M-SEMIREGULAR SUBALGEBRAS IN HYPERFINITE FACTORS

BY

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1. Introduction. The general study of algebras of operators on Hilbert space has led to the investigation of rings of operators, also called $W^*$-algebras or von Neumann algebras. If the center of a ring (center in the algebraic sense) consists only of scalar multiples of the identity, then the ring is a factor. Since every ring can be decomposed into factors [6], the study of rings is, in a sense, reduced to a study of factors. In this paper we are concerned with the maximal abelian subalgebras of type II$_1$ factors, or continuous factors which have a finite trace defined on them [2]. For the present, we restrict ourselves to the study of hyperfinite factors, that is, those which are generated by a sequence of factors $ℳ_α$ of type I$_α$, with $ℳ_α$ $ḍ$ $ℳ_β$ $ḍ$ $ℳ_γ$ $ḍ$ $⋯$. (The factor $ℳ_α$ is isomorphic to the algebra of $n$ by $n$ matrices.) Since all hyperfinite factors are algebraically isomorphic [5, §4.7], while the concept of a subring of a finite factor is purely algebraic [5, §1.6], any construction used will yield general results.

Dixmier has defined three types of maximal abelian subalgebras $R$ in a factor $ℳ$: regular, semiregular, and singular [3]. These depend on the properties of $N(R)$, the ring generated by $\{V : VRV^*=R, V \text{ unitary}, V \in ℳ\}$. In other words, $N(R)$ is the normalizer of $R$ in $ℳ$. Later, Anastasio defined an additional type, $M$-semiregular ($M=1, 2, 3, \ldots$), which coincides with the semiregular type when $M=1$. Extending the notation $N(D)$ to any subring $D ⊂ ℳ$, and letting $N_k(D) = N[N_{k-1}(D)]$, we have a chain $R ⊂ N(R) ⊂ N^2(R) ⊂ \cdots ⊂ N^M(R) = ℳ$. We say that a maximal abelian subalgebra $R$ is $M$-semiregular if $N^k(R)$ is not a factor for $k < M$, but $N^M(R)$ is a factor [1]. Anastasio constructed infinite sequences of non-isomorphic 2-semiregular and 3-semiregular subalgebras in a hyperfinite factor. (The 1-semiregular case had already been done [7].) In this paper we propose to show the existence of $M$-semiregular subalgebras for every positive integer $M≠1$.

We use the notation and results of [7]. Let $ℳ_α$ be the full $2^α$ by $2^α$ matrix algebra over the complex numbers, and $\{pE_{ij} : i, j=0, 1, \ldots, 2^α−1\}$ the matrix units which generate it. By letting $pE_{ij} = p\sum E_{2i_1,2j_1} + pE_{2i_1+1,2j_1+1}$, we imbed $ℳ_α$ in $ℳ_{α+1}$. Then $\bigcup_{α=1}^∞ ℳ_α = ℳ$ is a *-algebra. The normalized matrix trace on $ℳ$ makes it into a pre-Hilbert space $ℳ$: If $A, B \in ℳ$, let $(A, B) = Tr(B^*A)$, so that $(A, A)^{1/2} = [A]$, the Hilbert space or metric norm of $A$. If $A$ is in $ℳ$, then $A$ acting

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by left multiplication is a bounded operator on $\mathcal{H}$, so it can be extended to the Hilbert space closure $\mathcal{H}$. If $\mathfrak{A}$ is the weak closure of $\mathcal{M}$, then it is well known that $\mathfrak{A}$ is a hyperfinite factor [2].

2. M-semiregular subalgebras. The following general construction leads to a large variety of maximal abelian subalgebras of $\mathfrak{A}$.

**Definitions 2.1.** Let $\{U_t : t=1, 2, \ldots\}$ be a set of selfadjoint unitaries such that:
1. $U_t \in \mathcal{M}$;
2. $U_t$ is zero except for 2 by 2 blocks along the main diagonal. Let $Y_t=U_1U_2 \cdots U_t$, and for $A \in \mathfrak{A}$, define $A^{(t)}=Y_tAY_t^*$ and $A^{(t)}=Y_t^*AY_t$. For fixed $t$, the mappings $A \to A^{(t)}$ and $A \to A^{(t)}$ are $*$-automorphisms of $\mathfrak{A}$ and inverses of each other. Because of the form of $U_t$, the matrix unit $e_{ij}$ commutes with $U_t$ for all $t \geq p$. Thus if $A$ is a diagonal matrix in $\mathfrak{A}$, then $A^{(t)}=A^{(t+1)}$, and so $\lim_{t \to \infty} A^{(t)}=A^{(\infty)}$ exists in $\mathfrak{A}$, hence in $\mathfrak{M}$.

In general, for $A \in \mathfrak{A}$, the limit $A^{(\infty)}$ does not exist. The mapping $A \to A^{(\infty)}$ is thus an isomorphism of some proper subalgebra of $\mathfrak{A}$ into $\mathfrak{M}$. This subalgebra, the domain of the mapping, we call $\mathfrak{D}$. If $E$ is the set of diagonal matrices, then $E \subseteq \mathfrak{D}$, as seen above. The ring $(E^{(\infty)})^{-}$ is the maximal abelian subalgebra $R$ which we study in this paper. (Cf. [7, pp. 285-286], for the proof that $R$ is maximal abelian.) In Lemma 2.2 we will show that $E^{(\infty)} \subseteq \mathfrak{D}$, and that $(E^{(\infty)})^{-}=(E^{(\infty)})^{-} \subseteq R$.

**Lemma 2.2.** If $F=E^{-}$, then $F \subseteq \mathfrak{D}$, and $F^{(\infty)}=(E^{(\infty)})^{-} \subseteq R$.

**Proof.** Suppose $A \in F$. Then there is a sequence $A_n \in E \cap \mathfrak{M}$, $A_n \to A$, with $A_n^{(\infty)} \in \mathfrak{M}$. Let $\varepsilon>0$ be given, and choose $n$ such that $\|A_n-A\|<\varepsilon/2$. Consider

$$\|A^{(t)} - A^{(t)}\| \leq \|Y_sAY_s^* - Y_tAY_t^*\| + \|Y_sA_nY_s^* - Y_tA_nY_t^*\| + \|Y_tA_nY_t^* - Y_tA_nY_t^*\|.$$ 

Choose $s, t$ such that both are greater than or equal to $n$. Then $Y_sA_nY_s^* = A_n^{(s)} = A_n,$ and $Y_tA_nY_t^* = A_n^{(t)} = A_n$. Hence $\|Y_sA_nY_s^* - Y_tA_nY_t^*\|=0$ if $s, t \geq n$. Since $Y_s$ and $Y_t$ are unitary, the other two norms equal $\|A-A_n\|$, and so the sum is less than $\varepsilon$. Therefore $A^{(t)}$ is Cauchy in the metric topology.

Now $A \in \mathfrak{A}$ and so $\|A\|<\infty$. Since $\|A^{(t)}\|=\|A\|$, $A^{(t)}$ is a bounded sequence. By [5, p. 723], $A^{(t)}$ is then Cauchy in the strong topology also, so its limit exists in $\mathfrak{A}$. Therefore $F \subseteq \mathfrak{D}$.

We next show that $F^{(\infty)}=(E^{(\infty)})^{-}$ or $R$. Let $A \in F$, $A_n \in E \cap \mathfrak{M}$, $A_n \to A$. Let $\varepsilon>0$ be given, and choose $n$ so that both $\|A_n-A\|<\varepsilon/2$ and $\|A^{(n)}-A^{(n)}\|<\varepsilon/2$. Then

$$\|A_n^{(\infty)} - A^{(\infty)}\| \leq \|A_n^{(\infty)} - A^{(n)}\| + \|A^{(n)} - A^{(\infty)}\| + \|A^{(n)} - A^{(\infty)}\|.$$

$$= \|A_n^{(\infty)} - A^{(n)}\| + \|A_n - A\| + \|A^{(n)} - A^{(\infty)}\| \leq 0 + \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Therefore $A^{(\infty)} \in (E^{(\infty)})^{-}$, and so $F^{(\infty)} \subseteq R$.

On the other hand, if $G \subseteq R$, there is a sequence $A_n \in E \cap \mathfrak{M}$, $A_n^{(\infty)} \to G$, with $\|A_n^{(\infty)}\|=\|A_n^{(n)}\|=\|A_n\| \leq \|G\|$ [4]. Since $A_n^{(\infty)}$ is metrically Cauchy, so is $A_n$, which
is also strongly Cauchy because of the bound on the norm. Hence $A_n$ has a limit $A \in F$. By another standard argument, if $\varepsilon > 0$ be given, there exists $N$ such that $\| Y_t A Y_t^* - G \| < \varepsilon$ when $t \geq N$. Therefore $G = \lim_{t \to \infty} Y_t A Y_t^*$ and so $G \in F^{(*)}$. Hence $R \subset F^{(*)}$, and $F^{(*)} = R$.

Remark. The normalizer of $R$ in $\mathfrak{A}$ results in a similar way from the mapping $A \mapsto A^{(\infty)}$. The subalgebra $r(\mathfrak{G}_0)$, to be defined later, has the property that $E \subset r(\mathfrak{G}_0) \subset \mathfrak{G}$, and $r(\mathfrak{G}_0)^{(*)} = N(R)$. (It appears that $r(\mathfrak{G}_0) = \mathfrak{G}$, but we do not need this fact and have not proved it.)

By means of various choices of the sequence $\{U_i\}$, in §3 we construct a maximal abelian subalgebra $R_n$ for each $n = 1, 2, 3, \ldots$, where $R_n$ is $M$-semiregular, $M = n + 1$. The chain $R_n \subset N(R) = P_n \subset N^2(R_n) \subset \cdots \subset N^{n+1}(R_n) = N^M(R_n) = \mathfrak{A}$ is such that $N^k(R_n)$ is not a factor for $k = 1, 2, \ldots, n < M$, while $N^M(R_n)$ is the factor $\mathfrak{A}$.

Furthermore, the subalgebras $R_n$ are not conjugate under any $*$-automorphism of $\mathfrak{A}$. The integer $n$ determines the number of normalizers between $R_n$ and $\mathfrak{A}$ in the chain, and this is an automorphism invariant (cf. [7, pp. 282 and 305]).

Note. For convenience of notation, we often work with $N^k(P_n) = N^{k+1}(R_n)$, $k = 0, 1, \ldots, n$.

3. Detailed construction of $M$-semiregular subalgebras. In the construction of $M$-semiregular subalgebras, we use the following notations and definitions.

Definitions 3.1. We regard $n = 1, 2, 3, \ldots$ as fixed, and let

$$\Gamma = \{p : p = (3c+1)n, c = 0, 1, 2, \ldots, \}$$

an infinite set of positive integers. We define $\mathfrak{G}_n = \{rE_{rs} : p \in \Gamma\}$. In the following paragraphs, we define a decomposition of $\mathfrak{G}_n$ into $2^n$ disjoint subsets $K_y$ ($0 \leq y \leq 2^n - 1$), so that $\mathfrak{G}_n = \bigcup K_y$.

Let $\Theta_n$ be the set of all $n$-tuples $(a_1, a_2, \ldots, a_n)$, where $a_k = 0$ or 1. This is a commutative group under the operation of coordinate-wise addition modulo 2. If $y = 0, 1, \ldots, 2^n - 1$ and $y = \sum_{i=1}^n a_i 2^{n-i}$, we identify it with its binary expansion $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, so that we can consider $\gamma$ as an element of $\Theta_n$. The sum $\gamma_1 + \gamma_2$ is then defined by addition in $\Theta_n$.

We determine the set $K_y$ in which $rE_{rs}$ is contained as follows: For any index $r$ ($0 \leq r \leq 2^{(3c+1)n} - 1$), let $r = \sum_{k=0}^c r_k 2^{kn}$. (Congruence is modulo 3 in this and in the following summations.) For $k = 0$, we have $0 \leq r_k < 2^2$, and so $r_k = \sum_{i=1}^n k_i 2^{n-1}$ with $(k_1, \ldots, k_n) \in \Theta_n$. Designate this element of $\Theta_n$ by $\psi(r_k)$. For $k = 1$, $0 \leq r_k < 2^{2n}$, and we let $\sigma(r_k) = 2(r_k \mod 2^n)$, so that $\psi(\sigma(r_k))$ is defined. Let

$$\Delta(r) = \sum_{k=0, k=0}^{3c} \psi(r_k) + \sum_{k=1, k=1}^{3c-2} \psi(\sigma(r_k)),$$

where the addition is coordinate-wise (mod 2), so that $\Delta(r) \in \Theta_n$. Then $K_y = \{rE_{rs} : \Delta(r) + \Delta(s) = \gamma\}$ and we say that $K_y = K(r_{rs})$. Since this is independent of $p$, we sometimes write $K_y = K(r, s)$. 

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Definitions 3.2. We also define the following sets of matrix units, again subsets of \( \mathcal{C}_n : \mathcal{C}_0 = \mathcal{N}_0 = K_0 \). For \( j = 1, 2, \ldots, n, \mathcal{C}_j = \bigcup \{ K_r : r = (a_1, \ldots, a_n, 0, 0, \ldots, 0) \} \) and \( \mathcal{N}_j = \mathcal{C}_j \sim \mathcal{C}_{j-1} \). If we let
\[
\mathfrak{M}^D = 1E_{00} \mathfrak{M}^1 E_{00} + 1E_{11} \mathfrak{M}^1 E_{11},
\]
then we define \( \mathcal{C}_j^D = \mathcal{C}_j \cap \mathfrak{M}^D \) and \( \mathcal{N}_j^D = \mathcal{N}_j \cap \mathfrak{M}^D \), while \( \mathcal{C}_j' = \mathcal{C}_j \sim \mathcal{C}_j^D \) and \( \mathcal{N}_j' = \mathcal{N}_j \sim \mathcal{N}_j^D \). We let \( r(\mathcal{C}_j^D) \) be the ring generated by the matrix units in \( \mathcal{C}_j^D \), while \( R(\mathcal{C}_j^D) \) is the ring generated by \( \{ F : F = (pE_{rs})^D \} \) with \( pE_{rs} \in \mathcal{C}_j^D \).

Lemma 3.3. Suppose \( p \in \Gamma, p + 3nE_{rs} \in K_r \). Let \( r = r'2^{3n} + r_22^n + r_0 \) and \( s = s'2^{3n} + s_12^n + s_0 \). Then
\[
\gamma = \Delta(r) + \Delta(s) = (\Delta(r') + \Delta(s')) + \sigma + (\Delta(r_0) + \Delta(s_0)),
\]
where \( \sigma = \psi(\sigma(r_1)) + \psi(\sigma(s_1)). \)

Proof. This follows by computation from Definitions 3.1, since \( \Delta(r) \) can be written as \( \Delta(r') + \psi(\sigma(r_1)) + \psi(r_0) \), and the same for \( \Delta(s) \).

Construction 3.4. In constructing the maximal abelian subalgebra \( R_n \) according to \( \S 2.1 \), the sequence \( \{ U_t : t = 1, 2, 3, \ldots \} \) is to be as follows: Let
\[
B_1 = \begin{bmatrix}
2^{-1/2} & 2^{-1/2} \\
2^{-1/2} & -2^{-1/2}
\end{bmatrix}
\]
Let \( B_i \) be in \( \mathfrak{M}_n \), with all entries zero except for 2 by 2 blocks like \( B_1 \) along the main diagonal.

For \( n > 1 \), \( U_t = I \) if \( t < n \). If \( p \in \Gamma \) and if \( \Delta(r) = (a_1, a_2, \ldots, a_n) \), define:
\[
pE_{rr} U_{p+1} = pE_{rr} \quad \text{if} \quad a_n = 0,
\]
\[
pE_{rr} B_{p+1} = pE_{rr}B_{p+1} \quad \text{if} \quad a_n = 1,
\]
\[
\vdots
\]
\[
pE_{rr} U_{p+n-1} = pE_{rr} \quad \text{if} \quad a_{j-1} = 0,
\]
\[
pE_{rr} B_{p+n-1} = pE_{rr}B_{p+n-1} \quad \text{if} \quad a_{j-1} = 1,
\]
\[
\vdots
\]
\[
U_{p+n+1} = I.
\]
\[
pE_{rr} U_{p+n+2} = pE_{rr} \quad \text{if} \quad a_{n+1} = 0,
\]
\[
pE_{rr} B_{p+n+2} = pE_{rr}B_{p+n+2} \quad \text{if} \quad a_{n+1} = 1,
\]
\[
\vdots
\]
\[
pE_{rr} U_{p+n+j} = pE_{rr} \quad \text{if} \quad a_j = 0,
\]
\[
pE_{rr} B_{p+n+j} = pE_{rr}B_{p+n+j} \quad \text{if} \quad a_j = 1,
\]
\[
\vdots
\]
\[
pE_{rr} U_{p+2n} = pE_{rr} \quad \text{if} \quad a_{n} = 0,
\]
\[
pE_{rr} B_{p+2n} = pE_{rr}B_{p+2n} \quad \text{if} \quad a_{n} = 1,
\]
\[
U_{p+2n+1} = \cdots = U_{p+3n-2} = I,
\]
\[
1E_00 U_{p+3n-1} = 1E_00,
\]
\[
1E_{11} U_{p+3n-1} = 1E_{11} B_{p+3n-1},
\]
\[
U_{p+3n} = I.
\]
For \( n = 1, p \in \Gamma, \) and if \( \Delta(r) = (a_i), \) define:

\[
E_{rr} U_{p+1} = E_{rr} \quad \text{if } a_1 = 0,
\]

\[
E_{rr} B_{p+1} \quad \text{if } a_1 = 1,
\]

\[
E_{00} U_{p+2} = E_{00},
\]

\[
E_{11} U_{p+2} = E_{11} B_{p+2},
\]

\[
U_{p+3} = I.
\]

**Remark.** With this construction we aim to show that \( N^{j+1}(R) = N^j(P) = R(\mathcal{D}_0) \) for \( j = 0, 1, \ldots, n-1, \) and that none of these is a factor. However,

\[
N^n(P) = \mathcal{A} = R(\mathcal{D}_n \cup \mathcal{C}_n).
\]

(For \( n = 1, \) the following three propositions hold with slight adaptations. Then nothing else is needed until Theorems 3.14 and 3.15.)

**Theorem 3.5.** \( N(R) = P = R(\mathcal{D}_0). \)

**Proof.** If \( p \in \Gamma, \) \( E_{rs} \in \mathcal{C}_0, \) then \( \Delta(r) + \Delta(s) = (0, 0, \ldots, 0). \) So computation with the definitions of §3.4 shows that

\[
U_{p+3n} \cdots U_{p+1} E_{rs} U_{p+1} \cdots U_{p+3n} = E_{rs}.
\]

If \( q \in \Gamma, q > p, \) then \( q = p + 3hn \) for some integer \( h. \) Since \( E_{rs} \) is a sum \( \sum_{\nu} E_{rs}, \)

with all terms of the sum in \( \mathcal{C}_0, \) we have

\[
U_q \cdots U_{p+1} E_{rs} U_{p+1} \cdots U_q = E_{rs} \in \mathcal{C}_0.
\]

But if \( E_{rs} \in \mathcal{N}^j (j \geq 1), \) then

\[
U_{p+n-j+1} \cdots E_{rs} \cdots U_{p+n-j+1} = E_{rs} B_{p+n-j+1}.
\]

Also, if \( E_{rs} \in \mathcal{C}_0, \)

\[
U_{p+3n-1} \cdots E_{rs} \cdots U_{p+3n-1} = E_{rs} B_{p+3n-1}.
\]

Hence our construction satisfies the conditions of [7, §4.1], with \( \mathcal{C}_0^p \) taking the place of \( K_0. \) Also, \( d \leq 3n - 1 \) is surely sufficient. Thus we can apply [7, Lemma 4.3] in order to conclude that any unitary \( V \) leaving \( R \) invariant is the metric limit of a sequence \( V_m \) in \( \mathcal{M} \) such that if \( V_m \in \mathcal{M} (p \in \Gamma), \) then \( V_m^{[p]} = \sum \alpha_{cd} E_{cd} \) with \( E_{cd} \in \mathcal{C}_0. \) So if \( V \in N(R), \) then \( V \in R(\mathcal{C}_0), \) and we have \( N(R) \subset R(\mathcal{C}_0). \)

On the other hand, consider a unitary \( V \) in \( \mathcal{M} (p \in \Gamma) \) such that \( V^{[p]} = \sum \pm E_{rs} \) with \( E_{rs} \in \mathcal{C}_0^p \) and signs arbitrary. It is straightforward to show that \( V \) leaves \( R \) invariant. Since the collection of all unitaries of this type is sufficient to generate \( R(\mathcal{C}_0), \) we have \( R(\mathcal{C}_0^p) \subset N(R). \)

Therefore \( N(R) = P = R(\mathcal{C}_0). \)

**Remark.** The preceding proof also implies that \( r(\mathcal{C}_0) \) is in \( \mathcal{D} \) and that \( R(\mathcal{C}_0) = r(\mathcal{C}_0) \) \( (\infty) \). For if \( F = \sum \alpha_{rs} E_{rs} \) with \( E_{rs} \in \mathcal{C}_0, \) then \( F^{(p)} = F^{(p+h)} \) for any \( h > 0. \) Hence \( \lim_{p \to \infty} F^{(p)} = F^{(\infty)} \) exists and \( F \in \mathcal{D}. \) Using this information about \( F \in r(\mathcal{C}_0) \cap \mathcal{M}, \) Lemma 2.2 and its proof can be rephrased to show that \( r(\mathcal{C}_0) \subset \mathcal{D}, \)
and that $\mathcal{R}(\mathfrak{C}_n^O)$, which is defined as the closure of $[\mathfrak{C}_n^O \cap \mathfrak{M}]^{(n)}$, can also be regarded simply as $\mathfrak{C}_n^O$.

**Lemma 3.6.** Let $p \in \Gamma$, $A^{[p]} = \sum \alpha_{cd}^p E_{cd}$ with $p E_{cd}$ in $\mathcal{N}_l$ ($0 \leq j \leq n$), $\mathfrak{C}_{n-1}$, or $\mathcal{N}'_n$. Then if $q \in \Gamma$, $q > p$, $A^{(q)} = \sum \beta_{rs} E_{rs}$ with $s E_{rs}$ also in $\mathcal{N}_l$, $\mathfrak{C}_{n-1}$, or $\mathcal{N}'_n$ respectively.

**Proof.** The case $\mathcal{N}_l^O$ has already been dealt with, since $\mathcal{N}_l^O = \mathfrak{C}_0^O$.

We first consider $q = p + 3n$. Then $A^{(q)} = U_{p+3n} \cdots A^{[0]} \cdots U_{p+3n}$, and because of linearity it is sufficient to consider one term of $A^{[p]}$, say $p E_{cd}$.

If $1 \leq j \leq n$ and $p E_{cd} \in \mathcal{N}_l^O$, then Definition 3.4 shows that

$$U_{p+3n} \cdots p E_{cd} \cdots U_{p+3n} = \sum \delta_{rs} p^{3n} E_{rs}$$

is in $\mathfrak{C}_{n-1}$. Consider one term $p^{3n} E_{rs}$. With $r = c \cdot 2^{3n} + r_1 2^n + r_0$ and $s = d \cdot 2^{3n} + s_1 2^n + s_0$, we thus have $r_0 = s_0$ and $r_1 \equiv s_1$ (mod $2^{n-1}$). So $\sigma(r_1) = \sigma(s_1)$ (mod $2^n - 1$); and therefore $\psi(\sigma(r_1)) + \psi(\sigma(s_1)) = (\ldots, 1, 0)$, while $\psi(s_0) = (0, 0, \ldots)$. Hence, applying Lemma 3.3, $p^{3n} E_{rs} \in \mathcal{N}_l$ as was $p E_{cd}$.

Now the action of the unitaries $U_l$ surely preserves $\mathfrak{C}_n$, and therefore $p^{3n} E_{rs}$ is in $\mathcal{N}_n^O$.

Next suppose $p E_{cd} \in \mathfrak{C}_{n-1}^O$ and consider $U_{p+3n} \cdots p E_{cd} \cdots U_{p+3n}$. The product is in $\mathfrak{C}_{n-1}^O$, by Definition 3.4, so one term $p^{3n} E_{rs} \in \mathcal{N}_n^O$ with $r = d \cdot 2^{3n} + r_1 2^n + r_0$ and $s = c \cdot 2^{3n} + s_1 2^n + s_0$ with $r_0 \equiv s_0$ (mod 2). Thus $\psi(r_0) + \psi(s_0) = (\ldots, 1, 0)$, and we can have $p^{3n} E_{rs} \in \mathcal{N}_n$ if and only if $\psi(\sigma(r_1)) + \psi(\sigma(s_1)) = (\ldots, 1, 0)$. But by definition, $\sigma(r_1) \equiv 0$ (mod 2), so this cannot happen. As before, the action of the $U_l$'s preserves $\mathfrak{C}_n$. Therefore $p^{3n} E_{rs} \in \mathfrak{C}_{n-1}^O$.

If $p E_{cd} \in \mathfrak{C}_{n-1}^O$, then this time the computations of the preceding paragraph lead to the conclusion that the terms of $U_{p+3n} \cdots p E_{cd} \cdots U_{p+3n}$ are in $\mathfrak{C}_{n-1}^O$. (Here $\psi(\sigma(r_1)) + \psi(\sigma(s_1)) = (\ldots, a_{n-1}, 0)$.)

If $q \in \Gamma$, $q > p$, then $q = p + 3hn$ for some integer $h$, and the desired result follows by induction.

**Lemma 3.7.** For $j = 1, 2, \ldots, n$, $R(\mathfrak{C}_{j-1}^O) \not\subseteq R(\mathfrak{C}_j^O)$.

**Proof.** The inclusions are trivial and we need only show that they are proper inclusions.

Let $F$ be a matrix unit in $\mathcal{N}_l^O$ (resp. $\mathfrak{C}_n^O$), so that $F \in \mathfrak{M}_p (p \in \Gamma)$ and $F^{(p)} = E_{ab}$ in $\mathcal{N}_l^O$ (resp. $\mathfrak{C}_n^O$). Suppose that $F$ is also in $R(\mathfrak{C}_{p-1}^O)$ ($R(\mathfrak{C}_p^O)$). Then there is a sequence $F_m \in \mathfrak{M}_q$ converging strongly to $F$.

Let $F_m \in \mathfrak{M}_q$, and choose $q \in \Gamma$ such that $F_m \in \mathfrak{M}_q$. Then by Lemma 3.6, $F^{(q)} = \sum \alpha_{cd}^q E_{cd}$ with $E_{cd} \in \mathcal{N}_l^O$ ($\mathfrak{C}_n^O$). Choose $F_m$ such that $\|F_m - F\| < 1/2^p$ and choose $q \in \Gamma$ such that $F_m \in \mathfrak{M}_q$. Then by Lemma 3.6, $F^{(q)} = \sum \alpha_{cd}^q E_{cd}$ with $E_{cd} \in \mathcal{N}_l^O$ ($\mathfrak{C}_n^O$).

**Case 1.** $F \in \mathcal{N}_l^O$. Since $F^{(q)} = (F_m, F) = 0$, we have $\|F_m - F\|^2 = \|F_m\|^2 + \|F\|^2 > 1/2^p$, a contradiction. Therefore $F \notin R(\mathfrak{C}_{j-1}^O)$.

**Case 2.** $F \in \mathfrak{C}_n^O$. Here

$$F^{(q)} = (I_{E_{ii}} F_{m}^{(q)} E_{ii} + 1 E_{ij} F_{m}^{(q)} E_{ij}, 1 E_{ji} F_{m}^{(q)} E_{ji})$$

$$= 0 \quad \text{where } i, j = 0 \text{ or } 1, i \neq j.$$

So again $\|F_m - F\|^2 > \|F_m - F\|^2 > 1/2^p$, a contradiction, and therefore $F \notin R(\mathfrak{C}_n^O)$.
Definition 3.8. We define the following projections in $\mathcal{C}_n^\otimes$: For $k = 2, \ldots, n$ and $s = 0, 1, \ldots, 2^k - 1$, let $P_k(s)$ be the operator such that $P_k(s)[p + 3n] = 2^n + 3nE_{s''} + h, \ s'' = 2^3s, \ h = 0 \ (mod \ 2^3)$ and $0 \leq h \leq 2^{3n} - 1$. Let $P'(s)$ be the operator such that $P'(s)[p + 3n] = 2^n + 3nE_{s''} + h, \ s'' = 2^3s, \ h = 0 \ (mod \ 2^3)$ and $0 \leq h \leq 2^{3n} - 1$.

Lemma 3.9. Suppose $W = \mathcal{W}(p \in \mathcal{P})$ is such that $WW = \mathcal{V}^{p + 3n} + \mathcal{V}^{p + 3n}$, with $\mathcal{V}^{p + 3n} = \sum \beta E_{rs}(pE_{rs} \in \mathcal{C}_n^\otimes)$ and $X^{p + 3n} = \sum \alpha E_{rs}(pE_{rs} \in \mathcal{C}_n^\otimes)$. Let $E_{rt}$ be a fixed matrix unit in $\mathcal{C}_{n-2}^\otimes$ with $K(r, t) = K_r$. Then

\[ pE_{rt}[U_{p + 3n} \cdots W^{p + 3n}] \sum_{s = 0}^{2^k - 1} P'(s)[p + 3n][U_{p + 3n} \cdots W^{p + 3n}] = A(r, t)[p + 3n] + Q(r, t)[p + 3n], \]

where $(A, Q) = 0$ and

\[ Q[p + 3n] = \sum_{s = 0}^{2^k - 1} \alpha E_{rs} C(y)[p + 3n]E_{ab} \]

with $p + 3nE_{ab}$ in $\mathcal{N}^\otimes_{n - 2}$ or $\mathcal{N}^\otimes_{n}$, $C(y)$ a nonzero integer.

Proof. The following statements are verified by calculations similar to those of [7, pp. 295-301]. Suppose $K(pE_{rs}) = K_r$ and $K(pE_{st}) = K_s$, with both matrix units in $\mathcal{C}_n^\otimes$, $\alpha + \beta = \gamma$. If $\alpha = (a_1, a_2, \ldots, a_n)$ and $\beta = (b_1, b_2, \ldots, b_n)$, define $\omega_1 = \omega(\alpha) = 2(\sum_{i=2}^{n} a_i) + a_1 + 1, \ \omega_2 = \omega(\beta)$, and $\mu(\alpha, \beta) = 2(\sum_{i=2}^{n} a_ib_i) + a_1b_1$. Then the nonzero entries of the product $U_{p + 3n} \cdots pE_{rs} \cdots U_{p + 3n}$ have numerical value $\pm (2^{-1/2})^{\omega_1}$, and similarly for $E_{st}$. Let $r_0 = 2^{3n-2}r, s_0 = 2^{3n-2}s, t_0 = 2^{3n-2}t$. Then $2^n$ is the number of distinct $\delta$'s such that

\[ p + 3n - 2E_{r_0}r_0[U_{p + 3n} \cdots pE_{rs} \cdots U_{p + 3n}] \]

and

\[ p + 3n - 2E_{s_0}r_0[U_{p + 3n} \cdots pE_{rs} \cdots U_{p + 3n}] \]

are both nonzero.

Using the preceding, a matrix calculation shows that

\[ [U_{p + 3n} \cdots pE_{rs} \cdots U_{p + 3n}]P'(s)[p + 3n][U_{p + 3n} \cdots pE_{rs} \cdots U_{p + 3n}] \]

has a term of the form $C(y)[p + 3n]E_{r'' + 1} + 2$ and a term of the form $C(y)[p + 3n]E_{t'' + 1} + 2$, where $r'' = 2^3r, t'' = 2^3t$, and $C(y) = 2^n(2^{-1/2})^{\omega_1} + \omega_2$. It is straightforward to show that $C(y)$ depends only on $\gamma$ and on the fact that $pE_{rs}$ and $E_{st}$ are in $\mathcal{C}_n^\otimes$. By Lemma 3.3, if $K_r \subset \mathcal{C}_n^\otimes$, then $K(r'', t'' + 2)$ is in $\mathcal{N}_{n-1}$. If $K_r \subset \mathcal{N}_{n-1}$ or $\mathcal{N}_n$, then so is $K(r'' + 2, t'' + 2)$. Also, since $pE_{rs} \in \mathcal{C}_n^\otimes$, so are these matrix units. Now the product $(**)$ of the lemma equals

\[ \sum_{s = 0}^{2^k - 1} [U_{p + 3n} \cdots pE_{rs} \cdots U_{p + 3n}]P'(s)[p + 3n][U_{p + 3n} \cdots pE_{rs} \cdots U_{p + 3n}] \]

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Suppose \( K_r \subset \mathcal{N}_{n-2} \) and \( s \) such that \( pE_{rs} \) and \( pE_{st} \) are both in \( \mathcal{G}_n \). The summand corresponding to this \( s \) includes the term \( \alpha_{rs} \beta_{st} C(\gamma) p_{rs}^{+3n} E_{r^*+s,t^*+t} \), which is in \( \mathcal{N}_{n-1} \).

Considering the summands corresponding to other \( s \), we could not have one matrix unit in \( \mathcal{G}_n \), the other in \( \mathcal{G}_D \), since \( pE_{rs} \in \mathcal{G}_D \). But if both are in \( \mathcal{G}_n \), then the product is in \( \mathcal{M}_{p+2n} \) so there is no element in position \( (r^*, t^*+2) \).

Suppose \( K_r \subset \mathcal{N}_n \) or \( \mathcal{N}_{n-1} \). If \( s \) is such that \( pE_{rs} \) and \( pE_{st} \) are both in \( \mathcal{G}_n \), then the summand includes the term \( \alpha_{rs} \beta_{st} C(\gamma) p_{rs}^{+3n} E_{r^*+s,t^*+t} \), which is in the same class as \( K_r \). Again, if \( s \) is such that both matrix units are in \( \mathcal{G}_D \) there is no element in position \( (r^*+2, t^*+2) \).

So if we let \( Q \) be as stated in the lemma, with \( (a, b) = (r^*, t^*+2) \) or \( (r^*+2, t^*+2) \) according to \( K_r \), then \( (A, Q) = 0 \) and \( p_{rs}^{+3n} E_{ab} \in \mathcal{N}_{n-1}^D \) or \( \mathcal{N}_n^D \).

**Lemma 3.10.** Suppose \( W \in \mathcal{M}_{p} \) \((p \in \Gamma)\) is such that \( W^{[p]} = V^{[p]} + X^{[p]} \), with \( V^{[p]} = \sum r_{rs} p_{rs}^{+3n} \) \((pE_{rs} \in \mathcal{G}_D)\) and \( X^{[p]} = \sum r_{rs} p_{rs}^{+3n} \) \((pE_{rs} \in \mathcal{N}_n^D)\). Let \( pE_{rt} \) be a fixed matrix unit in \( \mathcal{G}_n^D \) with \( K(r, t) = K_r \). Then

\[
W^{[p]} U_{p+3n} \ldots W^{[p]} U_{p+3n} \sum_{s=0}^{2p-1} P_k(s) [W^{[p]} U_{p+3n} \ldots W^{[p]} U_{p+3n} pE_{rt}]
\]

\[= A(r, t)^{[p+3n]} + Q(r, t)^{[p+3n]}, \]

where

\[
Q^{[p+3n]} = \sum_{s=0}^{2p-1} r_{rs}^* \beta_{st} D_k(\gamma) p_{rs}^{+3n} E_{ab}
\]

with \( p_{rs}^{+3n} E_{ab} \in \mathcal{N}_{k-1}^D \), \( D_k(\gamma) \) a nonzero integer.

**Proof.** The proof is like that of the preceding lemma, with the following changes: \( \omega_1 = \omega(\alpha) = 2(\sum_{i=0}^{n} a_i) + a_1 \) (and a similar change in \( \omega_2 \)), \( \mu(\alpha, \beta) = 2(\sum_{i=0}^{2n-k} a_i b_i) + a_i b_k + a_i b_1 \), \( r_0 = 2^{n+k-1} r, s_0 = 2^{n+k-1} s, t_0 = 2^{n+k-1} t \). Then \( 2^s \) is the number of distinct \( \delta \)'s such that

\[
p_{rs}^{+n+k-1} E_{r_0 t_0} U_{p+3n} \ldots pE_{rs} U_{p+3n} p_{rs}^{+n+k-1} E_{s_0 t_0}^0 \delta_0 + \delta
\]

and

\[
p_{st}^{+n+k-1} E_{s_0 t_0} U_{p+3n} \ldots pE_{st} U_{p+3n} p_{st}^{+n+k-1} E_{t_0 s_0}^0 \delta_0 + \delta
\]

are both nonzero. The expression

\[[U_{p+3n} \ldots pE_{rs} \ldots U_{p+3n}] P_k(s) [p_{rs}^{+3n}] U_{p+3n} \ldots pE_{rt} \ldots U_{p+3n} \]

has a term of the form \( D_k(\gamma) p_{rs}^{+3n} E_{r^*+s} \) and a term of the form \( D_k(\gamma) p_{rs}^{+3n} E_{r^*+s} \) where \( r^* = 2^n r, t^* = 2^n t, \pi = 2^{n-k}, \) and \( D_k(\gamma) = 2^n (2^{n-k}) a_1 + a_2 \). Here \( D_k(\gamma) \) depends only on \( \gamma \) and on \( k \). By Lemma 3.3, if \( K_r \subset \mathcal{G}_k \), then \( K(r^*, t^*+\pi) \) is in \( \mathcal{N}_{k-1} \); if \( K_r \subset \mathcal{N}_k \), then \( K(r^*+\pi, t^*+\pi) \) is in \( \mathcal{N}_{k-1} \).

It can be verified, as in the preceding lemma, that \( (A, Q) = 0 \) if we take \( Q \) as stated, with \( (a, b) = (r^*, t^*+\pi) \) or \( (r^*+\pi, t^*+\pi) \) according to \( K_r \).
Lemma 3.11. If the results of Lemmas 3.9 and 3.10 hold for \( q = p + 3n \) (i.e., \( q \in \Gamma \)). Also,

\[
\| Q \|^2 \geq \left| \sum_{s=0}^{2^p-1} \alpha_s \tilde{a}_s \right|^2 / 2^p + 5n.
\]

Proof. We first obtain bounds for \( C(\gamma) \) and \( D_k(\gamma) \). In both cases, we have \( \mu \geq 0 \) and \( \omega_1 + \omega_2 \leq 2(2n-1) + 2 = 4n \). Hence \( C(\gamma) \) or \( D_k(\gamma) = 2^n(2^{-1/2})(\omega_1 + \omega_2) \geq (2^{-1/2})4n = 1/2^n \).

\[
\| Q^{(p+3n)} \|^2 \geq \| C(\gamma) \|^2 \left| \sum_{s=0}^{2^p-1} \alpha_s \tilde{a}_s \right|^2 / 2^p + 3n
\]

and similarly in the case of \( D_k(\gamma) \).

Now the unitaries \( U_{p+3n+1}, \ldots, U_{p+3hn} \) preserve the orthogonality of \( A \) and \( Q \) and the norm of \( Q \). Also, by Lemma 3.6, matrix units in \( \mathcal{A}_r \) \((j=1,2,\ldots,n)\) are left in that class under the action of the unitaries \( U_r \).

Lemma 3.12. For \( j=1,2,\ldots,n-1 \), let \( \gamma_j = \{ V : V[R(\mathcal{A}_r)]V^* = R(\mathcal{A}_r) \}, \) \( V \) unitary, \( V \in \mathcal{A}_r \). If \( V \in \gamma_j \), then there is a sequence \( V_m \in \mathcal{A}_r \) converging metrically to \( V \) such that if \( V_m \in \mathcal{M}_p (p \in \Gamma) \), \( V_m^{(p)} = \sum \beta_{rs}^p E_{rs} \) with \( \beta_{rs}^p \in \mathcal{D}_r \). Thus, \( N(R(\mathcal{D}_r)) \subset R(\mathcal{A}_r) \).

Proof. (i) Since \( V \in \mathcal{A}_r \), \( \| V \| \leq 1 \), there is a sequence \( W_m \in \mathcal{M}_p \), \( \| W_m \| \leq 1 \), converging strongly and metrically to \( V \) [4]. If \( W_m \in \mathcal{M}_p \), let \( W_m^{(p)} = V_m^{(p)} + X_m^{(p)} \), where \( V_m^{(p)} = \sum \beta_{rs}^p E_{rs} \) and \( X_m^{(p)} = \sum \alpha_{rs}^p E_{rs} \). Because of the orthogonality of \( V_m \) and \( X_m \), \( X_m \) itself is Cauchy in the metric topology. Now \( \| W_m \| \leq 1 \) implies \( \| V_m \| \leq 1 \) because of the definition of \( \mathcal{D}_r \). Since \( X_m = W_m - V_m \), we have \( \| X_m \| \leq 2 \), and so \( X_m \) is also Cauchy in the strong topology [5, p. 723]. Let \( X_m \to X \). Suppose \( \lim_m \| X_m \| \neq 0 \); then \( \lim_m \| X_m X_m^* \| \neq 0 \) also. Hence \( \| X_m X_m^* \|^2 > 2^{5n} \epsilon^2 \) for all \( m \) and some \( \epsilon > 0 \). (Recall that \( n \) is fixed and related only to \( R = R_n \)).

Choose \( W_m \) so that \( \| W_m - V \| < \epsilon/4 \). Suppose \( W_m \in \mathcal{M}_p \). Then

\[
\left\| X_m^{(p)} X_m^{(p)*} \right\|^2 = (1/2^p) \sum \left| \sum_{s=0}^{2^p-1} \alpha_s \tilde{a}_s \right|^2 > 2^{5n} \epsilon^2.
\]

(The outer summation is over pairs \((r,t)\) such that \( E_{rt} \in \mathcal{D}_r \), since \( E_{rs} \in \mathcal{D}_r \).)

Fix \( p \) from here on.

Consider \( \sum_{s=0}^{2^p-1} P(s)^{p+3n} \), which has its matrix units in \( \mathcal{D}_r \). Then \( \sum P(s) \) is in \( R(\mathcal{A}_r) \) for any \( j \geq 1 \), and if \( V \in \gamma_j \), \( V \sum P(s) V^* \in \gamma_j \). So there exists a sequence \( T_j \in \mathcal{M}_q \), \( \| T_j - T \| \to 0 \), and \( T_j \in \mathcal{M}_q \) (\( q \in \Gamma \)) implies \( T_j^{(p)} = \sum \eta_{rn} E_{rn} \) with \( E_{rn} \) in \( \mathcal{D}_r \). Choose \( T_j \) such that \( \| V(\sum P(s)) V^* - T_j \| < \epsilon/2 \). Since \( \sum P(s) \) is a projection, of norm at most one,

\[
\left\| W_m \left( \sum P(s) \right) W_m^* - V \left( \sum P(s) \right) V^* \right\| < \epsilon/2,
\]
and thus it follows that

$$\left[ W_m \left( \sum_s P'(s) \right) W_m^* - T \right] < \epsilon.$$ 

On the other hand, we can apply Lemmas 3.9 and 3.11 with $W_m$ replacing $W$. Take $q$ to be such that $q \in \Gamma$, $q \geq p + 3n$, and $T \in \mathcal{M}_q$. Since $Q_{\pi}^{(a)} = \sum \lambda_{cd}^a E_{cd}$ ($E_{cd} \in \mathcal{N}_{n-1}$) and $T_{\pi}^{(a)} = \sum \eta_{ih}^a E_{ih}$ ($E_{ih} \in \mathcal{D}_{j-1}$, where $j-1 < n-1$), we have $(T_{\pi}^{(a)}, Q_{\pi}^{(a)}) = 0$. Therefore

$$\left[ \sum P'(s) W_m^* - p E_{rt} T_{\pi}^{(a)} p E_{tt} \right]^2 
\geq \left[ Q(r, t) \right]^2 \geq \left| \sum \alpha_{rs} \alpha_{tt} \right|^2 / 2^{n+5n}.$$ 

Finally, we have:

$$\epsilon^2 \geq \sum_{r,s} \left| \sum_s P'(s) W_m^* - \sum_{s} P'(s) W_m^* - T \right|^2 \geq \left| \sum \alpha_{rs} \alpha_{tt} \right|^2 / 2^{n+5n} > \epsilon^2,$$

which is a contradiction.

Therefore $\lim_{m} \|X_m^r\| = 0$ and so $\lim_{m} \|V_m - V\| = 0$, where $\|V_m\| \leq 1$ and $V_m \in \mathcal{M}_x$ ($x \in \Gamma$) implies $V_{\pi}^{(a)} = \sum \beta_{rs} E_{rs}$ with $E_{rs} \in \mathcal{D}_n$.

(ii) To show: Suppose $j < k \leq n$ and suppose there exists $W_m \in \mathcal{M}_x$ such that $\|W_m\| \leq 1$, $\lim_{m} \|W_m - V\| = 0$, and $W_m \in \mathcal{M}_p$. Then there exists $V_m$ with the same properties except that $V_{\pi}^{(a)} = \sum \beta_{rs} E_{rs}$ with $E_{rs} \in \mathcal{D}_{k-1}$. Then there exists $V_m$ with the same properties except that $V_{\pi}^{(a)} = \sum \beta_{rs} E_{rs}$ with $E_{rs} \in \mathcal{D}_{k-1}$.

We let the assumed $W_{\pi}^{(a)} = V_{\pi}^{(a)} + X_{\pi}^{(a)}$, where the matrix units of the two summands are in $\mathcal{D}_{k-1}$ and $\mathcal{N}_{k}$ respectively. The argument proceeds much as in part (i), with $\sum P_k(s)$ replacing $\sum P'(s)$, so that Lemmas 3.10 and 3.11 apply. Since $V(\sum P_k(s)) V^* = T$ in $\mathcal{R}(\mathcal{D}_{j-1})$ and since $j-1 < k-1$, the desired orthogonality holds between $Q$ (in $\mathcal{N}_{k-1}$) and $T_{\pi}$ (the sequence of matrices converging to $T$). We are led to conclude that $\lim_m \| X_m \| = 0$, and that $V$ is the metric limit of $V_m$.

Since we can extend this as far as $k = j+1$ by a finite induction process, the lemma is proved.

**Theorem 3.13.** For $j = 1, 2, \ldots, n-1$, if $R(\mathcal{V}_j)$ is the ring generated by $\mathcal{V}_j$ as defined in Lemma 3.12, then $R(\mathcal{V}_j) = R(\mathcal{D}_j)$. Thus, $N(R(\mathcal{D}_{j-1}^p)) = R(\mathcal{D}_j^p)$.

**Proof.** By Lemma 3.12, $R(\mathcal{V}_j) \subset R(\mathcal{D}_j^p)$.

For the reverse inclusion, take $T \in R(\mathcal{D}_{j-1}^p)$. Let $V_{\pi}^{(a)} = \sum \pm E_{rs}$ with $E_{rs} \in \mathcal{D}_{j-1}$ and signs arbitrary. Then $V_{\pi}^* T V_{\pi}^*$ is in $R(\mathcal{D}_{j-1}^p)$ since all three operators are.
Next let $V_{2}^{[p]} = \sum \pm pE_{rs}$ with $pE_{rs}$ in $\mathcal{N}_{2}$. Take a sequence $T_{m} \in \mathfrak{M}$, $T_{m} \to T$, and if $T \in \mathfrak{M}$, $T_{m}^{[q]} = \sum \beta_{cd}E_{cd}$ with $qE_{cd}$ in $\mathfrak{E}_{2}$. If $z = \max \{p, q\}$, then

$$V_{2}^{[q]} T_{m}^{[q]} V_{2}^{[p]} = \left[ \sum \delta_{rs}^z E_{rs} \right] \left[ \sum \beta_{cd}^z E_{cd} \right] \left[ \sum \delta_{rs}^z E_{rs} \right],$$

where the matrix units of the first sum are in $\mathfrak{N}_{2}$, those of the second in $\mathfrak{E}_{2}$, and those of the third in $\mathfrak{N}_{1}$, by Lemma 3.6. Calculating by means of §3.1, we see that each matrix unit of this product is in $\mathfrak{E}_{2}$. Hence $V_{2} T_{m} V_{2}^{*}$ is in $R(\mathfrak{E}_{2})$, and so is its strong limit $V_{2} TV_{2}^{*}$.

But all unitaries of the form $V_{1}$ or $V_{2}$ are sufficient to generate $R(\mathfrak{E}_{2})$. Therefore $R(\mathfrak{E}_{2}) \subset R(\mathfrak{E}_{2})$, and hence $R(\mathfrak{E}_{2}) = R(\mathfrak{E}_{2})$.

**Theorem 3.14.** If $\mathfrak{E}_{n} = \{V : R(\mathfrak{E}_{n-1})V^{*} = R(\mathfrak{E}_{n-1}), V$ unitary, $V \in \mathfrak{M}\}$, then $R(\mathfrak{E}_{n}) = R(\mathfrak{E}_{n}) = \mathfrak{M}$. Thus, $N(\mathfrak{E}_{n-1}) = \mathfrak{M}$.

**Proof.** Obviously $R(\mathfrak{E}_{n}) \subset R(\mathfrak{E}_{n})$.

For the reverse inclusion, let $T$ be in $R(\mathfrak{E}_{n-1})$. Consider in turn four types of unitaries $V_{i}^{[p]} = \sum \pm pE_{rs}$ ($i = 1, 2, 3, 4$ and signs arbitrary). For $i = 1$, the matrix units are to be in $\mathfrak{E}_{2}$; for $i = 2$, in $\mathfrak{E}_{2}$; for $i = 3$, in $\mathfrak{E}_{1}$; for $i = 4$, in $\mathfrak{N}_{2}$. By Lemma 3.6, these classes are preserved under the unitaries $U_{i}$. So calculations like those in the proof of Theorem 3.13 show that $V_{i} TV_{i}^{*}$ is in $R(\mathfrak{E}_{n-1})$ for $i = 1, 2, 3, 4$.

But all unitaries of these types are sufficient to generate $R(\mathfrak{E}_{n})$, or $\mathfrak{M}$. Therefore $R(\mathfrak{E}_{n}) \subset R(\mathfrak{E}_{n})$, and $R(\mathfrak{E}_{n}) = \mathfrak{M}$.

**Remark.** Theorems 3.13 and 3.14, together with Theorem 3.5 and Lemma 3.7, show that for each $R_{n}$, $n = 1, 2, 3, \ldots$, we have $R_{n} \supset N(\mathfrak{M}) \supset \cdots \supset N^{n+1}(R_{n}) = \mathfrak{M}$. In order to prove that $R_{n}$ is $M$-semiregular ($n + 1 = M$), we need only show that $N(R_{n})$, $N^{2}(R_{n})$, $\ldots$, $N^{n}(R_{n})$ are not factors. ($N^{n+1}(R_{n}) = N^{M}(R_{n})$ is the factor $\mathfrak{M}$.)

**Theorem 3.15.** For $k = 1, 2, \ldots, n$, $N^{k}(R_{n})$ is not a factor.

**Proof.** If $k \neq n$, $N^{k}(R_{n}) = N^{k-1}(P_{n}) = R(\mathfrak{E}_{k-1}) = R(\mathfrak{E}_{k-1})$. Consider the projection $1E_{00} = 1E_{00} \in R_{n} \subset N^{k}(R_{n})$. If $A$ is any operator in $N^{k}(R_{n})$, there is a sequence $A_{m} \to A$ such that if $A_{m} \in \mathfrak{M}$, $A_{m}^{[p]} = \sum \alpha_{rs} E_{rs}$ with $pE_{rs}$. Then

$$1E_{00} A_{m}^{[p]} 1E_{00} = \sum \alpha_{rs} 1E_{00}^{*} E_{rs} 1E_{00}$$

$$= \sum \alpha_{rs} E_{rs} \quad \text{(by definition of } \mathfrak{E}_{k-1})$$

Thus $1E_{00} A_{m} 1E_{00} = A_{m}$, and taking strong limits, $1E_{00} A 1E_{00} = A$.

Therefore $1E_{00}$ commutes with $N^{k}(R_{n})$, $1E_{00} \neq \alpha I$, $1E_{00} \in N^{k}(R_{n})$, and so $N^{k}(R_{n})$ is not a factor.

**Bibliography**


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