A CRITERION FOR RINGS OF ANALYTIC FUNCTIONS

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1. **Introduction.** It is a well-known result of Bers [2] that two domains in the complex plane are conformally equivalent if the rings of all analytic functions on these domains are isomorphic over the complex numbers. Rudin [15] and Royden [12] extended this result to arbitrary open Riemann surfaces, and many related theorems have been obtained by others.

At the Michigan Conference in 1953, Kakutani proposed the following problem (originally suggested by S. B. Myers): to give conditions or "axioms" for an abstract ring to be isomorphic to a ring of analytic functions. Then, provided the underlying surface is canonically determined, the results of Bers, Rudin, and Royden would follow as corollaries.

This paper presents one solution to the Myers-Kakutani problem. Of course there are many possible solutions—the essential characteristic of this kind of problem is that it is very indefinite. Ideally a proposed set of axioms should be "elegant," that is, it should not rely on "artificial hypotheses" which happen to hold for rings of analytic functions.

The solution given here breaks into two parts. First we postulate, rather artificially, that R is equivalent to a ring of complex functions on the space $\Omega(R)$ of its maximal principal ideals, and that $\Omega(R)$ has a certain pseudo-topological structure. Then we prove some theorems which allow us to put an analytic structure on the pair $\langle R, \Omega(R)\rangle$. This is the main step, and the key results are Lemmas 2 and 3 in §3.

Lemma 2 is equivalent to the following: *Let D be the unit disk $\{|z|<1\}$, and let R be a ring of continuous complex-valued functions on D which contains the polynomials. Suppose that* (1) *the maximal ideal $P=\{f\in R \mid f(p)=0\}$ corresponding to every point $p\in D$ is principal, and* (2) *for all such P, $\bigcap_{n=1}^{\infty} P^n=(0)$. Then R is a subring of the ring of analytic functions.* [The hypothesis (2) is necessary, as example 3, §4 shows.]

Lemma 2 can be strengthened somewhat. Consider a ring R of continuous complex-valued functions on a topological space D, *and let R satisfy* (1) *and* (2) *above*. Theorem 2, §3 asserts that if D contains (as not necessarily open subsets) sufficiently many two-dimensional disks which are uniformized by functions $f\in R$, then D must be a surface. Lemma 2 then implies the existence of a complex analytic structure on $\langle R, D\rangle$.

For comparison, we describe very briefly a recent result of I. Kra [10] which is

Received by the editors October 28, 1966.

(1) This work was partially supported by NSF Grant GP 4033.

along the same lines as our Theorem 2. Kra starts with a ring of continuous complex functions which separates points on an arbitrary connected locally compact Hausdorff space D, and adds the hypotheses: (1′) same as our no. 1 above, and (2′) every nonconstant function in R gives an open mapping into the complex plane. Then it follows that D is a Riemann surface, and the functions in R are analytic. (The hypothesis (2′) was used by Rudin [14]; in the case where D is the unit disk and R contains the polynomials, (2′) is sufficient without (1′).)

§4 in our paper contains some counterexamples. In §5 we discuss fields of meromorphic functions. All the results of §3 have natural analogues, the ideal theoretic conditions used in §3 being replaced by the hypothesis of discrete valuations.

We wish to mention that these results grew out of joint work done with Professor Paul C. Rosenbloom, and the present paper owes much to this collaboration.

2. **Preliminaries.** Throughout this paper a *ring* R will mean a commutative ring which is also a complex algebra with identity. That is, R contains a subring C isomorphic to the complex numbers, and the identity $1 \in C$ also acts as an identity on R. (R is not endowed with either a norm or a topology.) A *homomorphism* of R means a C-homomorphism. If Ω is a Riemann surface, then the ring of all complex-valued analytic functions on Ω will be denoted by $R[\Omega]$. Here the "surface" Ω is not required to be connected.

DEFINITION 1. A subring $R \subset R[\Omega]$ is *full* if (a) R separates points on Ω, (b) for every point $p \in \Omega$, the maximal ideal $P = \{f \in R \mid f(p) = 0\}$ is principal, and (c) conversely every maximal principal ideal in R is of this form.

In Theorem 1 we characterize algebraically the full subrings of $R[\Omega]$. By a theorem of Florack [6], the ring $R[\Omega]$ itself is full provided Ω is connected and not compact. (The needed theorem asserts that, for all $p \in \Omega$, there exists a function in $R[\Omega]$ with a simple zero at p and no other zeros.)

PROPOSITION 1. *Let R be a full subring of $R[\Omega]$. Take a point $p \in \Omega$, and let P be the corresponding maximal principal ideal. Then the function $f \in R$ which generates P has a simple zero at p.*

Proof. Let f have a zero of order k. Then every $g \in R$ has a zero of order nk for some n. Define $t \equiv f^{1/k}$ in a neighborhood of p. Then the Taylor series expansion of any $g \in R$ involves only kth powers of t (using t as a local coordinate, and noting that $t^k, t^{2k}, \ldots \in R$). Since R separates points, $k = 1$, Q.E.D.

Now let R be an abstract ring: we wish to represent R as a full subring of $R[\Omega]$ for some Riemann surface Ω.

Let $\Omega = \Omega(R)$ denote the set of all maximal principal ideals in R. Our first conditions are:

(A) For any $P \in \Omega$, the set $P + C \equiv \{f + \alpha \mid f \in P, \alpha \in C\} = R$; furthermore $\bigcap \{P \mid P \in \Omega\} = (0)$.

This means that R is isomorphic to a ring of complex-valued functions on Ω.

(B) For all $P \in \Omega$, $\bigcap_{n=1}^{\infty} P^n = (0)$.

The final assumption which we need is that every element $P \in \Omega$ is contained in at least one "disk":

DEFINITION 2. A *disk* in $\langle R, \Omega \rangle$ is a C-homomorphism ϕ of R into the ring of all *continuous* complex functions on the open disk $D = \{|z| < 1\}$ such that: (a) for each point $z_0 \in D$ the maximal ideal $P = \{h \in R \mid \phi(h)(z_0) = 0\}$ is principal (i.e., $P \in \Omega$); (b) the image set $\phi(R)$ contains the function $w(z) = z$. An ideal $P \in \Omega$ is *contained in* ϕ if it corresponds by (a) to some point $z_0 \in D$.

There is a more concrete representation: a disk ϕ is uniquely determined by the pair $\langle f, \Delta \rangle$, where $\Delta \subset \Omega$ is the set of ideals contained in ϕ, and $f \in R$ is any element such that $\phi(f) = z$. Then f maps Δ bijectively onto $\{|z| < 1\}$, and for all $g \in R$, $g \circ f^{-1}$ is continuous on $f(\Delta)$. Sometimes we will refer to the set Δ by itself as a "disk."

If Δ is a disk, a subset $\Delta_1 \subset \Delta$ will be called *open in* Δ if $f(\Delta_1)$ is open in $f(\Delta)$ (this condition is independent of the uniformizer f).

The next definition applies only to Theorem 2.

DEFINITION 2a. Let R be a ring of continuous complex-valued functions on a topological space Ω. A *toplogical disk* in $\langle R, \Omega \rangle$ is a pair $\langle f, \Delta \rangle$ with $f \in R$, $\Delta \subset \Omega$ such that f maps Δ one to one and bicontinuously onto the open disk $\{|z| < 1\}$.

(Note that the set Δ is not required to be open in Ω.)

For the following, the term "disk" may be interpreted either in the sense of Definition 2 or Definition 2a.

DEFINITION 3. A subset $E \subset \Omega$ is *diskwise connected* if for every $p, q \in E$ there is a sequence of disks $\langle f_0, \Delta_0 \rangle, \ldots, \langle f_n, \Delta_n \rangle$ such that $p \in \Delta_0$, $q \in \Delta_n$, $\Delta_i \subset E$ and $\Delta_i \cap \Delta_{i-1} \neq \varnothing$ for all i. If Ω is a topological space, then we will say that Ω is *locally diskwise connected* if the diskwise connected open sets form a basis for the topology.

3. **Algebraic and topological conditions which imply an analytic structure.** Recall that Ω denotes the set of all maximal principal ideals in R, and we have made the assumptions (A) that R is naturally isomorphic to a ring of complex functions on Ω, and (B) for all $P \in \Omega$, $\bigcap_{n=1}^{\infty} P^n = (0)$.

THEOREM 1. *Let the ring R satisfy* (A) *and* (B) *above, and suppose that every point $P \in \Omega$ is contained in at least one disk* (*Definition* 2). *Then the set Ω can be made into a Riemann surface in such a way that the elements of R become analytic functions. Ω is connected if and only if the pair $\langle R, \Omega \rangle$ is diskwise connected. The topological and conformal structure of Ω is uniquely determined by R.*

The next theorem illustrates the topological restrictions imposed on a space Ω by conditions analogous to (A) and (B).

THEOREM 2. *Let R be a ring of continuous complex-valued functions on a topological space Ω; we assume that R contains the constants and that R separates points on Ω. Suppose Ω is locally diskwise connected* (*Definitions* 2a *and* 3) *and*:

(1) *For every point $p \in \Omega$, the maximal ideal $P = \{f \in R \mid f(p) = 0\}$ is principal*;

(2) $\bigcap_{n=1}^{\infty} P^n = (0)$.

Then Ω is the underlying topological space of a Riemann surface on which the functions in R become analytic.

COROLLARY 1. *Let Ω be a topological surface (orientable or not). Let R be a ring of continuous complex-valued functions on Ω containing the constants and separating points on Ω, and suppose that for every point $p \in \Omega$ there is a function $f \in R$ which is univalent in a neighborhood of p. Then, if R satisfies* (1) *and* (2) *above, the surface Ω can be given a conformal structure in such a way that the functions in R become analytic. (Thus in particular Ω is orientable and noncompact.)*

Proof of Theorem 1. The proof uses three lemmas. The first lemma is used to prove the second; it involves a topological variation of the argument principle. The main step is contained in Lemma 2, which asserts that every disk has an analytic structure. Lemma 3 then guarantees that the intersection of two disks is an "open" subset of each disk, i.e., that the disks generate a topology.

LEMMA 1. *Let $u(z)$ and $v(z)$ be continuous complex functions on the closed disk $\{|z| \leqq a\}$, $a > 0$. Suppose that for some integer n, $z = u(z) \cdot v(z)^n$ and $u(0) \neq 0$. Then $n = 1$.*

Proof. Without loss of generality we may assume that $u(z) \neq 0$ throughout $\{|z| \leqq a\}$; otherwise replace this disk by a smaller one. For convenience suppose that $a = 1$, $u(0) = 1$. Define $h(\rho, z) \equiv u(\rho z) \cdot v(z)^n$, $0 \leqq \rho \leqq 1$. Then $h(0, z) = v(z)^n$ and $h(1, z) = z$. Let γ_ρ be the closed curve $\gamma_\rho(\theta) \equiv h(\rho, e^{i\theta})$. The curves γ_ρ are all freely homotopic in the punctured plane $\{0 < |z| < \infty\}$; thus the winding number of γ_ρ about the origin is independent of ρ. But $\gamma_1(\theta) = e^{i\theta}$, $\gamma_0(\theta) = v(e^{i\theta})^n$, so the winding number of γ_1 is $+1$ and that of γ_0 is a multiple of n. Q.E.D.

LEMMA 2. *Let $\langle f, \Delta \rangle$ be a disk in the sense of Definition 2. Then every element $h \in R$ is an analytic function of f in the region $f(\Delta)$. (Recall that f maps Δ bijectively onto the disk $\{|z| < 1\}$, and that every $g \in R$ is continuous as a function of f.)*

Proof. We need to demonstrate the existence of the limit $(dh/df)(P) = \lim_{Q \to P} [h(Q) - h(P)]/[f(Q) - f(P)]$ for all $h \in R$, $P \in \Delta$. Every $P \in \Omega$ is simultaneously a "point" in Ω and a maximal principal ideal in R. For each $P \in \Delta$ ($\Delta \subset \Omega$), let t_P be a generator of this ideal. Then for any $h \in R$ the quotient $[h - h(P)]/t_P$ is also in R. Corresponding to each point $P \in \Delta$ we define a functional D_P on R by

$$D_P(h) \equiv ([h - h(P)]/t_P)(P).$$

Now it is easy to verify that the limit $(dh/df)(P)$ defined above exists and equals $D_P(h)/D_P(f)$ provided that $D_P(f) \neq 0$.

By assumption $\bigcap_{n=1}^{\infty} P^n = (0)$. Hence there is some integer n such that $[f - f(P)] = g t_P{}^n$, $g \in R$, $g(P) \neq 0$. From Lemma 1 (with $[f - f(P)] = z$) it follows that $n = 1$, whence $D_P(f) = g(P) \neq 0$. This proves Lemma 2.

REMARK. Our hypotheses do not imply any cohesion between the generators of neighboring principal ideals $P \neq Q$. Thus we have established the existence of the derivative (dh/df) without simultaneously proving its continuity.

LEMMA 3. *Let* $\langle f_1, \Delta_1 \rangle$ *and* $\langle f_2, \Delta_2 \rangle$ *be disks in the sense of Definition* 2. *Then* $\Delta_1 \cap \Delta_2$ *is open in* Δ_1 (*with respect to the topology induced by* f_1).

REMARK. Recall that the space Ω is not originally endowed with a topology. However, for each *disk* $\langle f, \Delta \rangle$, the set $\Delta \subset \Omega$ has a topology which is induced by the uniformizing function f.

Proof. Take any point $P \in \Delta_1 \cap \Delta_2$; we have to prove that $\Delta_1 \cap \Delta_2$ contains a neighborhood of P in Δ_1. Let $t \in P$ be a generator of the principal ideal P (t is independent of Δ_1, Δ_2). We will show that, in terms of the disk topology, there exist open subsets $U_1 \subset \Delta_1$ and $U_2 \subset \Delta_2$ with $P \in U_1 \cap U_2$ such that:

(i) The functions $t|U_1$ and $t|U_2$ are one to one.

(ii) If $P_1 \in U_1$, $P_2 \in U_2$, and $t(P_1) = t(P_2)$, then $P_1 = P_2$.

Proof of (i). By Lemma 2 above, $t|\Delta_1$ is an analytic function of f_1 on the domain $f_1(\Delta_1)$. Since t generates the ideal $P = \{h \in R \mid h(P) = 0\}$, and f_1 has a simple zero at P, the function $t|\Delta_1$ must have a simple zero at P. Thus $t|\Delta_1$ is univalent in some neighborhood of P in Δ_1; similarly for Δ_2.

Proof of (ii). Take neighborhoods V_1 and V_2 of P in Δ_1, Δ_2 on which the function t is one to one. Consider any $h \in R$. Now $h|V_1$ and $h|V_2$ are analytic functions of f_1 and f_2, and hence they are also analytic functions of t (since t is analytic). It follows that $h|V_1$ and $h|V_2$ have Taylor series expansions in powers of t near $t=0$. Thus let $h|V_1 = a_0 + a_1 t + a_2 t^2 + \cdots$ in some neighborhood of $t=0$ $(=P)$ in V_1. Then $a_0 = h(P)$ and $a_n = h^{(n)}(P)$ where $h^{(n)}$ is defined inductively by

$$h^{(n)} = [h^{(n-1)} - h^{(n-1)}(P)]/t.$$

But the functions $h^{(n)}$ are determined by h and t alone; they are the same for $h|V_1$ as for $h|V_2$. Hence, near P, the value of any $h \in R$ is determined by the value of t. Since R separates points on Ω, this proves (ii).

Let U_1, U_2 be neighborhoods of P in Δ_1, Δ_2 which satisfy (i) and (ii) above. Then, since $t(U_2)$ is an open set in the complex plane ($t|\Delta_2$ is analytic), and the restriction $t_1 = t|U_1$ is continuous, the inverse image $t_1^{-1}[t(U_2)]$ is open in U_1. But (ii) implies that $t_1^{-1}[t(U_2)]$ is just $U_1 \cap U_2$, and hence $U_1 \cap U_2$ is a neighborhood of P in Δ_1.

Q.E.D.

We now make Ω into a Riemann surface by declaring that the disks Δ form a basis for the topology of Ω. Lemma 3 insures that the disks intersect in the proper manner, and Lemma 2 implies that the pair $\langle R, \Omega \rangle$ has a complex analytic structure. The connected components of Ω are precisely the diskwise connected sets (Definition 3).

As for uniqueness of the topological and conformal structure: the disks are defined purely algebraically (see Definition 2), and Lemma 3 guarantees that every

abstract "disk" corresponds to an open set in terms of the given topology on Ω. The conformal structure is determined by the requirement that the topological local uniformizers (locally univalent functions) be analytic. This completes the proof of Theorem 1.

Proof of Theorem 2. This is almost a corollary of the above proof. For every *topological disk* in Ω is also a *disk* in the sense of Definition 2 (once the assumptions (1) and (2) in Theorem 2 are made).

What we need to show is that every disk is open in the topology of Ω. Choose a topological disk $\langle f, \Delta\rangle$ (see Definition 2a; here f maps Δ bicontinuously onto $\{|z|<1\}$). Let $D_r \subset \Delta$ and $K_r \subset \Delta$ be the inverse images under f of $\{|z|<r\}$ and $\{|z|=r\}$ $(0<r<1)$.

Now by Lemma 3, Δ intersects any other disk Δ' in a relatively open subset of Δ'. Hence $D_r \cap \Delta'$ is also relatively open in Δ'; on the other hand, by compactness, $[D_r \cup K_r] \cap \Delta'$ is relatively closed. Now since Δ' is connected, any disk Δ' which intersects D_r but not K_r must be contained in D_r. The same thing holds automatically for any diskwise connected set. Thus if Ω is locally diskwise connected, D_r is open in Ω. Q.E.D.

Proof of Corollary 1. This is an immediate corollary of Theorem 2. It also follows directly from Lemma 2 above.

REMARK. It is easy to verify that, if Ω is connected and not compact, the subrings $R \subset R[\Omega]$ characterized by Theorem 1 are precisely the full subrings of $R[\Omega]$ (see Proposition 1, §2).

An Application. Let Ω_1 and Ω_2 be connected noncompact Riemann surfaces, and let R_1 and R_2 be full subrings of $R[\Omega_i]$, $i=1, 2$. Then if R_1 and R_2 are isomorphic over the complex numbers, Ω_1 and Ω_2 are conformally equivalent.

Proof. Since the rings $R_1 \cong R_2$ are full, there is obviously a one-to-one correspondence between the points of Ω_1 and Ω_2; moreover these points are related to the rings R_1, R_2 exactly as in Theorem 1 above. Therefore equivalence of the topological and conformal structures follows from the uniqueness part of Theorem 1. Q.E.D.

(For $R_i = R[\Omega_i]$, $i=1, 2$ this result is due to Rudin [15] and Royden [12]. Royden also classifies the homomorphisms of $R[\Omega_1]$ into $R[\Omega_2]$. However his proof requires that the rings R_i satisfy a hypothesis such as: for any sequence $p, p_1, p_2, \ldots$ of points in Ω_i, $p_n \to p$ if and only if $f(p_n) \to f(p)$ for all $f \in R_i$. We derive the topology on Ω_i via Lemma 3; this method is closely related to a theorem of Heins [7, Theorem A].)

4. Counterexamples.

EXAMPLE 1. Let Ω be the real line and let R be the ring of all complex-valued real-analytic functions on Ω.

EXAMPLE 2. Let Ω be the real line. Let R be the ring of all complex-valued functions f on Ω such that (a) f is real-analytic except at a finite set of points $\{x_i\}$,

where $\{x_i\}$ depends on f; (b) near each x_i, f is of the form $f(x)=a_i(x)+b_i(x)(x-x_i)^{1/3}+c_i(x)(x-x_i)^{2/3}$ with a_i, b_i, c_i real-analytic.

The rings in Examples 1 and 2 satisfy Conditions (A) and (B) of Theorem 1, but of course they possess no "disks." Note that in Example 2, there is no manifold structure on E^1 in terms of which all the functions in R become analytic (in contrast to the situation for surfaces; cf. Corollary 1).

EXAMPLE 3. Let Ω be the euclidean space E^n. Let R be the ring of all complex-valued functions f on Ω such that (a) f is C^1 except at a finite set of points $\{x_i\}$, where $\{x_i\}$ depends on f; (b) near each x_i, f is of the form

$$f(x) = \sum_{k=-m}^{m} c_k(x)[\log |x-x_i|]^{-k},$$

with $c_k \in C^1$ for all k, and $c_k(x_i)=0$ for $k<0$.

(Of course the functions c_k also depend on x_i, $c_k=c_k(x, x_i)$; $|x-x_i|$ denotes the euclidean distance between x_i and x.)

Here the maximal ideal at every point $p \in \Omega$ is principal; it is generated by $[\log |x-p|]^{-1}$. For $f(p)=0$ only if $c_0(p)=0$ above (with $x_i=p$), whence

$$f(x)[\log |x-p|] \in R.$$

Thus R satisfies all the hypotheses of Theorem 2 except Condition (2). Clearly $\langle R, \Omega\rangle$ has no complex analytic structure.

5. **Fields of meromorphic functions.** Results analogous to Theorems 1 and 2 can be proved for fields $F \supset C$. Here we give conditions which imply that an abstract field is isomorphic to a field of meromorphic functions.

DEFINITION 4 (CF. DEFINITION 2). Let $F \supset C$ be a field. A *disk* on F is a function ϕ defined on $F \times D$, where D is the open unit disk, taking values in $C \cup \{\infty\}$ such that:

(a) for each point $p \in D$, the function $\phi_p(f) \equiv \phi(f, p)$ is a C-place on F having a discrete valuation;

(b) if $f \in F$ is held fixed, then the transformation $p \to \phi(f, p)$ is a continuous mapping of the disk D into the Riemann sphere with its usual topology;

(c) for any $f \in F$ there exists some $p \in D$ such that $\phi(f, p) \neq \infty$;

(d) there exists an element $z \in F$ such that $\phi(z, p)=p$ for all points $p \in D$.

Thus ϕ defines an isomorphism of F onto a "field of continuous extended-complex valued functions" on the unit disk (ϕ is an isomorphism because (c) implies $\phi(f, p)=0$ for all p only if $f=0$).

EXAMPLE. Let $F \supset C$ be any field of meromorphic functions on the unit disk which contains the function $f(z)=z$. Then the mapping $\phi(f, p)=f(p)$ defines a *disk* on F.

THEOREM 3. *Let ϕ be a disk defined on a field $F \supset C$. For each element $f \in F$, let $f(p)$ be the function on $D=\{|z|<1\}$ defined by $f(p)=\phi(f, p)$. Then $f(p)$ is meromorphic.*

Sketch of proof. This is simply another version of Lemma 2. Here the hypothesis of discrete valuations replaces the ideal theoretic conditions (A) and (B).

A little care is needed because the functions in F may possess singularities. First, if z is the function of Part (d) above, it can be shown via Lemma 1 that $(z-p)$ has valuation $+1$ at all points p. Then, holding p fixed, $(dh/dz)(p)$ exists for all $h \in F$ such that $h(p) \neq \infty$. Finally, if $h(p)=\infty$, then $1/h$ is finite throughout some neighborhood of p.

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