=1}^{n_k} \psi(J_{m_k}^1) - T^C(f, I_k) \right| \right| \sum_{k=N+1}^\infty \left| \sum_{m=1}^{n_k} \psi(J_{m_k}^1) \right| + \sum_{k=N+1}^\infty |T^C(f, I_k)| < Ne\varepsilon/3N + 2 \sum_{N+1}^\infty O(T^C; f; I_k) < \varepsilon.$$ 

Thus $\int \psi$ exists and $\int \psi = \sum_{n=1}^\infty T^C(f, I_k)$. The continuity of $\int \psi$ is clear.

We now show that (2) implies (1). Let $\delta = \delta(1) > 0$ be such that

$$\left| \sum_{I' \in S_I} \psi(I') - \int_I \psi \right| < 1$$ 

whenever $S_I$ is a partition of $I$ with $|S_I| < \delta$, and also such that

$$\left| \sum_{I' \in S} \psi(I') - \sum_{I' \in S'} \int_{I'} \psi \right| < 1$$ 

whenever $S$ is a finite family of nonoverlapping subintervals of $I$ with $\sup_{I' \in S} |I'| < \delta$ (see [1], 70).
Choose $N = N(\delta)$ such that $\sum_{k=1}^{\infty} |I_k| < \delta/2$. We shall divide $\{I_k\}_{k=1}^{\infty}$ into two classes, $\mathcal{S}^+$ and $\mathcal{S}^-$, as follows: put $I_k$ in $\mathcal{S}^+$ if there exists $\{I_i^{(j)}\}$, with $I_i^{(j)} \subseteq I_k$, such that $T^c(f, I_i^{(j)}) \to O(T^c; f; I_k)$ and put $I_k$ in $\mathcal{S}^-$ otherwise. Write $\mathcal{S}^+ = \{I_n\} = \{J_j\}$. Choose $J'_j$ with $J'_j \subseteq J_k$, such that $O(T^c; f; J'_j) - T^c(f, J'_j) < 1/2^j$. We shall show that $\sum O(T^c; f; J'_j) < \infty$. A similar argument yields the same inequality for the sum of the oscillations over the members of $\mathcal{S}^-$. The choice of $J'_j$ induces a partition of $J_j$, $\{J'_j, J'_j^*, J'_j^{**}\}$. Append to the family $\{J_j\}$ the family $\{I_1, \ldots, I_N\}$. For each $k = 1, \ldots, N$, choose a partition $\{I_k^1, \ldots, I_k^{n(k)}\}$ (enumerated from left to right) of $I_k$ such that $|I_k^j| < \delta$ for all $j$ and such that

$$
\left| T^c(f, I_k) - \sum_{j=1}^{n(k)-1} T^c(f, I_k^j) \right| < 1/2N.
$$

This can be done since $T^c(f, I)$ is additive and continuous on $I_k$ for all $k$. For any positive integer $p$, consider

$$
H = I - \left( \bigcup_{J_j} J_j \right) \cup \left( \bigcup_{J_i} J_i \right).
$$

If $H$ is nonvoid, it consists of a finite number of intervals. We shall now form a partition $S$ of $I$. Put $\{J'_j, J'_j^*, J'_j^{**}\}_{j=1}^p \cup \{I_k^1, \ldots, I_k^{n(k)}\}_{k=1}^N$ into $S$. Suppose $I' = [a', b']$ and $I'' = [a'', b'']$ are two of these intervals with $b' < a''$ such that none of the above intervals intersects $(b', a'')$. Then, since the sum of the lengths of the contiguous intervals contained in $H$ is $< \delta/2$, it follows that we can partition $(b', a'')$ by a sequence of intervals all of whose end-points are in $E$, and all of which have length $< \delta/2$. We put this partition into $S$. We do this for all pairs $I'$ and $I''$ satisfying the above conditions. We thereby obtain a partition $S$ of $I$ such that $|S| < \delta$. Then

$$
\left| \int_I \psi - \sum_{I' \in S} \psi(I') \right| < 1.
$$

But

$$
\sum_{I' \in S} \psi(I') = \sum_{j=1}^p \sum_{k=2}^{n(j)-1} T(f, I_k^j) + \sum_{j=1}^p T(f, J'_j).
$$

Thus

$$
\sum_{j=1}^p T(f, J'_j) \leq \left| \sum_{j=1}^p \sum_{k=2}^{n(j)-1} T(f, I_k^j) \right| + 1 + \left| \int_I \psi \right|
$$

$$
\leq 1 + \left| \int_I \psi \right| + N/2N + \sum_{j=1}^p \left| T^c(f, I_k^j) \right| < 2 + \left| \int_I \psi \right| + \sum_{j=1}^p \left| T^c(f, I_k^j) \right|.
$$

Since $O(T^c; f; J'_j) - T^c(f, J'_j) < 1/2^j$ for all $i$,

$$
\sum_{j=1}^p O(T^c; f; J'_j) \leq \sum_{j=1}^p 1/2^j + 2 + \left| \int_I \psi \right| + \sum_{j=1}^p \left| T^c(f, I_k) \right|
$$

$$
< 3 + \left| \int_I \psi \right| + \sum_{j=1}^p \left| T^c(f, I_k) \right|.
$$
Since $3 + \left| \int \psi + \sum_{k} |T^c(f, I_k)| \right|$ is an absolute finite constant and $p$ was arbitrary, $\sum_{k} O(T^c; f; I_k) < \infty$. As we noted above, a similar argument can be used to establish the result for $J^-$. Thus (2) implies (1).

The following example illustrates the fact that $T^H \subsetneq T^H^*$ and $T^C \subsetneq T^H^*$.

Let

$$\{I_n\} = \left\{ \left[ \frac{1}{n+1}, \frac{1}{n} \right] \right\} n = 1, 2, \ldots$$

Define

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x^2} & \text{if } 0 < x \leq \frac{1}{2}, \\ \frac{1}{2} \sin 4(1-x) & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Extend $f$ by the formula

$$f(x+n) = f(x)$$

for all integers $n$. Define

$$f_n(x) = \begin{cases} \frac{1}{2n+1} f((n+1)x) & \text{if } x \in I_n, \\ 0 & \text{if } x \notin I_n, \end{cases}$$

and

$$F(x) = \sum_{n=1}^{\infty} f_n(x)$$

for all $x \in I$. $F(x)$ is clearly continuous on $I$ and differentiable a.e. Let $g(x) = F'(x)$. Let $E = \{0\} \cup \{1/n\}$, $n = 2, 3, \ldots$ Clearly $g$ is $L^c$-integrable on $I_n$ for all $n$. By Theorem 4, to show $g$ is $L^{H^*}$-integrable on $I$ we need only show that $\sum O(F; I_k) < \infty$. Since $O(F; I_k) < 1/2^k$, this is immediate. Thus $g$ is $L^{H^*}$-integrable. However, since $E$ is the set of $L$-singular points of $g$, and since $g$ is not $L$-integrable on $I_n$ for any $n$, it follows that $g$ is not $L^{H^*}$-integrable. It is clear that $g$ is not $L^c$-integrable.

Finally we shall show that $T^{H^*} \subsetneq T^{CH}$. Let $E$ be the Cantor set. Let $\{I_n\} = \{[a_n, b_n]\}$ be the sequence of intervals contiguous to $E$. Let $f(x)$ be as in the previous example and

$$F_n(x) = \begin{cases} \frac{1}{2n+1} f\left( \frac{2n+1(x-a_n)}{|I_n|} \right) & \text{if } x \in I_n, \\ 0 & \text{otherwise}, \end{cases}$$

$n = 1, 2, \ldots$ Define

$$F(x) = \sum F_n(x).$$
Clearly \( g(x) = F'(x) \) exists a.e. Moreover, each point of \( E \) is an \( \mathcal{L}^{c} \)-singular point of \( g \), and no other point is singular. In addition, \( O(F; I_n) < 1/2^n \). Thus \( \sum O(F; I_n) < \infty \), implying that \( g \) is \( \mathcal{L}^{CH} \)-integrable. However, \( I_n \) contains \( 2^{n+1} + 1 \) \( \mathcal{L} \)-singular points, namely \( \{a_n + j(|I_n|/2^{n+1})\} \), \( j = 0, 1, \ldots, 2^{n+1} \). In addition,

\[
O\left(F; \left[a_n + j \frac{|I_n|}{2^{n+1}}, a_n + (j+1) \frac{|I_n|}{2^{n+1}} \right]\right) = \frac{1}{2^{n+1}} O(f; [0, 1])
\]

for all \( j \) and \( n \). Then, if \( H = E \cup \bigcup_{n=1}^{\infty} \bigcup_{j=0}^{2^{n+1} - 1} \{a_n + j(|I_n|/2^{n+1})\} \), we note that \( H \) is closed and that \( \sum O(F; I_n) = \infty \), where \( \{I_n\} \) is the sequence of intervals contiguous to \( H \). Moreover, if \( P \subseteq I \) is a closed set such that \( g \) is \( \mathcal{L} \)-integrable on \( P \) and on all \( I' \subseteq I \) with \( I' \cap P = \emptyset \), then \( \sum O(F; I') = \infty \), where \( \{I\} \) is the sequence of intervals contiguous to \( P \). For, let

\[
I_{n_j} = [a_n + j(|I_n|/2^{n+1}), a_n + (j+1)(|I_n|/2^{n+1})],
\]

\( j = 0, 1, \ldots, 2^{n+1} - 1 \), and write \( a_{n_j} = a_n + j(|I_n|/2^{n+1}) \). Let \( c_{n_j} \) be the midpoint of \( I_{n_j} \). Note that \( F \) is monotone increasing on \( [c_{n_j}, a_{n_j} + 1] \) and that \( g \) is positive on \( [c_{n_j}, a_{n_j} + 1] \). Let \( \{I_{n_j}\} \) be the sequence of intervals contiguous to \( P \cup \{c_{n_j}, a_{n_j} + 1\} \) in \( [c_{n_j}, a_{n_j} + 1] \). Then either

\[
\int_{P \cap [c_{n_j}, a_{n_j} + 1]} g \, dx \geq \frac{1}{8} |\sin 4|
\]

or

\[
\sum_{k} O(F; I_{n_k}) \geq \frac{1}{8} |\sin 4|.
\]

If

\[
\int_{P \cap [c_{n_j}, a_{n_j} + 1]} g \, dx \geq \frac{1}{8} |\sin 4|
\]

for infinitely many pairs \( n, j \), then \( \int_{P} g \, dx = \infty \) contradicting the assumption that \( g \) is \( \mathcal{L} \)-integrable on \( P \). Thus, assuming that \( g \) is \( \mathcal{L} \)-integrable on \( P \),

\[
\sum_{k} O(F; I_{n_k}) \geq |\sin 4|/8
\]

for infinitely many pairs \( n, j \), implying \( \sum O(F; I_j') = \infty \). Thus \( g \) is not \( \mathcal{L}^{H*} \)-integrable although it is \( \mathcal{L}^{CH*} \)-integrable.

**References**


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