ISOTOPY CLASSES OF IMBEDDINGS

BY

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1. Introduction. The deleted product space $X^*$ of a space $X$ is $X \times X - \Delta$. If $X$ is a finite polyhedron, let

$$P(X^*) = \bigcup \{ \sigma \times \tau \mid \sigma \text{ and } \tau \text{ are simplexes of } X \text{ and } \sigma \cap \tau = \emptyset \}.$$  

Hu [3] has shown that $X^*$ and $P(X^*)$ are homotopically equivalent. In [4], the author has shown that if $Y$ is a triod, then $P(Y^*)$ is a circle, and that up to homeomorphism the triod is the only tree (finite, contractible, 1-dimensional polyhedron) with this property. It is also shown in [4] that if $X$ is a tree, then $H_1(X^*, Z)$ where $Z$ is the integers, is a free abelian group. T. R. Brahana suggested to the author that if $X$ is a tree, then there might be a connection between the number of generators of $H_1(X^*, Z)$ and the number of isotopy classes of imbeddings of the triod in $X$ and that we might be able to extend this to higher dimensions.

In §2, we obtain a formula for computing the number of isotopy classes of imbeddings of the triod in a tree and show that there is a definite relation between this number and the 1-dimensional Betti number of the deleted product of the tree.

We show that up to homeomorphism there are at least two finite, contractible, 2-dimensional polyhedra, $C$ and $\theta$, which have the property that $P(C^*)$ and $P(\theta^*)$ are homeomorphic to the 2-sphere. There is at least one more finite, contractible, 2-dimensional polyhedron $\Lambda$ whose deleted product has the homotopy type of the 2-sphere. However $C$ can be imbedded in both $\theta$ and $\Lambda$, and in §4, we prove a collection of theorems which give a combinatorial method for computing the number of isotopy classes of imbeddings of $C$ in a finite, contractible, 2-dimensional polyhedron.

The connection between the number of isotopy classes of imbeddings of $C$ in a finite, contractible, 2-dimensional polyhedron $X$ and the 2-dimensional Betti number of the deleted product of $X$ is to be investigated in a forthcoming paper.

2. Imbedding the triod. Throughout this section, let $Y$ denote a triod, let $y_0$ denote the vertex of $Y$ of order 3, and let $X$ denote a tree which is not an arc.

Gottlieb [2] defined a branch point as follows: Let $S$ be a pathwise connected space. A point $x$ of $S$ is a branch point of $S$ if $S - \{x\}$ has at least three path-components.

**Theorem 1.** If $f: Y \to X$ is an imbedding, then $f(y_0)$ is a branch point of $X$ and $f(Y - \{y_0\})$ intersects exactly three path-components of $X - \{f(y_0)\}$.

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Proof. Let $P_1, P_2$, and $P_3$ denote the three path-components of $Y - \{y_0\}$, and let $Q_i, (i = 1, 2, 3)$ denote the path-component of $X - \{f(y_0)\}$ which contains $f(P_i)$. Suppose $Q_i = Q_i$ for some $i \neq j$. Let $p_i \in f(P_i)$ and $p_j \in f(P_j)$. Then there exists a homeomorphism $h : I \to X - \{f(y_0)\}$ such that $h(0) = p_i$ and $h(1) = p_j$. Now there exists a homeomorphism $g : I \to Y$ such that $g(0) = f^{-1}(p_i)$ and $g(1) = f^{-1}(p_j)$. Since $g(t) = y_0$ for some $t \in I$, $f(g(I)) \cup h(I)$ contains a simple closed curve. This contradicts the fact that $X$ is a tree.

THEOREM 2. If $f : Y \to X$ is an imbedding and $H : Y \times I \to X$ is an isotopy such that $H(y, 0) = f(y)$ for all $y \in Y$, then $H(y_0, t) = f(y_0)$ for all $t \in I$.

Proof. Define a path $\sigma : I \to X$ by $\sigma(t) = H(y_0, t)$. Suppose there exists $t_1 \in I$ such that $\sigma(t) \neq f(y_0)$. Then there exists $t_2 \in I$ such that $\sigma(t_1)$ is not a vertex of $X$. Thus $H(y_0, t_1) = \sigma(t_1)$ is not a branch point of $X$. But $h_1 : Y \to X$ defined by $h_1(y) = H(y, t_1)$ is an imbedding. Therefore, by Theorem 1, $h_1(y_0)$ is a branch point.

THEOREM 3. If $f, g : Y \to X$ are imbeddings, then $f$ is isotopic to $g$ if and only if $f(y_0) = g(y_0)$ and, for each path-component $P$ of $Y - \{y_0\}$, $f(P)$ and $g(P)$ are contained in the same path-component of $X - \{f(y_0)\}$.

Proof. Suppose there exists an isotopy $H : Y \times I \to X$ such that $H(y, 0) = f(y)$ and $H(y, 1) = g(y)$ for each $y \in Y$. Then $f(y_0) = g(y_0)$ by Theorem 2. Suppose there exist a path-component $P$ of $Y - \{y_0\}$ and path-components $Q_1$ and $Q_2$ of $X - \{f(y_0)\}$ such that $Q_1 \neq Q_2$, $f(P) \subset Q_1$, and $g(P) \subset Q_2$. Let $y_1 \in P$. Define a path $\sigma : I \to X$ by $\sigma(t) = H(y_1, t)$. Then $\sigma(0) = f(y_1) \in Q_1$ and $\sigma(1) = g(y_1) \in Q_2$. Since $Q_1$ and $Q_2$ are path-components of $X - \{f(y_0)\}$ and $Q_1 \neq Q_2$, $\sigma(t_1) = f(y_0)$ for some $t_1 \in I$. Now $h_1 : Y \to X$ defined by $h_1(y) = H(y, t_1)$ is an imbedding. But $h_1(y_1) = H(y_1, t_1) = \sigma(t_1) = f(y_0)$, and $h_1(y_0) = H(y_0, t_1) = f(y_0)$. Thus we have a contradiction.

If $f(y_0) = g(y_0)$, and, for each path-component $P$ of $Y - \{y_0\}$, $f(P)$ and $g(P)$ are contained in the same path-component of $X - \{f(y_0)\}$, then it is clear that $f$ is isotopic to $g$.

THEOREM 4. If $m$ is the order of a vertex of maximum order in $X$, and, for each $j = 3, 4, \ldots, m$, $p_j$ is the number of vertices of order $j$, then the number of isotopy classes of imbeddings of $Y$ in $X$ is $\sum_{j=3}^m j(j-1)(j-2)p_j$.

Proof. Let $p$ be the number of vertices of $X$ of order $\geq 3$, and for each $i = 1, 2, \ldots, p_i$, let $n_i$ be the order of the $i$th vertex. Then it follows from Theorems 1, 2, and 3 that the number of isotopy classes of imbeddings of $Y$ in $X$ is $\sum_{j=1}^n n_i(n_i-1)(n_i-2)$. But $\sum_{j=1}^p n_i(n_i-1)(n_i-2) = \sum_{j=3}^m j(j-1)(j-2)p_j$.

THEOREM 5. If $m$ is the order of a vertex of maximum order in $X$, and, for each $j = 3, 4, \ldots, m$, $p_j$ is the number of vertices of order $j$, then $H_1(X^*, Z)$ is the free abelian group on $\sum_{j=3}^m [(j-2)(j-1)p_j] - 1$ generators.
Proof. Let \( p \) be the number of vertices of \( X \) of order \( \geq 3 \), and for each \( i = 1, 2, \ldots, p \), let \( n_i \) be the order of the \( i \)th vertex. By Theorem 3.4 of [5], \( H_1(X^*, Z) \) is the free abelian group on \( \sum_{i=1}^{p} [(n_i - 1)^2 - (n_i - 1)] - 1 \) generators. But

\[
\sum_{i=1}^{p} [(n_i - 1)^2 - (n_i - 1)] - 1 = \sum_{j=3}^{\infty} [(j-1)(j-2)p_j] - 1.
\]

Thus, by comparing the formulas in Theorems 4 and 5, we see that there is a definite relation between the number of isotopy classes of imbeddings of \( Y \) in \( X \) and the 1-dimensional homology group of the deleted product of \( X \).

3. The 2-dimensional analog of the triod. For each \( i = 1, 2, 3 \), let \( \sigma_i \) be a 2-simplex, and let \( r \) be a 1-simplex. Throughout the remainder of this paper, let \( C \) denote the polyhedron, consisting of these simplexes and their faces, which satisfies the following conditions:

1. \( r \) is not a face of \( \sigma_i \) for any \( i \),
2. there is a vertex \( c_0 \) which is a vertex of \( r \) and of \( \sigma_i \) for each \( i \),
3. for each \( i < j \), \( \sigma_i \cap \sigma_j \) is a 1-simplex \( r_{ij} \), and
4. \( r_{ij} \neq r_{km} \) unless \( i = k \) and \( j = m \).

The polyhedron \( C \) is a cone with a "sticker" attached to its vertex \( c_0 \). We will continue to let \( r \) denote the 1-simplex described above. Also we will denote \( C - r \cup \{c_0\} \) by \( D \). Note that \( D \) is a disk.

Theorem 6. The polyhedron \( P(C^*) \) is homeomorphic to the 2-sphere.

Proof. For each \( i = 1, 2, 3 \), let \( r_i \) denote the 1-face of \( \sigma_i \) which does not have \( c_0 \) as a vertex. For each \( i < j \), let \( c_{ij} \) denote the other vertex of \( r_{ij} \), and let \( c \) denote the other vertex of \( r \). The polyhedron \( P(C^*) \) consists of the following 2-cells and their faces:

- \( \sigma_1 \times c_{23} \), \( c_{12} \times \sigma_3 \), \( r_{12} \times r_3 \), \( r_2 \times r \)
- \( \sigma_1 \times c \), \( c_{13} \times \sigma_2 \), \( r_{13} \times r_2 \), \( r_3 \times r_{12} \)
- \( \sigma_2 \times c_{13} \), \( c_{23} \times \sigma_1 \), \( r_{23} \times r_1 \), \( r_0 \times r \)
- \( \sigma_2 \times c \), \( c \times \sigma_1 \), \( r_{1} \times r_{23} \), \( r \times r_1 \)
- \( \sigma_3 \times c_{12} \), \( c \times \sigma_2 \), \( r_{1} \times r \), \( r \times r_2 \)
- \( \sigma_3 \times c \), \( c \times \sigma_3 \), \( r_2 \times r_{13} \), \( r \times r_3 \)

The proof now consists of only routine verifications, and hence it is omitted.

For each \( i = 1, 2, 3 \), let \( \tau_i \) be a 2-simplex, and suppose there is a 1-simplex \( s \) which is a face of \( \tau_i \) for each \( i \). Let \( u \) and \( v \) denote the vertices of \( s \), and for each \( i \), let \( u_i \) denote the vertex of \( \tau_i \) which is different from \( u \) and \( v \). Also for each \( i \), denote the 1-faces of \( \tau_i \) different from \( s \) by \( s_{11} \) and \( s_{12} \) so that \( s_{11} \cap s_{12} \neq \emptyset \neq s_{12} \cap s_{12} \) but \( s_{11} \cap s_{12} = \emptyset \) for \( i \neq j \). Let \( \theta \) denote the polyhedron consisting of these simplexes.

Theorem 7. The polyhedron \( P(\theta^*) \) is homeomorphic to the 2-sphere.
Proof. The polyhedron $P(\theta^*)$ consists of the following 2-cells and their faces:

\[
\begin{align*}
\tau_1 \times U_2 & \quad U_1 \times \tau_2 & \quad S_{11} \times S_{22} & \quad S_{22} \times S_{11} \\
\tau_1 \times U_3 & \quad U_1 \times \tau_3 & \quad S_{11} \times S_{22} & \quad S_{22} \times S_{21} \\
\tau_2 \times U_1 & \quad U_2 \times \tau_1 & \quad S_{12} \times S_{21} & \quad S_{91} \times S_{12} \\
\tau_2 \times U_3 & \quad U_2 \times \tau_3 & \quad S_{12} \times S_{31} & \quad S_{31} \times S_{22} \\
\tau_3 \times U_1 & \quad U_3 \times \tau_1 & \quad S_{21} \times S_{12} & \quad S_{92} \times S_{11} \\
\tau_3 \times U_2 & \quad U_3 \times \tau_2 & \quad S_{21} \times S_{22} & \quad S_{92} \times S_{21}
\end{align*}
\]

Again the proof now consists of only routine verifications.

Suppose we add a 2-simplex $\sigma_4$ to the polyhedron $C$ so that $r$ and $r_{12}$ are faces of $\sigma_4$. Let $r_4$ denote the remaining 1-face of $\sigma_4$, and let $\Lambda$ denote the polyhedron obtained in this manner.

**Theorem 8.** The polyhedron $P(\Lambda^*)$ has the homotopy type of the 2-sphere.

Proof. The polyhedron $P(\Lambda^*)$ consists of the following cells and their faces:

\[
\begin{align*}
\sigma_3 \times r_4 & \quad \sigma_1 \times c & \quad c \times \sigma_1 & \quad r_1 \times r \\
\sigma_4 \times r_3 & \quad \sigma_2 \times c_{13} & \quad c \times \sigma_2 & \quad r_2 \times r_{13} \\
r_4 \times \sigma_3 & \quad \sigma_2 \times c & \quad r_{10} \times r_3 & \quad r_3 \times r \\
r_3 \times \sigma_4 & \quad c_{13} \times \sigma_2 & \quad r_{23} \times r_1 & \quad r \times r_1 \\
\sigma_1 \times c_{23} & \quad c_{23} \times \sigma_1 & \quad r_1 \times r_{23} & \quad r \times r_2
\end{align*}
\]

It is now a routine matter to verify that $P(\Lambda^*)$ is homotopically equivalent to the 2-sphere.

It is clear that $C$ can be imbedded in both $\theta$ and $\Lambda$.

4. Imbedding $C$. Throughout this section, let $X$ denote a finite, contractible, 2-dimensional polyhedron.

**Definition 1.** A point $x \in X$ is called a c-point of $X$ if there exist 2-simplexes, $\tau_1, \tau_2, \ldots, \tau_n$, of $X$ and a simplex $\tau$ of $X$ such that:

1. $\tau$ is not a face of $\tau_i$ for any $i$,
2. $x$ is a vertex of $\tau$ and of $\tau_i$ for each $i$,
3. $\tau_n \cap \tau_1$ is a 1-simplex $s_n$,
4. for each $i=1, 2, \ldots, n-1$, $\tau_i \cap \tau_{i+1}$ is a 1-simplex $s_i$, and
5. $\tau_i \cap \tau_j = x$ unless $i$ and $j$ satisfy the conditions of either (3) or (4).

**Note.** By a collection of 2-simplexes satisfying Definition 1, we mean the 2-simplexes $\tau_1, \tau_2, \ldots, \tau_n$, i.e. we do not include $\tau$ even though it may be a 2-simplex.

**Theorem 9.** If $f: C \to X$ is an imbedding, then $f(c_0)$ is either a c-point of $X$ or an interior point of a 1-simplex which is a face of at least three 2-simplexes.

**Proof.** First suppose $f(c_0)$ is not a vertex of $X$. Then $f(c_0)$ is an interior point of either a 1-simplex or a 2-simplex. Since the interior of $C$ is not homeomorphic to a subset of an open disk, it is easy to see that $f(c_0)$ cannot be either an interior point of a 2-simplex or an interior point of a 1-simplex which is a face of less than three 2-simplexes.
Now suppose \( f(c_0) \) is a vertex of \( X \). Since \( f \) is an imbedding, there is an arbitrarily small neighborhood \( U \) of \( f(c_0) \) such that \( U \) contains a subset which is homeomorphic to \( C \). Therefore \( f(c_0) \) is a c-point.

**Notation.** If \( t \) is a point of a 1-simplex \( s = \langle x, y \rangle \), then there exists a number \( \lambda \) such that \( 0 \leq \lambda \leq 1 \) and \( t = \lambda x + (1 - \lambda) y \). Let \( \{x, t\} = \{z = \mu x + (1 - \mu) y \mid \lambda \leq \mu \leq 1\} \).

**Notation.** If \( P \) is a locally finite polyhedron and \( v \) is a vertex of \( P \), let \( St(v, P) \) denote the open star of \( v \) in \( P \).

**Theorem 10.** If \( f: C \to X \) is an imbedding and \( f(c_0) \) is a c-point of \( X \), then there exists a unique collection \( C_t \) of 2-simplexes of \( X \) satisfying Definition 1 such that (1) \( f(C - r) \) intersects the interior of every simplex in \( C_t \) and (2) there exists a neighborhood \( U \) of \( f(c_0) \) such that \( f(C - r) \cap \text{int}(r) \cap U = \emptyset \). Moreover there is a point \( t_1 \) in \( r \) \( (t_1 \neq c_0) \) such that \( f([c_0, t_1]) \cap \bigcup \{r_1 \mid r_1 \in C_t\} = \{f(c_0)\} \).

**Proof.** Since \( f \) is continuous, there is a neighborhood \( V \) of \( c_0 \) such that \( f(V) = \text{St}(f(c_0), X) \). Let \( C' \) be a subset of \( V \) which is homeomorphic to \( C \), and let \( \Gamma = \{\tau \mid \tau \text{ is a 2-simplex of } X \text{ and } \text{int}(\tau) \cap f(C' - r) \neq \emptyset\} \).

Since \( f(C') \) is homeomorphic to \( C \) and \( f(c_0) \) is a vertex, \( \Gamma \) contains a collection \( C_t \) of 2-simplexes satisfying Definition 1. Suppose there exists a 2-simplex \( \tau \in \Gamma - C_t \). Let \( \Gamma' = \{\tau \mid \tau \in \Gamma - C_t\} \). If \( \bigcup \{\tau \mid \tau \in \Gamma'\} \cap \bigcup \{\tau \mid \tau \in C_t\} = \{f(c_0)\} \), then \( f(C' - r) \) is not connected. Therefore there exist 1-simplexes \( s_1, s_2, \ldots, s_p \) such that \( f(c_0) \) is a vertex of \( s_k \) for each \( k \) and \( \bigcup \{\tau \mid \tau \in \Gamma'\} \cap \bigcup \{\tau \mid \tau \in C_t\} = \bigcup_{k=1}^p s_k \). Since \( f(C' - r) \subset \bigcup \{\tau \mid \tau \in \Gamma\} \), \( f(C' - r) - \bigcup_{k=1}^p s_k \) is not connected. Therefore

\[
(C' - r) - f^{-1}\left(\bigcup_{k=1}^p s_k\right)
\]

is not connected, and hence \( p > 1 \). Let \( K_1, K_2, \ldots, K_n \) be the components of \( (C' - r) - f^{-1}(\bigcup_{k=1}^p s_k) \), and suppose \( K_1, K_2, \ldots, K_n \) are ordered so that \( K_i \) and \( K_{i+1} \) have a common limit point different from \( c_0 \). \( K_n \) and \( K_1 \) have a common limit point different from \( c_0 \), and \( f(K_i) \subset \bigcup \{\tau \mid \tau \in C_t\} \). Note that no three of the \( K_i \)'s can have a common limit point different from \( c_0 \). Without loss of generality, we may assume that \( f(K_2) \subset \bigcup \{\tau \mid \tau \in \Gamma\} \). Let \( p_1 \) be a common limit point of \( K_1 \) and \( K_2 \) such that \( p_1 \in C' \) and \( p_1 \neq c_0 \). There exists \( j (1 \leq j \leq p) \) such that \( f(p_1) \in s_j - \{f(c_0)\} \). Since \( C_t \) satisfies Definition 1, there exists \( i (3 \leq i \leq n) \) such that \( f(K_i) \subset \bigcup \{\tau \mid \tau \in C_t\} \) and \( f(K_i) \) and \( f(K_i) \) have a common limit point \( q_i \) in \( s_i - \{f(c_0)\} \). Thus \( f([c_0, q_1] \cap [f(c_0), f(p_1)]) \) contains a point \( x \) different from \( f(c_0) \). But \( x \in [f(c_0), q_1] \cap [f(c_0), f(p_1)] \) and \( x \neq f(c_0) \) implies that \( f^{-1}(x) \) is a limit point of \( K_1, K_2, \) and \( K_i \) which is different from \( c_0 \). Therefore \( \Gamma = C_t \).

Now let \( W \) be a neighborhood of \( c_0 \) such that \( \overline{W} \subset C' \), let \( U' \) be a neighborhood of \( f(c_0) \) which does not intersect \( f(C - W) \), and let \( U = U' \cap \text{St}(f(c_0), X) \). Let \( r \) be any simplex which is not a face of a simplex of \( C_t \). Then \( f(C' - r) \cap \text{int}(r) = \emptyset \).
Since $W \subset C'$, $f(C-C') \subset f(C-W)$. Let $x \in f(C-r)$. Then either $x \in f(C'-r)$ or $x \in f(C-C')$. If $x \in f(C'-r)$, then $x \notin \text{int}(\tau)$. If $x \in f(C-C')$, then $x \notin U$. Therefore $f(C-r) \cap \text{int}(\tau) \cap U = \emptyset$.

Suppose there is another collection $C'_r$ of 2-simplexes of $X$ satisfying Definition 1 and conditions (1) and (2) of the theorem. Then either there is a 2-simplex in $C_r$ which is not in $C'_r$ or there is a 2-simplex in $C'_r$ which is not in $C_r$. Suppose $\tau$ is in $C_r$ but not in $C'_r$. Then, since $\tau$ is not in $C'_r$, there is a neighborhood $U$ of $f(c_0)$ such that $f(C-r) \cap U \cap \text{int}(\tau) = \emptyset$. But this contradicts the fact that $\tau$ is in $C_r$.

If $f(r) \cap \bigcup \{\tau \mid \tau \in C_j\} = \{f(c_0)\}$, then we can take $t_r$ to be any point of $r$ other than $c_0$. Suppose $f(r) \cap \bigcup \{\tau \mid \tau \in C_j\} \neq \{f(c_0)\}$. Let $c$ be the other vertex of $r$, let

$$A = \{x = \mu c_0 + (1-\mu)c \mid 0 \leq \mu < 1 \text{ and } f(x) \in \bigcup \{\tau \mid \tau \in C_j\}\},$$

and let $\lambda = \inf \{\mu \mid \mu c_0 + (1-\mu)c \in A\}$. Suppose $\lambda = 1$. There exists a neighborhood $U$ of $f(c_0)$ such that each point of $U \cap \bigcup \{\tau \mid \tau \in C_j\}$ is the image of a point of $(C-r) \cup \{c_0\}$ under $f$. Since $f$ is continuous, there exists a neighborhood $V$ of $c_0$ such that $f(V) \subset U$. Since $\lambda = 1$, there is a point $c' \in r \cap V$ such that $c' \neq c_0$ and $f(c') \in \bigcup \{\tau \mid \tau \in C_j\}$. Thus $f(c') \in \bigcup \{\tau \mid \tau \in C_j\} \cap U$, and hence $f$ is not one-to-one. Therefore $\lambda < 1$. Let $\lambda'$ be a number such that $\lambda < \lambda' < 1$, and let

$$t_r = \lambda' c_0 + (1-\lambda')c.$$
Note. We extend Definition 2 in the obvious way so that we can talk about a chain of 2-simplexes joining either two c-lines or a c-line and a simplex.

Theorem 11. Let \( f, g: C \to X \) be imbeddings such that \( f(c_0) = g(c_0) \) is a c-point of \( X \). If \( f \) is isotopic to \( g \) under an isotopy \( H \) such that \( H(c_0, t) = f(c_0) \) for each \( t \in I \), then \( C_f = C_g \) and either \( s_f = s_g \) or there exists a chain \( \tau_1, \tau_2, \ldots, \tau_n \) of 2-simplexes joining \( s_f \) and \( s_g \) such that \( f(c_0) \) is a vertex of \( \tau_i \) for each \( i \) and \( \tau_i \cap \tau_{i+1} \) is not a face of a simplex of \( C_f \) for any \( i \).

Proof. Let \( H: C \times I \to X \) be an isotopy such that \( H(w, 0) = f(w) \) and \( H(w, 1) = g(w) \) for each \( w \in C \) and \( H(c_0, t) = f(c_0) \) for each \( t \in I \). For each \( t \in I \), let \( h_t: C \to X \) be the imbedding defined by \( h_t(w) = H(w, t) \).

Suppose \( C_f \neq C_g \), and let \( t' = \text{lub} \{ t \mid C_{h_t} = C_f \} \). Suppose \( C_{h_{t'}} = C_f \). Let \( \{ t_i \}_{i=1}^\infty \) be a sequence of points such that \( t_i < t_i' \) for each \( i \), \( t_i > t_{i+1} \) for each \( i \), and \( \lim_{i \to \infty} t_i = t' \). For each \( i \), there exists a 2-simplex \( \eta_i \) of \( C_f \) such that \( \eta_i \neq C_{h_{t_i}} \).

Since \( C_f \) has only a finite number of simplexes, there is a 2-simplex \( \eta \) such that \( \eta_i = \eta \) for an infinite number of \( i \)'s. Let \( V' \) be a neighborhood of \( (c_0, t') \) such that \( H(V') \subseteq \text{St}(f(c_0), X) \). There exists a connected neighborhood \( M' \) of \( c_0 \) and a neighborhood \( N' \) of \( t' \) such that \( M' \times N' \subseteq V' \). Let \( c_1 \in M' \cap D \) such that \( H(c_1, t') \in \text{int}(\eta) \). Let \( V \) be any neighborhood of \( (c_1, t') \). There exists a neighborhood \( M \) of \( c_1 \) and a connected neighborhood \( N \) of \( t' \) such that

\[
M \times N \subseteq V \cap (M' \times N').
\]

There exists \( i \) such that \( t_i \in N \) and \( \eta_i = \eta \). Since

\[
M' \times \{ t_i \} \subseteq V', H(M' \times \{ t_i \}) \subseteq \text{St}(f(c_0), X).
\]

Therefore, since \( M' \times \{ t_i \} \) is connected, \( c_0 \in M' \), and \( c_1 \in M' \cap D \), \( H(c_1, t_i) \in C_{h_{t_i}} \).

Therefore \( H(V) \notin \text{int}(\eta) \), and hence \( H \) is not continuous. If \( C_{h_{t'}} \neq C_f \), then \( t' > 0 \), and, using essentially the same argument, we can show that \( H \) is not continuous.

Therefore \( C_f = C_g \).

Suppose \( s_f \neq s_g \). For each \( t \in I \), there exists a neighborhood \( V_t \) of \( c_0 \) and a neighborhood \( W_t \) of \( t \) such that \( H(V_t \times W_t) \subseteq \text{St}(f(c_0), X) \). Let \( V_{t_1} \times W_{t_1}, V_{t_2} \times W_{t_2}, \ldots, V_{t_n} \times W_{t_n} \) be a finite subcollection of \( \{ V_t \times W_t \mid t \in I \} \) which covers \( \{ c_0 \} \times I \). Let \( V = \bigcap_{t=1}^n V_{t_i} \). There \( V \) is a neighborhood of \( c_0 \) and \( H(V \times I) \subseteq \text{St}(f(c_0), X) \). Let \( t_H \in r(t_H \neq c_0) \) such that \( [c_0, t_H] \subseteq V \), and let \( c_1 \in [c_0, t_H] \cap [c_0, t_f] \cap [c_0, t_3] \) such that \( c_1 \neq c_0 \).

We assert that there exists a neighborhood \( N \) of \( 1 \) such that if \( t \in N \), then \( t_H \) can be chosen so that

\[
h_t([c_0, t_H] \cap [c_0, c_1]) \cap \{ \tau \mid \tau \in A_\beta \} - \{ f(c_0) \} \neq \emptyset.
\]

First suppose

\[
g(\partial(C - r \cup \{ f(c_0) \})) \cap \{ \tau \mid \tau \in C_\beta \} = \emptyset.
\]
For each \( x \in \partial (C - r \cup \{ f(c_0) \}) \), there exists a neighborhood \( M_x \) of \( x \) and a neighborhood \( N_x \) of \( 1 \) such that \( H(M_x \times N_x) \cap \bigcup \{ \tau \mid \tau \in C_g \} = \emptyset \). Let \( M_x_1 \times N_x_1 \cdot M_x_2 \times N_x_2 , \ldots , M_x_n \times N_x_n \) be a finite subcollection of
\[
\{ M_x \times N_x \mid x \in \partial (C - r \cup \{ f(c_0) \}) \}
\]
which covers \( \partial (C - r \cup \{ f(c_0) \}) \times \{ 1 \} \), and let \( N' = \bigcap_{i=1}^n N_x_i \). Then \( N' \) is a neighborhood of \( 1 \), and \( H(\partial (C - r \cup \{ f(c_0) \}) \times N') \cap \bigcup \{ \tau \mid \tau \in C_g \} = \emptyset \). If \( t \in N' \), then each point of \( \bigcup \{ \tau \mid \tau \in C_g \} \) is the image under \( h_t \) of some point of \( C - r \cup \{ f(c_0) \} \). Therefore \( h_t([c_0, c_1]) \cap \bigcup \{ \tau \mid \tau \in C_g \} - \{ f(c_0) \} = \emptyset \) if \( t \in N' \). Thus we may assume that for \( t \in N' \), \([c_0, c_1]\subset [c_0, t_{hi}] \). Let \( B_g = \{ \tau \mid f(c_0) \text{ is a vertex of } \tau \} \). Then \( W = \bigcup \{ \text{int}(\tau) \mid \tau \in B_g \} \) is an open set such that \( g(c_1) \in W \subseteq \bigcup \{ \tau \mid \tau \in A_g \} - \{ f(c_0) \} \). Therefore there exists a neighborhood \( N' \) of \( 1 \) such that if \( t \in N' \), then \( h_t(c_1) \in W \). Let \( N = N' \cap N'' \). If \( t \in N \), then
\[
h_t([c_0, c_1]) \cap \{ \tau \mid \tau \in A_g \} - \{ f(c_0) \} \neq \emptyset
\]
because \([c_0, c_1]\subset [c_0, t_{hi}] \) and \( h_t(c_1) \in W \). Now suppose
\[
g(\partial (C - r \cup \{ f(c_0) \})) \cap \bigcup \{ \tau \mid \tau \in C_g \} \neq \emptyset.
\]
Let
\[
e = d[g(\partial (C - r \cup \{ f(c_0) \})) \cap \bigcup \{ \tau \mid \tau \in C_g \}, f(c_0)],
\]
where \( d \) is a metric for \( X \). Then \( \epsilon > 0 \), and by an argument similar to the one above, there exists a neighborhood \( N' \) of \( 1 \) such that if \( t \in N' \), then
\[
d[h_t(\partial (C - r \cup \{ f(c_0) \})) \cap \bigcup \{ \tau \mid \tau \in C_g \}, f(c_0)] > \epsilon / 2.
\]
Let \( U' \) be the \( \epsilon / 2 \)-neighborhood of \( f(c_0) \), and let \( U = \text{St}(f(c_0), X) \cap U' \). There exists a neighborhood \( M \) of \( c_0 \) and neighborhood \( N' \) of \( 1 \) such that \( H(\text{M} \times N') \subseteq U \). Let \( c' \in r(c' \neq c_0) \) such that \([c_0, c'] \subset M \cap [c_0, c_1]\). If \( t \in N' \cap N'' \), then
\[
h_t([c_0, c']) \cap \{ \tau \mid \tau \in C_g \} = \{ f(c_0) \}.
\]
Thus if \( t \in N' \cap N'' \), we may assume that \([c_0, c'] \subset [c_0, t_{hi}] \). Therefore, by an argument similar to the one above, we can show that there exists a neighborhood \( N \) of \( 1 \) such that if \( t \in N \), then \( t_{hi} \) can be chosen so that
\[
h_t([c_0, t_{hi}] \cap [c_0, c_1]) \cap \{ \tau \mid \tau \in A_g \} - \{ f(c_0) \} \neq \emptyset.
\]
Now we return to the proof of the theorem. There exist 2-simplexes \( \tau_1, \tau_2, \ldots , \tau_m \) in \( X \) such that \( H([c_0, c_1] \times I) \cap \tau_i \neq \emptyset \) for each \( i = 1, 2, \ldots , m \), and
\[
H([c_0, c_1] \times I) \subset \bigcup_{i=1}^m \tau_i.
\]
Obviously some subcollection of \( \tau_1, \tau_2, \ldots , \tau_m \) is a chain joining \( s_p \) and \( s_q \). Suppose that for each such subcollection \( \tau_1, \tau_2, \ldots , \tau_n \), \( \tau_i \cap \tau_{i+1} \) is a face of a simplex of
C, for some i = 1, 2, . . . , n − 1. For each t ∈ I, some subcollection of τ₁, τ₂, . . . , τₘ is a chain joining sℏᵢ and sᵢ. Let

$$\Gamma = \{t \mid \text{if } \tau₁, \tau₂, . . . , \tauₙ \text{ is any subcollection of } \tau₁, \tau₂, . . . , \tauₘ \text{ which is a chain joining } sℏᵢ \text{ and } sᵢ, \text{ then } \tauᵢ \cap \tauᵢ₊₁ \text{ is a face of some simplex of } C, \text{ for some } i = 1, 2, . . . , n − 1\},$$

and let t' = glb{t | t ∈ Γ}. Suppose t' = 1. Observe that if ρ, ρ' ∈ Aₜᵢ for some t, and ρ can be joined to sᵢ by a subcollection τ₁, τ₂, . . . , τₙ of τ₁, τ₂, . . . , τₘ so that τᵢ ∩ τᵢ₊₁ is not a face of C, for any i = 1, 2, . . . , n − 1, then ρ' can be joined to sᵢ by such a subcollection of τ₁, τ₂, . . . , τₘ. If

$$h₝([c₀, ℏᵢ] \cap [c₀, c₁]) \cap \bigcup \{τ \mid τ ∈ Aₜᵢ\} - \{f(c₀)\} \neq \emptyset,$$

then Aₜᵢ and Aₜ have a common simplex and hence each simplex in Aₜᵢ can be joined to sᵢ by a subcollection τ₁, τ₂, . . . , τₙ of τ₁, τ₂, . . . , τₘ so that τᵢ ∩ τᵢ₊₁ is not a face of C, for any i = 1, 2, . . . , n − 1. Therefore, if t' = 1,

$$h₝([c₀, ℏᵢ] \cap [c₀, c₁]) \cap \bigcup \{τ \mid τ ∈ Aₜᵢ\} - \{f(c₀)\} = \emptyset$$

for each t < 1. This contradicts the assertion and hence t' ≠ 1. By the assertion, there exists a neighborhood N of t' such that if t ∈ N, then ℏᵢ can be chosen so that

$$h₝([c₀, ℏᵢ] \cap [c₀, c₁]) \cap \bigcup \{τ \mid τ ∈ Aₜᵢ\} - \{f(c₀)\} \neq \emptyset.$$

If t ∈ N, then each simplex in Aₜᵢ can be joined to sₜᵢ by a subcollection τ₁, τ₂, . . . , τₙ of τ₁, τ₂, . . . , τₘ so that τᵢ ∩ τᵢ₊₁ is not a face of C, for any i = 1, 2, . . . , n − 1. Since there exist t ∈ N ∩ Γ, t' ∈ Γ. Therefore t' > 0, and hence there exist t ∈ N such that t < t'. Thus t' ∉ Γ.

The original proof of the following theorem was due to Ross Finney. The author is also indebted to the referee for suggesting a simpler proof.

**Theorem 12.** Let K be a locally finite polyhedron, and let v be a vertex of K. If h: I × I → K is a homeomorphism such that h(0) = v, then there exists an isotopy F: I × I → K such that F(x, 0) = h(x) for all x ∈ I, F | I × {1} is a homeomorphism of I onto an edge emanating from v, and F(0, t) = v for all t ∈ I.

**Proof.** If h(I) ∉ St(v, K), let x be the smallest number in I such that h(x) ∉ St(v, K). Then

$$H: I × I → K, (x, t) → h(x − tx + tx),$$

is an isotopy such that H(x, 0) = h(x) for all x ∈ I, H(x, 1) = h(x) ∈ St(v, K) for all x ∈ I, H(1, 0) = v, and H(0, t) = v for all t ∈ I. If h(I) ⊆ St(v, K), then it is easy to see that there is an isotopy H: I × I → K such that H(x, 0) = h(x) for all x ∈ I, H(x, 1) ∈ St(v, K) for all x ∈ I, and H(0, t) = v for all t ∈ I. Thus we may assume without loss of generality that h(I) ⊆ St(v, K) and h(x) ∈ ∂St(v, K) if and only if x = 1. Now define G: I × I → K by

$$G(x, t) = xh(1) + (1 − x)v, \quad t ≤ x ≤ 1,$$

$$= th(x/t) + (1 − t)v, \quad 0 ≤ x < t.$$
Then $G$ is an isotopy such that $G \mid I \times \{0\}$ is a homeomorphism of $I$ onto a line segment in $[St(v, K)]^{-}$ from $v$ to $h(1)$, $G(x, 1) = h(x)$ for all $x \in I$, and $G(0, t) = v$ for all $t \in I$.

**Notation.** Let $x_0$ be a $c$-point of $X$, let $C_p$ be a collection of 2-simplexes of $X$, and let $s_p$ be a $c$-line of $X$ such that $x_0$, $C_p$, and $s_p$ satisfy Definition 1. Let $\tau$ be a 2-simplex of $C_p$, let $s_l$ and $s_o$ denote the 1-faces of $\tau$ which have $x_0$ as a vertex, let $s_3$ denote the 1-face of $\tau$ which does not have $x_0$ as a vertex, and let

$$S = \bigcup \{s : s \text{ is a 1-face of a simplex of } C_p, x_0 \text{ is not a vertex of } s, \text{ and } s \text{ is not a face of } \tau\}.$$  

Using the same notation for the simplexes of $C$ as that used in §3, let $p, p' : C \to X$ be the homeomorphisms which satisfy the following properties:

1. $p$ maps $r$ linearly onto $s_p$,
2. $p$ maps $r_j$ linearly onto $s_{j-1}$ for each $j = 2, 3$,
3. $p$ maps each point of $s_i$ into the point of $\tau$ which has the same barycentric coordinates,
4. $p$ maps $r_2 \cup r_3$ linearly onto $S$,
5. if $L$ is a line segment from $c_0$ to $r_2 \cup r_3$, then $p$ maps $L$ linearly onto the line segment from $x_0$ to $p(L \cap (r_2 \cup r_3))$,
6. $p'$ maps $r$ linearly onto $s_p$,
7. $p'$ maps $r_j$ linearly onto $s_{j-1}$ for each $j = 2, 3$,
8. $p'$ maps $r_1$ linearly onto $S$,
9. if $L$ is a line segment from $c_0$ to $r_1$, then $p'$ maps $L$ linearly onto the line segment from $x_0$ to $p'(L \cap r_1)$,
10. $p'$ maps $r_2 \cup r_3$ linearly onto $s_3$, and
11. if $L$ is a line segment from $c_0$ to $r_2 \cup r_3$, then $p'$ maps $L$ linearly onto the line segment from $x_0$ to $p'(L \cap (r_2 \cup r_3))$.

**Note.** In the remainder of this paper, when we speak of $p$ and $p'$, we will mean homeomorphisms satisfying the above conditions. This means that $C_p = C_{p'}$ and $s_p = s_{p'}$.

**Theorem 13.** If $f : C \to X$ is an imbedding such that $f(c_0)$ is a $c$-point of $X$, $C_f = C_p$, and either $s_l = s_o$ or there exists a chain $\tau_1, \tau_2, \ldots, \tau_n$ of 2-simplexes joining $s_l$ and $s_o$ such that $f(c_0)$ is a vertex of $\tau_i$ for each $i$ and $\tau_i \cap \tau_{i+1}$ is not a face of a simplex of $C_f$ for any $i$, then $f$ is isotopic to either $p$ or $p'$ under an isotopy $H$ such that $H(c_0, t) = f(c_0)$ for each $t \in I$.

**Proof.** Since $f$ is continuous, there exists a neighborhood $V$ of $c_0$ such that $f(V) \subseteq \text{St}(f(c_0), X)$. Let $c' \in r$ such that $c' \neq c_0$ and $[c_0, c'] \subseteq V$, and let $c_1 \in [c_0, t_j] \cap [c_0, c']$ such that $c_1 \neq c_0$. There exists $\lambda (0 \leq \lambda < 1)$ such that $c_1 = \lambda c_0 + (1 - \lambda)c$. Let $(w, t) \in C \times I$. If $w \in r$, there exists $\mu$ $(0 \leq \mu \leq 1)$ such that $w = \mu c_0 + (1 - \mu)c$. Define $K : C \times I \to X$ by

$$K(w, t) = f(w), \quad \text{if } w \in D,$$

$$= f((\mu + t\lambda - t\lambda\mu)c_0 + (1 - \mu - \lambda t + t\lambda\mu)c), \quad \text{if } w \in r.$$
Then $K$ is an isotopy, $K(w, 0) = f(w)$, and $K_1$ is an imbedding of $C$ into $X$ such that $K_1(\tau) \subseteq (\mathrm{St}(f(c_0), X) - \bigcup \{\tau \mid \tau \in C_i\}) \cup \{f(c_0)\}$ and $K_1(w) = f(w)$ for all $w \in D$.

There exists a positive number $S$ such that if
\[
D' = \{x \mid x \in E \{\tau \mid \tau \in C_i\} \text{ and } d(f(c_0), x) < S\},
\]
then $D' \subseteq (\mathrm{int}(f(D))) \cup \bigcup \{\tau \mid \tau \in C_i\}$. Then $f^{-1}(D') \subseteq \mathrm{int}(D)$. Let $A_1$ be the annulus bounded by $f^{-1}(\partial D')$ and $\partial D$, and let $k_1$ be a homeomorphism of $A_1$ onto the annulus $\{z \mid 3 \leq |z| \leq 4\}$ in the plane which sends $f^{-1}(\partial D')$ onto $\{z \mid |z| = 3\}$. Let $D^* = \text{a disk with center at } c_0$ such that $D^* \subseteq \mathrm{int}(f^{-1}(D'))$, and let $A_2$ be the annulus bounded by $\partial D^*$ and $f^{-1}(\partial D')$. Let $k_2$ be a homeomorphism of $A_2$ onto the annulus $\{z \mid 1 \leq |z| \leq 2\}$ in the plane which sends $\partial D^*$ onto $\{z \mid |z| = 1\}$. Define $k_3$ mapping $D^*$ onto the disk $\{z \mid |z| \leq 1\}$ in the plane as follows: $k_3(c_0)$ is the origin, $k_3(w)$ is $k_3(w)$ if $w \in \partial D^*$, and if $L$ is a line segment from $c_0$ to $\partial D^*$, then $k_3$ maps $L$ linearly onto the line segment from the origin to $k_3(L \cap \partial D^*)$. Then $k_4 : f^{-1}(D') \to E^2$ defined by $k_4(w) = k_3(w)$, if $w \in f^{-1}(D') - \partial D^*$, and $k_4(w) = k_3(w)$, if $w \in \partial D^*$, is a homeomorphism of $f^{-1}(D')$ onto the disk $\{z \mid |z| \leq 2\}$. Define $k_5 : \{z \mid |z| \leq 2\} \to \{z \mid |z| \leq 3\}$ by $k_5$ of the origin is the origin, $k_5(z) = k_3(k_2^{-1}(z))$, if $|z| = 2$, and if $L$ is a line segment from the origin to $\{z \mid |z| = 2\}$, then $k_5$ maps $L$ linearly onto the line segment from the origin to the origin $k_5(L \cap \{z \mid |z| = 2\})$. Then $k : D \to \{z \mid |z| \leq 4\}$ defined by $k(z) = k_5(z)$, if $z \in D - f^{-1}(D')$, and $k(z) = k_5(k_4(z)$, if $z \in f^{-1}(D')$, is a homeomorphism which sends $\partial D$ onto $\{z \mid |z| = 4\}$ and $f^{-1}(\partial D')$ onto $\{z \mid |z| = 3\}$ and maps $c_0$ into the origin. Define $G : \{z \mid |z| \leq 4\} \times I \to \{z \mid |z| \leq 4\}$ by $G(z, t) = \frac{z - tz/4}{w}$. Define $F : D \times I \to X$ by $F(w, t) = f^{-1}G(k(w), t)$. Then $F$ is an isotopy, $F(w, 0) = f(w)$, and $F(w, 1) \in D'$. Since $F(c_0, t) = f(c_0)$ for all $t \in I$, we can extend $F$ to an isotopy $F^* : C \times I \to X$ by defining $F^*(w, t) = K_1(w)$ for all $w \in r$. Then $F^*(w, 0) = K_1(w)$ for all $w \in C$, and $F^*$ is an imbedding of $C$ into $X$ such that $F^*(D) = D'$.

Let $w \in D - \{c_0\}$, and let $L_1$ be the line segment from $c_0$ to $\partial D$ which passes through $w$. Then $F^*(L_1 \cap \partial D) \subseteq \partial D'$. Let $L_2$ be the line segment from $f(c_0)$ to $\partial(f(c_0) \cup \{\tau \mid \tau \in C_i\})$ which passes through $F^*(L_1 \cap \partial D)$, and let
\[
a = L_2 \cap \partial(f(c_0) \cup \{\tau \mid \tau \in C_i\})
\]
Let $e$ be a metric for $C$, and let $e$ be the $e$ radius of $D$. Define $J : C \times I \to X$ by
\[
J(w, t) = K_1(w), \text{ if } w \in r,
\]
$J(w, t) = F^*(w)$ (the point on $L_1$ whose distance from $c_0$ is $2e(w, c_0)/(2-t)$),
\[
J(w, t) = \begin{cases} \frac{2e(w, c_0) - 2e + et}{e}a + \frac{(3e - et - 2e(w, c_0))e}{e}F^*(L_1 \cap \partial D), & \text{if } w \in D \text{ and } e(w, c_0) \leq e(2-t)/2, \\
\end{cases}
\]
and
\[
J(w, t) = \begin{cases} \frac{(2e(w, c_0) - 2e + et)e}{e}a + \frac{(3e - et - 2e(w, c_0))e}{e}F^*(L_1 \cap \partial D), & \text{if } w \in D \text{ and } e(w, c_0) \geq e(2-t)/2. \\
\end{cases}
\]
Then $J$ is an isotopy, $J(w, 0) = F^*(w)$, if $w \in D$, and $J_1(D) = \bigcup \{\tau \mid \tau \in C_i\}$. 

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It is clear that there exists an isotopy $M^*: \partial D \times I \to \partial(\bigcup \{ \tau \mid \tau \in C_j \})$ such that $M^*(w, 0)=J_1(w)$ and $M^*$ is either $p|\partial D$ or $p'|\partial D$. Also it is clear that this isotopy can be extended to an isotopy $M': D \times I \to \bigcup \{ \tau \mid \tau \in C_j \}$ such that $M'_0=J_1$ and $M'(c_0, t)=f(c_0)$. Then we can extend $M'$ to an isotopy

$$M: C \times I \to \bigcup \{ \tau \mid \tau \in C_j \} \cup f([c_0, c_1])$$

by defining $M(w, t)=K_1(w)$ for all $w \in r$. Now by Alexander's Theorem [1], $M_1'$ is isotopic to either $p|D$ or $p'|D$ under an isotopy $N'$ such that $N'(c_0, t)=f(c_0)$. Again $N'$ can be extended to an isotopy $N: C \times I \to \bigcup \{ \tau \mid \tau \in C_j \} \cup f([c_0, c_1])$ by defining $N(w, t)=K_1(w)$ for all $w \in r$.

The desired result now follows immediately from Theorem 12.

**Theorem 14.** Let $f: C \to X$ be an imbedding such that $f(c_0)$ is a c-point of $X$. If $F: C \times I \to X$ is an isotopy such that $F(w, 0)=f(w)$ for each $w \in C$ and

$$t' = \text{glb}(t \mid F(c_0, t) \neq f(c_0)),$$

then there exists a neighborhood $V$ of $t'$ such that $F(c_0, t) \in \bigcup \{ \tau \mid \tau \in C_j \}$ whenever $t \in V$.

**Proof.** Suppose that for each neighborhood $R$ of $t'$, there exists $t \in R$ such that $F(c_0, t) \notin \bigcup \{ \tau \mid \tau \in C_j \}$. Observe that $F(c_0, t')=f(c_0)$. Let $V'$ be a neighborhood of $(c_0, t')$ such that $F(V') \subseteq \text{St}(f(c_0), X)$. There exists a connected neighborhood $M'$ of $c_0$ and a neighborhood $N'$ of $t'$ such that $M' \times N' \subseteq V'$. Let $c_1 \in M' \cap D$ such that $c_1 \neq c_0$. Then $F(c_1, t') \in \text{int}(\bigcup \{ \tau \mid \tau \in C_j \})$. Let $V$ be any neighborhood of $(c_1, t')$. There exists a neighborhood $M$ of $c_1$ and a connected neighborhood $N$ of $t'$ such that $M \times N \subseteq V \cap (M' \times N')$. There exists $t_1 \in N$ such that

$$F(c_0, t_1) \notin \bigcup \{ \tau \mid \tau \in C_j \}.$$

Since

$$M' \times \{t_1\} \subseteq V', F(M' \times \{t_1\}) \subseteq \text{St}(f(c_0), X).$$

Let $X'$ be a subdivision of $X$ such that $F(c_0, t_1)$ is a c-point of $X'$, and let $f_{t_1}=F|C \times \{t_1\}$. Since $M' \times \{t_1\}$ is connected, $c_0 \in M'$, and

$$c_1 \in M' \cap D, F(c_1, t_1) \in \text{int}(\bigcup \{ \tau \mid \tau \in C_{t_1} \}).$$

Therefore $F(V) \subseteq \text{int}(\bigcup \{ \tau \mid \tau \in C_j \})$, and hence $F$ is not continuous.

**Definition 4.** If $f, g: C \to X$ are imbeddings such that $f(c_0)$ and $g(c_0)$ are c-points of $X$, then we say that $f(C)$ and $g(C)$ are **combinatorially joined** if there exist a sequence $s_1, s_2, \ldots, s_a$ of 1-simplexes and three sequences $\tau_1, \tau_2, \ldots, \tau_b; \tau'_1, \tau'_2, \ldots, \tau'_a; \tau^*_1, \tau^*_2, \ldots, \tau^*_a$ of 2-simplexes such that:

1. $f(c_0)$ is a vertex of $s_1$ and $g(c_0)$ is a vertex of $s_a$,
2. $s_\beta \cap s_{\beta+1}$ is a vertex for each $\beta=1, 2, \ldots, a-1$,
(3) $s_\beta$ is a face of $\tau_1$ and $s_\alpha$ is a face of $\tau_\beta$.

(4) $\tau_1'$ and $\tau_1$ are simplexes of $C$ and $\tau_\alpha'$ and $\tau_\alpha$ are simplexes of $C_\alpha$.

(5) For each $i$, $j$, and $k$, $\tau_i \cap \tau_{i+1}$, $\tau_j \cap \tau_{j+1}$, and $\tau_k \cap \tau_{k+1}$ are $\rho$-simplexes ($\rho = 1, 2$), and

(6) For each $\beta = 1, 2, \ldots, \alpha$, we may choose $i(\beta)$, $j(\beta)$, and $k(\beta)$ such that:

(a) $j(1) = 1$, $k(1) = 1$, $j(\alpha) = m$, and $k(\alpha) = n$,

(b) For each $\beta = 1, 2, \ldots, \alpha - 1$, $i(\beta + 1) > i(\beta)$, $j(\beta + 1) > j(\beta)$, and $k(\beta + 1) > k(\beta)$,

(c) $\tau_{i(\beta)}$, $\tau_{j(\beta)}$, and $\tau_{k(\beta)}$ are distinct,

(d) $\tau_{i(\beta)} \cap \tau_{j(\beta)} \cap \tau_{k(\beta)} = s_\beta$,

(e) If $\alpha > 1, 2, \ldots, \alpha$, then $\tau_{i(\beta)} \cap \tau_{j(\beta)}$ is a $\rho$-simplex ($\rho = 1, 2$), then $\tau_{i(\beta)} \cap \tau_{j(\beta)} = \tau_{i(\beta)}$,

(f) If $\alpha > 1, 2, \ldots, \alpha$, then $\tau_{i(\beta)} \cap \tau_{j(\beta)}$ is a $\rho$-simplex ($\rho = 1, 2$), then $\tau_{i(\beta)} \cap \tau_{j(\beta)} = \tau_{i(\beta)}$,

(g) If $\alpha > 1, 2, \ldots, \alpha$, then $\tau_{i(\beta)} \cap \tau_{j(\beta)}$ is a vertex $v$, then, for each $\gamma = i(\beta)$, $j(\beta) + 1$, and each $\epsilon = k(\beta)$, $j(\beta) + 1$, and $\tau_{i(\beta)} \cap \tau_{j(\beta)} \cap \tau_{k(\beta)} = \{v\}$,

(h) If $\alpha > 1, 2, \ldots, \alpha$, then $\tau_{i(\beta)} \cap \tau_{j(\beta)}$ is a vertex $v$, then, for each $\gamma = i(\beta)$, $j(\beta) + 1$, and each $\epsilon = k(\beta)$, $j(\beta) + 1$, and $\tau_{i(\beta)} \cap \tau_{j(\beta)} \cap \tau_{k(\beta)} = \{v\}$,

(i) If $\alpha > 1, 2, \ldots, \alpha$, then $\tau_{i(\beta)} \cap \tau_{j(\beta)}$ is a vertex $v$, then, for each $\gamma = i(\beta)$, $j(\beta) + 1$, and each $\epsilon = k(\beta)$, $j(\beta) + 1$, and $\tau_{i(\beta)} \cap \tau_{j(\beta)} \cap \tau_{k(\beta)} = \{v\}$,

(j) If $\alpha > 1, 2, \ldots, \alpha$, then $\tau_{i(\beta)} \cap \tau_{j(\beta)}$ is a vertex $v$, then, for each $\gamma = i(\beta)$, $j(\beta) + 1$, and each $\epsilon = k(\beta)$, $j(\beta) + 1$, and $\tau_{i(\beta)} \cap \tau_{j(\beta)} \cap \tau_{k(\beta)} = \{v\}$,

(k) If $\alpha > 1, 2, \ldots, \alpha$, then, for each $\gamma = i(\beta)$, $j(\beta) + 1$, and each $\epsilon = k(\beta)$, $j(\beta) + 1$, and $\tau_{i(\beta)} \cap \tau_{j(\beta)} \cap \tau_{k(\beta)} = \{v\}$,

(l) If $\alpha < q$, then, for each $i = i(\alpha) + 1, \ldots, q$, $\tau_{i(\beta)} \cap \tau_{j(\beta)} \cap \tau_{k(\beta)} = \{v\}$.

We say that $s_1, s_2, \ldots, s_\alpha$ and $\tau_1, \tau_2, \ldots, \tau_\alpha$ combinatorially join $f(C)$ and $g(C)$.

Theorem 15. Let $f, g: C \to X$ be imbeddings such that $f(c_0)$ and $g(c_0)$ are c-points of $X$. If $f$ is isotopic to $g$ under an isotopy $H$ such that $H(c_0, t) \neq f(c_0)$ for some $t \in I$, then $f(C)$ and $g(C)$ are combinatorially joined.

Proof. We may choose 1-simplexes $s_1, s_2, \ldots, s_\alpha$ in

$\{s | s$ is a 1-simplex and $H((c_0 \times I) \cap \text{int}(s)) \neq \emptyset\}$,

2-simplexes $\tau_1, \tau_2, \ldots, \tau_q$ in

$\{\tau | \tau$ is a 2-simplex and for some $t \in I$ arbitrarily small neighborhoods of $H(c_0, t)$ intersect $H((r-\{c_0\} \times \{t\}) \cap \tau)\}$,

and 2-simplexes $\tau_{i1}, \tau_{i2}, \ldots, \tau_{i\alpha}$; $\tau_{j1}, \tau_{j2}, \ldots, \tau_{j\alpha}$ in

$\{\tau | \tau$ is a 2-simplex and for some $t \in I$ arbitrarily small neighborhoods of $H(c_0, t)$ intersect $H((D-\{c_0\}) \times \{t\}) \cap \tau)\}$

so that they may be ordered in such a way as to satisfy Definition 4.

Theorem 16. The imbeddings $p$ and $p'$ are not isotopic.

Proof. Suppose $F: C \times I \to X$ is an isotopy between $p$ and $p'$. If $F(c_0, t) = x_0$ for all $t \in I$, then $F | \partial D \times I$ is an isotopy in $X - \{x_0\}$ between $p | \partial D$ and $p' | \partial D$, and therefore $X$ is not contractible. Hence there exists $t \in I$ such that $F(c_0, t) \neq x_0$. 

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Now there exists a sequence \( s_1, s_2, \ldots, s_a \) of 1-simplexes and three sequences
\[ \tau_1, \tau_2, \ldots, \tau_q; \tau'_1, \tau'_2, \ldots, \tau'_m; \tau''_1, \tau''_2, \ldots, \tau''_n \]
of 2-simplexes which combinatorially join \( p(C) \) and \( p'(C) \) and which have the following properties:

1. \( \text{int}(s_\beta) \cap F((c_0) \times I) \neq \emptyset \) for each \( \beta = 1, 2, \ldots, a, \)
2. for each \( i, j \), there exists \( t \in I \) such that arbitrarily small neighborhoods of \( F(c_0, t) \cap \tau_i \)
3. for each \( j \), there exists \( t \in I \) such that arbitrarily small neighborhoods of \( F(c_0, t) \cap \tau'_j \)
4. for each \( k \), there exists \( t \in I \) such that arbitrarily small neighborhoods of \( F(c_0, t) \cap \tau''_k \).

We will assume throughout the remainder of this proof that \( F \mid C \times \{0\} = p \) and show that \( F \mid C \times \{1\} \neq p' \). First suppose that \( p(c_0) \) is not a vertex of \( s_\beta \) for any \( \beta = 2, 3, \ldots, a-1 \). If \( s_1, s_2, \ldots, s_a \) does not contain a simple closed curve, then it is easy to see that \( F \mid C \times \{1\} \) is “essentially” \( p \) rather than \( p' \) because the isotopy has not “flipped” the disk \( \cup \{ \tau \mid \tau \in C_p \} \). If \( s_1, s_2, \ldots, s_a \) contains a simple closed curve, then \( F \mid C \times \{1\} \) is “essentially” either \( p \) or a rotation of \( p \) rather than \( p' \) because if the isotopy “flips” the disk \( \cup \{ \tau \mid \tau \in C_p \} \) then \( s_p = s_p' \) cannot be a face of \( \tau_q \). Now if \( p(c_0) \) is a vertex of \( s_\beta \) for some \( \beta = 2, 3, \ldots, a-1 \), then, in order to determine \( F \mid C \times \{1\} \), we examine some finite combination of the possibilities listed above. But it is obvious that this finite combination will “essentially” yield either \( p \) or a rotation of \( p \) rather than \( p' \). Therefore \( p \) is not isotopic to \( p' \).

**Theorem 17.** Let \( f, g: C \to X \) be imbeddings such that \( f(c_0) \) and \( g(c_0) \) are c-points of \( X \). If \( C_f = C_p \), if either \( s_1 = s_p \) or there exists a chain \( \tau_1, \tau_2, \ldots, \tau_n \) of 2-simplexes joining \( s_f \) and \( s_p \) such that \( f(c_0) \) is a vertex of \( \tau_i \) for each \( i \) and \( \tau_i \cap \tau_{i+1} \) is not a face of a simplex of \( C_f \) for any \( i \), and if \( f(C) \) and \( g(C) \) are combinatorially joined, then \( g \) is isotopic to either \( p \) or \( p' \).

**Proof.** By Theorem 13, there is a \( p_g \) such that \( g \) is isotopic to either \( p_g \) or \( p'_g \). It is clear that \( p(C) \) and \( p_g(C) \) are combinatorially joined. Therefore \( p_g \) is isotopic to either \( p \) or \( p' \), and hence \( g \) is isotopic to either \( p \) or \( p' \).

**Theorem 18.** If \( f: C \to X \) is an imbedding and \( f(c_0) \) is an interior point of a 1-simplex \( s \) of \( X \), then there exists a unique collection \( D_f \) consisting of two 2-simplexes of \( X \) which contain \( s \) as a face such that (1) \( f(C-r) \) intersects the interior of every simplex in \( D_f \) and (2) there exists a neighborhood \( U \) of \( f(c_0) \) such that if \( \tau \) is a simplex which is not a face of a simplex of \( D_f \), then \( f(C-r) \cap \text{int}(\tau) \cap U = \emptyset \). Moreover there is a point \( t_j \) in \( r \) (\( t_j \neq c_0 \)) and a 2-simplex \( \tau \in X-C_f \) such that
\[ f([c_0, t_j]-(c_0)) \subset \text{int}(\tau). \]

The proof is essentially the same as the proof of Theorem 10 and hence it is omitted.
Notation. If \( \tau \) is the 2-simplex such that \( f([c_0, t_1]-(c_0)) \subseteq \text{int}(\tau) \), let \( s_\tau \) denote the line segment in \( \tau \) from \( f(c_0) \) to the vertex of \( \tau \) which is not a vertex of \( s \).

Note. Since \( f(c_0) \) is a c-point of a subdivision of \( X \), we can obviously define imbeddings \( p \) and \( p' \) just as before so that \( p(c_0) = p'(c_0) = f(c_0) \) and show that \( f \) is isotopic to either \( p \) or \( p' \) but not both.

**Theorem 19.** Suppose \( f, g : C \to X \) are imbeddings such that \( f(c_0) \) and \( g(c_0) \) are interior points of 1-simplexes \( s_1 \) and \( s_2 \) respectively (\( s_1 \neq s_2 \)). Then \( f \) is isotopic to \( g \) if and only if there exist imbeddings \( h, k : C \to X \) such that \( h(c_0) \) and \( k(c_0) \) are c-points, \( f \) is isotopic to \( h \), \( g \) is isotopic to \( k \), and \( h \) is isotopic to \( k \).

**Proof.** Suppose \( F : C \times I \to X \) is an isotopy such that \( F(w, 0) = f(w) \) and \( F(w, 1) = g(w) \) for all \( w \in C \). Suppose that \( F(c_0, t) \) is not a c-point of \( X \) for any \( t \in I \). If \( t_1 = \text{lub}\{t \mid F(c_0, t) \in s_1\} \), then \( F \) is not continuous at \((c_0, t_1)\).

If the condition is satisfied, then \( f \) is isotopic to \( g \) because isotopy is an equivalence relation.

**Theorem 20.** Suppose \( f : C \to X \) is an imbedding such that \( f(c_0) \) is an interior point of a 1-simplex \( s \) of \( X \). Then there exists an imbedding \( g : C \to X \) such that \( g(c_0) \) is a c-point of \( X \) and \( f \) is isotopic to \( g \) if and only if there exists a vertex \( v \) of \( s \), 2-simplexes \( \tau_1, \tau_2, \ldots, \tau_n \) of \( X \), and a 1-simplex \( s_1 \) of \( X \) such that:

1. \( v, \tau_1, \tau_2, \ldots, \tau_n, \) and \( s_1 \) satisfy Definition 1,
2. \( \bigcup \{\tau \mid \tau \in D_1\} \subseteq \bigcup_{i=1}^{n} \tau_i \), and
3. either \( s_1 \) and \( s \) are in the same 2-simplex or there exists a chain \( \tau_1, \tau_2, \ldots, \tau_q \) of 2-simplexes such that \( s_1 \subseteq \tau_1, s_j \subseteq \tau_j, v \) is a vertex of \( \tau_j \) for each \( j \), and \( \tau_j \cap \tau_{j+1} \) is a 1-simplex which is not a face of \( \tau_i \) for any \( i \).

**Proof.** Suppose there exists an imbedding \( g : C \to X \) such that \( g(c_0) \) is a c-point of \( X \) and \( f \) is isotopic to \( g \). Let \( F : C \times I \to X \) be an isotopy such that \( F(w, 0) = f(w) \) and \( F(w, 1) = g(w) \) for all \( w \in C \). Let \( t_1 = \text{glb}\{t \mid F(c_0, t) \in \text{int}(\tau)\} \). Then \( F(c_0, t_1) \) is a vertex \( v \) of \( s \), and it is clear that the imbedding \( f_{t_1} : C \to X \) defined by \( f_{t_1}(w) = F(w, t_1) \) gives us a collection of simplexes satisfying the condition.

Suppose the condition is satisfied. By the note preceding Theorem 19, there is a \( p \) such that \( f \) is isotopic if either \( p \) or \( p' \). It is clear that \( p \), and hence \( p' \), is isotopic to an imbedding \( g : C \to X \) such that \( g(c_0) \) is a c-point of \( X \).

**Theorem 21.** Let \( s \) be a 1-simplex of \( X \) which does not have a c-point as vertex but which is a face of at least three 2-simplexes. If \( n \) is the number of 2-simplexes which have \( s \) as a face, then \( \{f : C \to X \mid f \text{ is an imbedding and } f(c_0) \text{ is an interior point of } s\} \) consists of \( 6C(n, 3) \) isotopy classes.

**Proof.** It is clear that if either \( D_f \neq D_g \) or \( f([c_0, t_1]) \) and \( g([c_0, t_1]) \) are in different simplexes, then \( f \) is not isotopic to \( g \). Thus the theorem follows since there exists \( p \) such that if \( D_f = D_p \) and \( f([c_0, t_1]) \) and \( p(r) \) are in the same simplex, then \( f \) is isotopic to either \( p \) or \( p' \) but not both.
Summary. Now it follows that in order to compute the number of isotopy classes of imbeddings of $C$ in $X$, it is sufficient to consider only the $c$-points and the 1-simplexes which do not have a $c$-point as vertex but which are faces of at least three 2-simplexes. Let $x_1, x_2, \ldots, x_m$ denote the $c$-points of $X$, and let $s_1, s_2, \ldots, s_n$ denote the 1-simplexes which do not have a $c$-point as vertex but which are faces of at least three 2-simplexes. For each $i = 1, 2, \ldots, m$, let $C_{i1}, C_{i2}, \ldots, C_{iq_i}$ be the collections of 2-simplexes having $x_i$ as a vertex and satisfying Definition 1. Suppose $1 \leq i \leq m$ and $1 \leq k \leq q_i$. For each 2-simplex $\tau$ such that $x_i$ is a vertex of $\tau$ and the 1-faces of $\tau$ which have $x_i$ as a vertex are faces of simplexes of $C_{ik}$, choose a line segment in $\tau$ from $x_i$ to the barycenter of the 1-face of $\tau$ opposite $x_i$, and let $s_{ik1}, s_{ik2}, \ldots, s_{ikq_{ik}}$ denote this collection of line segments together with all 1-simplexes having $x_i$ as a vertex which are not faces of simplexes of $C_{ik}$. Corresponding to $(C_{11}, s_{111})$, there are 2 isotopy classes of imbeddings of $C$ in $X$. Corresponding to $(C_{11}, s_{112})$, there are 2 isotopy classes of imbeddings of $C$ in $X$. We examine these to see if either is one of the 2 classes previously obtained. We will either get 2 new classes or no new classes. We continue this process. For each $(C_{ik}, s_{ik1})$, $i = 1, 2, \ldots, m$; $k = 1, 2, \ldots, q_i$; $\beta = 1, 2, \ldots, \alpha_{ik}$, there are 2 isotopy classes of imbeddings of $C$ in $X$. They are either both new or neither is new. Let $\gamma_i$ be the number of distinct isotopy classes of imbeddings of $C$ in $X$ obtained from $(C_{ik}, s_{ik1})$, $k = 1, 2, \ldots, q_i$; $\beta = 1, 2, \ldots, \alpha_{ik}$. For each $i = 2, 3, \ldots, m$, let $\gamma_i$ be the number of distinct isotopy classes of imbeddings of $C$ in $X$ obtained from $(C_{ik}, s_{ik1})$, $k = 1, 2, \ldots, q_i$; $\beta = 1, 2, \ldots, \alpha_{ik}$, which are different from those obtained from $(C_{ak}, s_{ak1})$, $a = 1, 2, \ldots, i-1$; $k = 1, 2, \ldots, q_a$; $\beta = 1, 2, \ldots, \alpha_{ak}$. For each $j = 1, 2, \ldots, n$, let $n_j$ be the number of 2-simplexes which have $s_j$ as a face. Then the number of isotopy classes of imbeddings of $C$ in $X$ is

$$\sum_{i=1}^{m} \gamma_i + 6 \sum_{j=1}^{n} C(n_j, 3).$$

Bibliography