M-SEMIREGULAR SUBALGEBRAS IN HYPERFINITE FACTORS

BY
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1. Introduction. The general study of algebras of operators on Hilbert space has led to the investigation of rings of operators, also called \( W^* \)-algebras or von Neumann algebras. If the center of a ring (center in the algebraic sense) consists only of scalar multiples of the identity, then the ring is a factor. Since every ring can be decomposed into factors [6], the study of rings is, in a sense, reduced to a study of factors. In this paper we are concerned with the maximal abelian subalgebras of type \( \text{II}_1 \) factors, or continuous factors which have a finite trace defined on them [2]. For the present, we restrict ourselves to the study of hyperfinite factors, that is, those which are generated by a sequence of factors \( \mathcal{M}_n \) of type \( \text{II}_1 \), with \( \mathcal{M}_n \nsubseteq \mathcal{M}_{n+1} \). (The factor \( \mathcal{M}_n \) is isomorphic to the algebra of \( n \) by \( n \) matrices.) Since all hyperfinite factors are algebraically isomorphic [5, §4.7], while the concept of a subring of a finite factor is purely algebraic [5, §1.6], any construction used will yield general results.

Dixmier has defined three types of maximal abelian subalgebras \( R \) in a factor \( \mathfrak{A} \): regular, semiregular, and singular [3]. These depend on the properties of \( N(R) \), the ring generated by \( \{ V : VRV^* = R, \text{V unitary}, V \in \mathfrak{A} \} \). In other words, \( N(R) \) is the normalizer of \( R \) in \( \mathfrak{A} \). Later, Anastasio defined an additional type, \( M \)-semiregular \( (M = 1, 2, 3, \ldots) \), which coincides with the semiregular type when \( M = 1 \). Extending the notation \( N(D) \) to any subring \( D \subseteq \mathfrak{A} \), and letting \( N_k(D) = N[N^{k-1}(D)] \), we have a chain \( R \supseteq N(R) \supseteq N^2(R) \supseteq \cdots \supseteq N^M(R) = \mathfrak{A} \). We say that a maximal abelian subalgebra \( R \) is \( M \)-semiregular if \( N^k(R) \) is not a factor for \( k < M \), but \( N^M(R) \) is a factor [1]. Anastasio constructed infinite sequences of non-isomorphic 2-semiregular and 3-semiregular subalgebras in a hyperfinite factor. (The 1-semiregular case had already been done [7].) In this paper we propose to show the existence of \( M \)-semiregular subalgebras for every positive integer \( M \neq 1 \).

We use the notation and results of [7]. Let \( \mathfrak{M}_p \) be the full \( 2^p \) by \( 2^p \) matrix algebra over the complex numbers, and \( \{ pE_{ij} : i, j = 0, 1, \ldots, 2^p - 1 \} \) the matrix units which generate it. By letting \( pE_{ij} = p + 1E_{2i, 2j} + \cdots + p + 1E_{2^p - 1, 2^p - 1} \), we imbed \( \mathfrak{M}_p \) in \( \mathfrak{M}_{p+1} \). Then \( \bigcup_{p=1}^\infty \mathfrak{M}_p = \mathfrak{M} \) is a *-algebra. The normalized matrix trace on \( \mathfrak{M} \) makes it into a pre-Hilbert space \( \mathfrak{H} \): If \( A, B \in \mathfrak{M} \), let \( (A, B) = \text{Tr} (B^*A) \), so that \( (A, A)^{1/2} = \|[A]\| \), the Hilbert space or metric norm of \( A \). If \( A \) is in \( \mathfrak{M} \), then \( A \) acting

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by left multiplication is a bounded operator on \( \mathcal{H} \), so it can be extended to the Hilbert space closure \( \mathcal{H} \). If \( \mathcal{A} \) is the weak closure of \( \mathcal{M} \), then it is well known that \( \mathcal{A} \) is a hyperfinite factor [2].

2. \( M \)-semiregular subalgebras. The following general construction leads to a large variety of maximal abelian subalgebras of \( \mathcal{M} \).

Definitions 2.1. Let \( \{U_t: t = 1, 2, \ldots \} \) be a set of selfadjoint unitaries such that:

1. \( U_t \in \mathcal{M} \); (2) \( U_t \) is zero except for 2 by 2 blocks along the main diagonal. Let \( Y_t = U_1 U_2 \cdots U_t \), and for \( A \in \mathcal{M} \), define \( A^{(t)} = Y_t A Y_t^* \) and \( A^{(t)} = Y_t^* A Y_t \). For fixed \( t \), the mappings \( A \to A^{(t)} \) and \( A \to A^{(t)} \) are *-automorphisms of \( \mathcal{M} \) and inverses of each other. Because of the form of \( U_t \), the matrix unit \( e_{i,j} \) commutes with \( U_t \) for all \( t > p \). Thus if \( A \) is a diagonal matrix in \( \mathcal{M}_p \), \( p \leq t \), then \( A^{(t)} = A^{(t+1)} \), and so \( \lim_{t \to \infty} A^{(t)} = A^{(\infty)} \) exists in \( \mathcal{M} \), hence in \( \mathcal{A} \).

In general, for \( A \in \mathcal{M} \), the limit \( A^{(\infty)} \) does not exist. The mapping \( A \to A^{(\infty)} \) is thus an isomorphism of some proper subalgebra of \( \mathcal{M} \) into \( \mathcal{A} \). This subalgebra, the domain of the mapping, we call \( \mathcal{D} \). If \( E \) is the set of diagonal matrices, then \( E \subset \mathcal{D} \), as seen above. The ring \( (E^{(\infty)})^- \) is the maximal abelian subalgebra \( \mathcal{R} \) which we study in this paper. (Cf. [7, pp. 285-286], for the proof that \( \mathcal{R} \) is maximal abelian.) In Lemma 2.2 we will show that \( E^- \subset \mathcal{D} \), and that \( (E^-)^{(\infty)} = (E^{(\infty)})^- = \mathcal{R} \).

Lemma 2.2. If \( F = E^- \), then \( F \subset \mathcal{D} \), and \( F^{(\infty)} = (E^{(\infty)})^- = \mathcal{R} \).

Proof. Suppose \( A \in F \). Then there is a sequence \( A_n \in E \cap \mathcal{M}_n \), \( A_n \to A \), with \( A_n^{(\infty)} \in \mathcal{M} \). Let \( \epsilon > 0 \) be given, and choose \( n \) such that \( \|A_n - A\| < \epsilon / 2 \). Consider

\[
\|A^{(t)} - A^{(n)}\| = \|Y_t A Y_t^* - Y_n A Y_n^*\| \\
\leq \|Y_t A Y_t^* - Y_n A Y_n^*\| + \|Y_n A Y_n^* - Y_t A Y_t^*\| + \|Y_t A Y_t^* - Y_n A Y_n^*\|.
\]

Choose \( s, t \) such that both are greater than or equal to \( n \). Then \( Y_s A Y_s^* = A^{(n)} = A^{(n)} \) and \( Y_t A Y_t^* = A^{(n)} = A^{(n)} \). Hence \( \|Y_s A Y_s^* - Y_t A Y_t^*\| = 0 \) if \( s, t \geq n \). Since \( Y_s \) and \( Y_t \) are unitary, the other two norms equal \( \|A - A_n\| \), and so the sum is less than \( \epsilon \). Therefore \( A^{(t)} \) is Cauchy in the metric topology.

Now \( A \in \mathcal{M} \) and so \( \|A\| < \infty \). Since \( \|A^{(t)}\| = \|A\| \), \( A^{(t)} \) is a bounded sequence. By [5, p. 723], \( A^{(t)} \) is then Cauchy in the strong topology also, so its limit exists in \( \mathcal{M} \). Therefore \( F \subset \mathcal{D} \).

We next show that \( F^{(\infty)} = (E^{(\infty)})^- \) or \( \mathcal{R} \). Let \( A \in F, A_n \in E \cap \mathcal{M}_n, A_n \to A \). Let \( \epsilon > 0 \) be given, and choose \( n \) so that both \( \|A_n - A\| < \epsilon / 2 \) and \( \|A^{(n)} - A^{(n)}\| < \epsilon / 2 \). Then

\[
\|A^{(n)} - A^{(\infty)}\| \leq \|A^{(n)} - A^{(n)}\| + \|A^{(n)} - A^{(n)}\| + \|A^{(n)} - A^{(n)}\| \\
= \|A^{(n)} - A^{(n)}\| + \|A_n - A\| + \|A^{(n)} - A^{(n)}\| \\
\leq 0 + \epsilon / 2 + \epsilon / 2 = \epsilon.
\]

Therefore \( A^{(n)} \in (E^{(\infty)})^- \), and so \( F^{(\infty)} \subset \mathcal{R} \).

On the other hand, if \( G \in \mathcal{R} \), there is a sequence \( A_n \in E \cap \mathcal{M}_n, A_n^{(\infty)} \to G \), with \( \|A_n^{(\infty)}\| = \|A_n^{(n)}\| = \|A_n\| \leq \|G\| \) [4]. Since \( A_n^{(\infty)} \) is metrically Cauchy, so is \( A_n \), which
is also strongly Cauchy because of the bound on the norm. Hence \( A_n \) has a limit \( A \in F \). By another standard argument, if \( \varepsilon > 0 \) be given, there exists \( N \) such that \( \| Y_t A Y^*_t - G \| < \varepsilon \) when \( t \geq N \). Therefore \( G = \lim_{t \to \infty} Y_t A Y^*_t \) and so \( G \in F(\alpha) \). Hence \( R \subset F(\alpha) \), and \( F(\alpha) = R \).

**Remark.** The normalizer of \( R \) in \( \mathcal{U} \) results in a similar way from the mapping \( A \to \mathcal{A}(\alpha) \). The subalgebra \( \mathcal{r}(\mathcal{U}_0) \), to be defined later, has the property that \( E \subset \mathcal{r}(\mathcal{U}_0) \subset \mathcal{U} \), and \( \mathcal{r}(\mathcal{U}_0)(\alpha) = N(R) \). (It appears that \( \mathcal{r}(\mathcal{U}_0) = \mathcal{U} \), but we do not need this fact and have not proved it.)

By means of various choices of the sequence \( \{U_i\} \), in §3 we construct a maximal abelian subalgebra \( R_n \) for each \( n = 1, 2, 3, \ldots \), where \( R_n \) is \( M \)-semiregular, \( M = n + 1 \). The chain \( R_n \subset N(R) = P_n \subset N^2(R_n) \subset \cdots \subset N^{n+1}(R_n) = N^M(R_n) = \mathcal{U} \) is such that \( N^k(R_n) \) is not a factor for \( k = 1, 2, \ldots, n < M \), while \( N^M(R_n) \) is the factor \( \mathcal{U} \).

Furthermore, the subalgebras \( R_n \) are not conjugate under any *-automorphism of \( \mathcal{U} \). The integer \( n \) determines the number of normalizers between \( R_n \) and \( \mathcal{U} \) in the chain, and this is an automorphism invariant (cf. [7, pp. 282 and 305]).

**Note.** For convenience of notation, we often work with \( N^k(P_n) = N^{k+1}(R_n) \), \( k = 0, 1, \ldots, n \).

### 3. Detailed construction of \( M \)-semiregular subalgebras.

In the construction of \( M \)-semiregular subalgebras, we use the following notations and definitions.

**Definitions 3.1.** We regard \( n = 1, 2, 3, \ldots \) as fixed, and let

\[ \Gamma = \{ p : p = (3c + 1)n, c = 0, 1, 2, \ldots \}, \]

an infinite set of positive integers. We define \( \mathcal{G}_n = \{ pE_{rs} : p \in \Gamma \} \). In the following paragraphs, we define a decomposition of \( \mathcal{G}_n \) into \( 2^n \) disjoint subsets \( K_y \) \((0 \leq y \leq 2^n - 1)\), so that \( \mathcal{G}_n = \bigcup_y K_y \).

Let \( \mathcal{Q}_n \) be the set of all \( n \)-tuples \( (a_1, a_2, \ldots, a_n) \), where \( a_k = 0 \) or \( 1 \). This is a commutative group under the operation of coordinate-wise addition modulo 2. If \( y = 0, 1, \ldots, 2^n - 1 \) and \( y = \sum_{i=1}^n a_i 2^{n-1} \), we identify it with its binary expansion \( (a_1, a_2, \ldots, a_n) \), so that we can consider \( y \) as an element of \( \mathcal{Q}_n \). The sum \( y_1 + y_2 \) is then defined by addition in \( \mathcal{Q}_n \).

We determine the set \( K_y \) in which \( pE_{rs} \) is contained as follows: For any index \( r \) \((0 \leq r \leq 2^{(3c + 1)n} - 1)\), let \( r = \sum_{k=0, k \neq 2}^{3c} r_k 2^{kn} \). (Congruence is modulo 3 in this and in the following summations.) For \( k = 0 \), we have \( 0 \leq r_k < 2^2 \), and so \( r_k = \sum_{j=1}^{n} k_j 2^{n-j} \) with \( (k_1, \ldots, k_n) \in \mathcal{Q}_n \). Designate this element of \( \mathcal{Q}_n \) by \( \psi(r_k) \). For \( k = 1, 0 \leq r_k < 2^n \), and we let \( \sigma(r_k) = 2(r_k \mod 2^n - 1) \), so that \( \psi(\sigma(r_k)) \) is defined. Let

\[ \Delta(r) = \sum_{k = 0, k \neq 0}^{3c} \psi(r_k) + \sum_{k = 1}^{3c-2} \psi(\sigma(r_k)), \]

where the addition is coordinate-wise (mod 2), so that \( \Delta(r) \in \mathcal{Q}_n \). Then \( K_y = \{ pE_{rs} : \Delta(r) + \Delta(s) = y \} \) and we say that \( K_y = K(pE_{rs}) \). Since this is independent of \( p \), we sometimes write \( K_y = K(r, s) \).
Definitions 3.2. We also define the following sets of matrix units, again subsets of \( \mathcal{G}_n : \mathcal{G}_0 = \mathcal{N}_0 = K_0 \). For \( j = 1, 2, \ldots, n \), \( \mathcal{G}_j = \bigcup \{ K_j : \gamma = (a_1, \ldots, a_n, 0, 0, \ldots, 0) \} \) and \( \mathcal{N}_j = \mathcal{G}_j \sim \mathcal{G}_{j-1} \). If we let

\[
\mathcal{M}^D = 1\mathcal{E}_0 \mathcal{E}_1 \quad \text{and} \quad \mathcal{M}_j = \mathcal{G}_j \cap \mathcal{M}_j^D \quad \text{and} \quad \mathcal{N}_j = \mathcal{N}_j \cap \mathcal{M}_j^D, \]

then we define \( \mathcal{G}_j^D = \mathcal{G}_j \cap \mathcal{M}_j^D \) and \( \mathcal{N}_j = \mathcal{N}_j \cap \mathcal{M}_j^D \). We let \( r(\mathcal{G}_j^D) \) be the ring generated by the matrix units in \( \mathcal{G}_j^D \), while \( R(\mathcal{G}_j^D) \) is the ring generated by \( \{ F : F = (p\mathcal{E}_r)^p \} \) with \( p\mathcal{E}_r \in \mathcal{G}_j^D \).

Lemma 3.3. Suppose \( p \in \Gamma \), \( p + 3n \mathcal{E}_r \in \mathcal{K}_j \). Let \( r = r'2^{3n} + r_2^{2n} + r_0 \) and \( s = s'2^{3n} + s_2^{2n} + s_0 \) (\( 0 \leq r_1, s_1 < 2^{2n} \), \( 0 \leq r_0, s_0 < 2^n \)). Then

\[
\gamma = \Delta (r) + \Delta (s) = (\Delta (r') + \Delta (s')) + \sigma + (\Delta (r_0) + \Delta (s_0)),
\]

where \( \sigma = \psi (\sigma (r_1)) + \psi (\sigma (s_1)) \).

Proof. This follows by computation from Definitions 3.1, since \( \Delta (r) \) can be written as \( \Delta (r') + \psi (\sigma (r_1)) + \psi (\sigma (r_0)) \), and the same for \( \Delta (s) \).

Construction 3.4. In constructing the maximal abelian subalgebra \( R_n \) according to §2.1, the sequence \( \{ U_t : t = 1, 2, 3, \ldots \} \) is to be as follows: Let

\[
B_1 = \begin{bmatrix} 2^{-1/2} & 2^{-1/2} \\ 2^{-1/2} & -2^{-1/2} \end{bmatrix}
\]

Let \( B_t \) be in \( \mathcal{M}_n \), with all entries zero except for 2 by 2 blocks like \( B_1 \) along the main diagonal.

For \( n > 1 \), \( U_t = I \) if \( t < n \). If \( p \in \Gamma \) and if \( \Delta (r) = (a_1, a_2, \ldots, a_n) \), define:

\[
p\mathcal{E}_{rr} U_{p+1} = p\mathcal{E}_{rr} \quad \text{if} \quad a_n = 0,
\]

\[
p\mathcal{E}_{rr} U_{p+2} = p\mathcal{E}_{rr} \quad \text{if} \quad a_1 = 1,
\]

\[
p\mathcal{E}_{rr} U_{p+n-j+1} = p\mathcal{E}_{rr} \quad \text{if} \quad a_j = 0,
\]

\[
p\mathcal{E}_{rr} B_{p+n-j+1} = p\mathcal{E}_{rr} \quad \text{if} \quad a_j = 1,
\]

\[
U_{p+n+1} = I.
\]

\[
p\mathcal{E}_{rr} U_{p+n+2} = p\mathcal{E}_{rr} \quad \text{if} \quad a_2 = 0,
\]

\[
p\mathcal{E}_{rr} B_{p+n+2} = p\mathcal{E}_{rr} \quad \text{if} \quad a_1 = 1,
\]

\[
p\mathcal{E}_{rr} U_{p+n+f} = p\mathcal{E}_{rr} \quad \text{if} \quad a_f = 0,
\]

\[
p\mathcal{E}_{rr} B_{p+n+f} = p\mathcal{E}_{rr} \quad \text{if} \quad a_f = 1,
\]

\[
p\mathcal{E}_{rr} U_{p+2n} = p\mathcal{E}_{rr} \quad \text{if} \quad a_n = 0,
\]

\[
p\mathcal{E}_{rr} B_{p+2n} = p\mathcal{E}_{rr} \quad \text{if} \quad a_n = 1,
\]

\[
U_{p+2n+1} = \ldots = U_{p+3n-2} = I,
\]

\[
1\mathcal{E}_{00} U_{p+3n-1} = 1\mathcal{E}_{00},
\]

\[
1\mathcal{E}_{11} U_{p+3n-1} = 1\mathcal{E}_{11} B_{p+3n-1},
\]

\[
U_{p+3n} = I.
\]
For \( n = 1, \, p \in \Gamma \), and if \( \Delta(r) = (a_t) \), define:

\[
\begin{align*}
{^pE}_r U_{p+1} = {^pE}_r & \quad \text{if } a_t = 0, \\
& = {^pE}_r B_{p+1} & \text{if } a_t = 1,
\end{align*}
\]

\[
\begin{align*}
{^1E}_{00} U_{p+2} = {^1E}_{00}, \\
{^1E}_{11} U_{p+2} = {^1E}_{11} B_{p+2}, \\
U_{p+3} = I.
\end{align*}
\]

**Remark.** With this construction we aim to show that \( N^{j+1}(R) = N^j(P) = R(\mathcal{D}_p^j) \) for \( j = 0, 1, \ldots, n-1 \), and that none of these is a factor. However,

\[
N^n(P) = \mathcal{U} = R(\mathcal{D}_n^0 \cup \mathcal{D}_n^1).
\]

(For \( n = 1 \), the following three propositions hold with slight adaptations. Then nothing else is needed until Theorems 3.14 and 3.15.)

**Theorem 3.5.** \( N(R) = P = R(\mathcal{D}_0^P) \).

**Proof.** If \( p \in \Gamma, \ {^pE}_r \in \mathcal{D}_p^0 \), then \( \Delta(r) + \Delta(s) = (0, 0, \ldots, 0) \). So computation with the definitions of §3.4 shows that

\[
U_p \cdots U_{p+3n-1} {^pE}_r U_p \cdots U_{p+3n-1} = {^pE}_r.
\]

If \( q \in \Gamma, \ q > p \), then \( q = p + 3hn \) for some integer \( h \). Since \( {^pE}_rs \) is a sum \( \sum_v {^pE}_r_s v_s \), with all terms of the sum in \( \mathcal{D}_0^0 \), we have

\[
U_q \cdots U_{p+1} {^pE}_r U_p \cdots U_{p+3n-1} = {^pE}_r \in \mathcal{D}_0^0.
\]

But if \( {^pE}_r \in \mathcal{N}^j \) (\( j \geq 1 \)), then

\[
U_p \cdots U_{p+n-j+1} {^pE}_r U_p \cdots U_{p+n-j+1} = {^pE}_r B_{p+n-j+1}.
\]

Also, if \( {^pE}_r \in \mathcal{D}_0^0 \),

\[
U_p \cdots U_{p+3n-1} {^pE}_r U_p \cdots U_{p+3n-1} = {^pE}_r B_{p+3n-1}.
\]

Hence our construction satisfies the conditions of [7, §4.1], with \( \mathcal{D}_0^0 \) taking the place of \( K_0 \). Also, \( d \leq 3n-1 \) is surely sufficient. Thus we can apply [7, Lemma 4.3] in order to conclude that any unitary \( V \) leaving \( R \) invariant is the metric limit of a sequence \( V_m \) in \( \mathcal{M} \) such that if \( V_m \in \mathcal{M}_p \) (\( p \in \Gamma \)), then \( V_m = \sum \alpha_c {^pE}_c \) with \( {^pE}_c \in \mathcal{D}_0^0 \). So if \( V \in N(R) \), then \( V \in R(\mathcal{D}_0^0) \), and we have \( N(R) \subset R(\mathcal{D}_0^0) \).

On the other hand, consider a unitary \( V \in \mathcal{M}_p \) (\( p \in \Gamma \)) such that \( V = \sum \pm {^pE}_r \) with \( {^pE}_r \in \mathcal{D}_0^0 \) and signs arbitrary. It is straightforward to show that \( V \) leaves \( R \) invariant. Since the collection of all unitaries of this type is sufficient to generate \( R(\mathcal{D}_0^0) \), we have \( R(\mathcal{D}_0^0) \subset N(R) \).

Therefore \( N(R) = P = R(\mathcal{D}_0^0) \).

**Remark.** The preceding proof also implies that \( r(\mathcal{D}_0) \) is in \( \mathfrak{D} \) and that \( R(\mathcal{D}_0^0) = r(\mathcal{D}_0^0)^{\infty} \). For if \( F = \sum \alpha_r {^pE}_r \) with \( {^pE}_r \in \mathcal{D}_0^0 \), then \( F^{(p-h)} = F^{(p-h)} \) for any \( h > 0 \). Hence \( \lim_{p \to \infty} F^{(p)} = F^{(\infty)} \) exists and \( F \in \mathfrak{D} \). Using this information about \( F \in r(\mathcal{D}_0^0) \cap \mathcal{M} \), Lemma 2.2 and its proof can be rephrased to show that \( r(\mathcal{D}_0^0) \subset \mathfrak{D} \).
and that $R(\mathcal{C}^D_p)$, which is defined as the closure of $[\mathcal{C}^D_p \cap \mathfrak{M}]^{(n)}$, can also be regarded simply as $r(\mathcal{C}^D_p)^{(n)}$.

**Lemma 3.6.** Let $p \in \Gamma$, $A^{[q]} = \sum \alpha_{cd} p E_{cd}$ with $p E_{cd} \in \mathcal{N}_p^D (0 \leq j \leq n)$, $\mathcal{C}_n^{-1}$, or $\mathcal{N}_n^-$. Then if $q \in \Gamma$, $q > p$, $A^{[q]} = \sum \beta_{rs} p E_{rs}$ with $p E_{rs} \in \mathcal{N}_p^D$, $\mathcal{C}_n^{-1}$, or $\mathcal{N}_n^-$ respectively.

**Proof.** The case $\mathcal{N}_0^D$ has already been dealt with, since $\mathcal{N}_0^D = \mathcal{C}^D_0$.

We first consider $q = p + 3n$. Then $A^{[q]} = U_p + 3n \cdots A^{[p]} \cdots U_p + 3n$, and because of linearity it is sufficient to consider one term of $A^{[p]}$, say $p E_{cd}$.

If $1 \leq j \leq n$ and $p E_{cd} \in \mathcal{N}_p^D$, then Definition 3.4 shows that

$$U_p + 3n \cdots p E_{cd} \cdots U_p + 3n = \sum \delta_{rs} p + 3n E_{rs}$$

is in $\mathcal{W}_{n+3}$. Consider one term $p + 3n E_{rs}$. With $r = c \cdot 2^{3n} + r_1 2^{2n} + r_0$ and $s = d \cdot 2^{3n} + s_1 2^{2n} + s_0$, we thus have $r_0 = s_0$ and $r_1 = s_1$ (mod $2^{2n-1}$) (and therefore $\psi(s_1) + \psi(s_1) = (a_1, \ldots, a_{n-1}, 0, 0, \ldots)$, while $\psi(s_0) + \psi(s_0) = (0, 0, 0, \ldots)$). Hence, applying Lemma 3.3, $p + 3n E_{rs} \in \mathcal{N}_n^-$ as was $p E_{cd}$.

Now the action of the unitaries $U_i$ surely preserves $\mathcal{C}_n^-$, and therefore $p + 3n E_{rs}$ is in $\mathcal{N}_n^-$.\]

Next suppose $p E_{cd} \in \mathcal{C}_n^-$ and consider $U_p + 3n \cdots p E_{cd} \cdots U_p + 3n$. The product is in $\mathcal{W}_{n+3}$, by Definition 3.4, so one term $p + 3n E_{rs}$ has $r = c \cdot 2^{3n} + r_1 2^{2n} + r_0$, $s = d \cdot 2^{3n} + s_1 2^{2n} + s_0$, with $r_0 = s_0$ (mod $2$). Thus $\psi(r_0) + \psi(s_0) = (\ldots, a_{n-1}, 0)$, and we can have $p + 3n E_{rs} \in \mathcal{N}_n$ only if $\psi(r_0) + \psi(s_0) = (\ldots, a_{n-1}, 1)$. By definition, $\sigma(r_1) \equiv 0$ (mod $2$), so this cannot happen. As before, the action of the $U_i$’s preserves $\mathcal{C}_n^-$. Therefore $p + 3n E_{rs}$ is in $\mathcal{C}_n^-$.

If $p E_{cd} \in \mathcal{C}_n^-$, then this computation of the preceding paragraph lead to the conclusion that the terms of $U_p + 3n \cdots p E_{cd} \cdots U_p + 3n$ are in $\mathcal{N}_n^-$. (Here $\psi(r_0) + \psi(s_0) = (\ldots, a_{n-1}, 0)$.)

If $q \in \Gamma$, $q > p$, then $q = p + 3hn$ for some integer $h$, and the desired result follows by induction.

**Lemma 3.7.** For $j = 1, 2, \ldots, n$, $R(\mathcal{C}_j^n)^an) \subset R(\mathcal{C}_j^D) \subset R(\mathcal{C}_j^-)$.\]

**Proof.** The inclusions are trivial and we need only show that they are proper inclusions.

Let $F$ be a matrix unit in $\mathcal{N}_p$ (resp. $\mathcal{C}_n^-$), so that $F \in \mathfrak{W}_p$ ($F \in \Gamma$) and $F^{(p)} = p E_{ab}$ in $\mathcal{N}_p^D$ (resp. $\mathcal{C}_n^-$). Suppose that $F$ is also in $R(\mathcal{C}_j^n)^an)$. Then there is a sequence $F_m \in \mathcal{W}_q$ converging strongly to $F$, such that if $F_m \in \mathfrak{W}_q (q \in \Gamma)$, $F_m^{[q]} = \sum \beta_{cd} E_{cd}$ with $p E_{cd} \in \mathcal{C}_n^-$ (resp. $\mathcal{C}_n^-$). Choose $F_m$ such that $\|F_m - F\| < 1/2p$ and choose $q \in \Gamma$ such that $F_m, F \in \mathfrak{W}_q$. Then by Lemma 3.6, $F^{[q]} = \sum \alpha_{ab} E_{cd}$ with $p E_{cd} \in \mathcal{N}_p^D$ (resp. $\mathcal{C}_n^-$).

**Case 1.** $F \in \mathcal{N}_p^D$. Since $\sigma^{[q]} F^{[q]} = (F_m, F) = 0$, we have $1/2p > \|F_m - F\|^2 \geq \|F_m\|^2$ $+ \|F\|^2 > 1/2p$, a contradiction. Therefore $F \notin R(\mathcal{C}_j^n)^an)$.\]

**Case 2.** $F \in \mathcal{C}_n^-$. Here

$$(F_m^{[q]}, F^{[q]}) = (1 E_{ii} F_m^{[q]} 1 E_{ii} + 1 E_{jj} F_m^{[q]} 1 E_{jj}, 1 E_{ii} F^{[q]} 1 E_{jj}) = 0$$

where $i, j = 0$ or $i = j$. So again $1/2p > \|F_m - F\|^2 > 1/2p$, a contradiction, and therefore $F \notin R(\mathcal{C}_j^-)$.
DEFINITION 3.8. We define the following projections in $\mathcal{C}_n^P$: For $k = 2, \ldots, n$ and $s = 0, 1, \ldots, 2^k - 1$, let $P_k(s)$ be the operator such that $P_k(s) = \sum_{h=0}^{2^k-1} P(s + 3h) E_{s + 3h}$, where $s = 2^k s'$, $h = 0 \pmod{2^{k-1} + 1}$ and $0 \leq h \leq 2^k - 1$. Let $P'(s)$ be the operator such that $P'(s) = \sum_{h=0}^{2^k-1} P(s + 3h) E_{s + 3h}$, where $s = 2^k s'$, $h = 0 \pmod{2^k}$ and $0 \leq h \leq 2^k - 1$.

LEMMA 3.9. Suppose $W \in \mathfrak{M}_p (p \in \Gamma)$ is such that $W^{(p)} = \sum_{r=0}^{p-1} E_{rr} (\mathcal{C}_n^P)$ and $X^{(p)} = \sum_{a} E_{aa} (\mathcal{C}_n^P \in \mathcal{C}_n^P)$. Let $pE_{rt}$ be a fixed matrix unit in $\mathcal{C}_n^P$ with $K(r, t) = K_r$. Then

$$pE_{rt} [U_{p+3n} \cdots W^{(p)} \cdots U_{p+3n}] \sum_{s=0}^{2^p-1} P(s)^{(p+3n)} [U_{p+3n} \cdots W^{(p)} \cdots U_{p+3n}] E_{st}$$

(\*)

$$= A(r, t)^{(p+3n)} + Q(r, t)^{(p+3n)},$$

where $(A, Q) = 0$ and

$$Q^{(p+3n)} = \sum_{s=0}^{2^p-1} \alpha_{rs} \beta_{ts} C(\gamma)^{(p+3n) E_{ab}}$$

with $pE_{ab}$ in $\mathcal{N}^P_{n-1}$ or $\mathcal{N}^P_n$, $C(\gamma)$ a nonzero integer.

Proof. The following statements are verified by calculations similar to those of \[7, pp. 295-301\].

Suppose $K(\gamma) = K_r$ and $K(pE_{at}) = K_a$, with both matrix units in $\mathcal{C}_n^P$, $\alpha + \beta = \gamma$. If $\alpha = (a_1, a_2, \ldots, a_n)$ and $\beta = (b_1, b_2, \ldots, b_n)$, define $\omega_1 = \omega(\alpha) = 2(\sum_{a=2}^{n} a_1) + a_1 + 1$, $\omega_2 = \omega(\beta)$, and $\mu(\alpha, \beta) = 2(\sum_{a=2}^{n} a_1) + a_1 + 1$. Then the nonzero entries of the product $U_{p+3n} \cdots pE_{rs} \cdots U_{p+3n}$ have numerical value $\pm (2^{-1/2})^{\omega_1}$, and similarly for $pE_{at}$. Let $r_0 = 2^3 - 2r$, $s_0 = 2^3 - 2s$, $t_0 = 2^3 - 2t$. Then $2^3$ is the number of distinct $s$'s such that

$$pE_{rs} [U_{p+3n} \cdots pE_{rs} \cdots U_{p+3n}] E_{s0} = 0$$

and

$$pE_{rs} [U_{p+3n} \cdots pE_{rs} \cdots U_{p+3n}] E_{t0} = 0$$

are both nonzero.

Using the preceding, a matrix calculation shows that

$$[U_{p+3n} \cdots pE_{rs} \cdots U_{p+3n}] P(s)^{(p+3n)} [U_{p+3n} \cdots pE_{rs} \cdots U_{p+3n}]$$

has a term of the form $C(\gamma)^{(p+3n)} E_{r0} + 2$ and a term of the form $C(\gamma)^{(p+3n)} E_{t0} + 2$, where $r_0 = 2^3 r$, $t_0 = 2^3 t$, and $C(\gamma) = 2^3 (2^{-1/2})^{\omega_1 + \omega_2}$. It is straightforward to show that $C(\gamma)$ depends only on $\gamma$ and on the fact that $pE_{rs}$ and $pE_{at}$ are in $\mathcal{C}_n^P$. By Lemma 3.3, if $K_r \subsetneq \mathcal{N}^P_{n-2}$, then $K(r, t) + 2$ is in $\mathcal{N}^P_{n-2}$. If $K_r \subsetneq \mathcal{N}^P_{n-1}$ or $\mathcal{N}_n$, then so is $K(r + 2, t + 2)$. Also, since $pE_{rt} \in \mathcal{N}^P_n$, so are these matrix units.

Now the product (\*) of the lemma equals

$$\sum_{s=0}^{2^p-1} [U_{p+3n} \cdots pE_{rs} \cdots U_{p+3n}] P(s)^{(p+3n)} [U_{p+3n} \cdots pE_{rs} \cdots U_{p+3n}]$$
Suppose $K_y \subset \mathcal{N}_n$ and $s$ such that $pE_{rs}$ and $pE_{st}$ are both in $\mathcal{G}$. The summand corresponding to this $s$ includes the term $\alpha_r \alpha_t C(\gamma) p + 3n E_{s+r,t+2}$, which is in $\mathcal{N}_{n-1}$. Considering the summands corresponding to other $s$, we could not have one matrix unit in $\mathcal{G}$, the other in $\mathcal{G}$, since $pE_{rs} \in \mathcal{G}$. But if both are in $\mathcal{G}$, then the product is in $\mathcal{M}_{p+2n}$ so there is no element in position $(r'', t'')$.

Suppose $K_y \subset \mathcal{N}_n \setminus \mathcal{N}_{n-1}$. If $s$ is such that $pE_{rs}$ and $pE_{st}$ are both in $\mathcal{G}$, then the summand includes the term $\alpha_r \alpha_t C(\gamma) p + 3n E_{s+r,t+2}$, which is in the same class as $K_y$. Again, if $s$ is such that both matrix units are in $\mathcal{G}$ there is no element in position $(r''+2, t''+2)$.

So if we let $Q$ be as stated in the lemma, with $(a, b) = (r'', t''+2)$ or $(r''+2, t''+2)$ according to $K_y$, then $(A, Q) = 0$ and $p+3n E_{ab} \in \mathcal{N}_{n-1} \setminus \mathcal{N}_{n}$.

**Lemma 3.10.** Suppose $W \in \mathcal{M}_p (p \in \Gamma)$ is such that $W^{[p]} = V^{[p]} + X^{[p]}$, with $V^{[p]} = \sum \beta_r E_{rs} (E_{rs} \in \mathcal{G}_{k-1})$ and $X^{[p]} = \sum \alpha_r p E_{rs} (p E_{rs} \in \mathcal{G}_{k}')$. Let $pE_{rt}$ be a fixed matrix unit in $\mathcal{G}_{k-1}$ with $K(r, t) = K_y$. Then

$$W^{[p]} \cdots U_{p+3n} = \sum_{s=0}^{2p-1} P_k(s) [U_{p+3n} \cdots W^{[p]} \cdots U_{p+3n}] pE_{rt}$$

$$= A(r, t)[p+3n] + Q(r, t)[p+3n],$$

where

$$Q[p+3n] = \sum_{s=0}^{2p-1} \alpha_r \alpha_t D_k(\gamma) p+3n E_{ab}$$

with $p+3n E_{ab} \in \mathcal{N}_{k-1}$, $D_k(\gamma)$ a nonzero integer.

**Proof.** The proof is like that of the preceding lemma, with the following changes: $\omega_1 = \omega(a) = 2(\sum_{i=2}^n a_i) + a_1$ (and a similar change in $\omega_2$), $\mu(\alpha, \beta) = 2(\sum_{i=2}^{n-1} a_i b_i) + a_i b_k + a_1 b_1$, $r_0 = 2^{n+k-1} r$, $s_0 = 2^{n+k-1} s$, $t_0 = 2^{n+k-1} t$. Then $2^\tau$ is the number of distinct $\delta$'s such that

$$p+n+k-1 E_{r_0 t_0} [U_{p+3n} \cdots pE_{rs} \cdots U_{p+3n}] p+n+k-1 E_{r_0 t_0} + \delta, t_0 + \delta$$

and

$$p+n+k-1 E_{s_0 t_0} [U_{p+3n} \cdots pE_{st} \cdots U_{p+3n}] p+n+k-1 E_{s_0 t_0} + \delta, t_0 + \delta$$

are both nonzero. The expression

$$[U_{p+3n} \cdots U_{p+3n}] pE_{rs} \cdots U_{p+3n}] P_k(s) [p+3n] [U_{p+3n} \cdots pE_{st} \cdots U_{p+3n}]$$

has a term of the form $D_k(\gamma) p+3n E_{r''+s, t''+s}$ and a term of the form $D_k(\gamma) p+3n E_{r''+t, t''+t}$ where $r'' = 2^\gamma r$, $t'' = 2^\gamma t$, $\pi = 2^n - k$, and $D_k(\gamma) = 2(2^{-1/2})^{n_1+w_2}$. Here $D_k(\gamma)$ depends only on $\gamma$ and on $k$. By Lemma 3.3, if $K_y \subset \mathcal{N}_{k-2}$, then $K(r'', t''+\pi)$ is in $\mathcal{N}_{k-1}$; if $K_y \subset \mathcal{N}_{k-1}$, then $K(r''+\pi, t''+\pi)$ is in $\mathcal{N}_{k-1}$.

It can be verified, as in the preceding lemma, that $(A, Q) = 0$ if we take $Q$ as stated, with $(a, b) = (r'', t''+\pi)$ or $(r''+\pi, t''+\pi)$ according to $K_y$. 

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Lemma 3.11. If the results of Lemmas 3.9 and 3.10 hold for \( q = p + 3n \) (i.e., \( q \in \Gamma \)). Also,

\[
\|Q\|^2 \geq \left| \sum_{s=0}^{2p-1} \alpha_s \bar{\alpha}_{ts} \right|^2 / 2p + 9n.
\]

**Proof.** We first obtain bounds for \( C(\gamma) \) and \( D_k(\gamma) \). In both cases, we have \( \mu \geq 0 \) and \( \omega_1 + \omega_2 \leq 2(2n-1) + 2 = 4n \). Hence \( C(\gamma) \) or \( D_k(\gamma) = 2^2(2^{-1/2})\omega_1 + \omega_2 \geq (2^{-1/2})4n = 1/2^{2n} \).

\[
\left| \sum_{s=0}^{2p-1} \alpha_s \bar{\alpha}_{ts} \right|^2 / 2p + 9n
\]

and similarly in the case of \( D_k(\gamma) \).

Now the unitaries \( U_{p+3n+1}, \ldots, U_{p+3hn} \) preserve the orthogonality of \( A \) and \( Q \) and the norm of \( Q \). Also, by Lemma 3.6, matrix units in \( N^P_j (j=1,2,\ldots,n) \) are left in that class under the action of the unitaries \( U_t \).

Lemma 3.12. For \( j=1,2,\ldots,n-1 \), let \( \gamma_j = \{ V : V[R(\mathcal{E}^P_{j+1})]V^* = R(\mathcal{E}^P_{j+1}) \}, \) \( V \) unitary, \( V \in \mathcal{A} \). If \( V \in \gamma_j \), then there is a sequence \( V_m \in \mathcal{A} \) converging metrically to \( V \) such that if \( V_m \in \mathcal{A}_p \) (\( p \in \Gamma \)), \( V_{m}^{(p)} = \sum \beta_{rs}^p E_{rs} \) with \( \beta_{rs}^p \in \mathcal{E}_n^P \). Thus, \( N(R(\mathcal{E}^P_{j+1})) \subset R(\mathcal{E}^P_j) \).

**Proof.** (i) Since \( V \in \mathcal{A}, \| V \| \leq 1 \), there is a sequence \( W_m \in \mathcal{A}, \| W_m \| \leq 1 \), converging strongly and metrically to \( V \) [4]. If \( W_m \in \mathcal{A}_p \), let \( W_m^{(p)} = V_{m}^{(p)} + X_m^{(p)} \), where \( V_{m}^{(p)} = \sum \beta_{rs}^p E_{rs} \) and \( X_m^{(p)} = \sum \alpha_s^p E_{rs} \). Because of the orthogonality of \( V_m \) and \( X_m \), \( X_m \) itself is Cauchy in the metric topology. Now \( \| W_m \| \leq 1 \) implies \( \| V_m \| \leq 1 \) because of the definition of \( \mathcal{E}_n^D \). Since \( X_m = W_m - V_m \), we have \( \| X_m \| \leq 2 \), and \( X_m \) is also Cauchy in the strong topology [5, p. 723]. Let \( X_m \to X \in \mathcal{A}. \) Suppose \( \| X_m \| \neq 0 \); then \( \lim_m \| X_m X_m^* \| \neq 0 \) also. Hence \( \| X_m X_m^* \|^2 > 2^{5n} \) for all \( m \) and some \( \epsilon > 0 \). (Recall that \( n \) is fixed and related only to \( R = R_n \).)

Choose \( W_m \) so that \( \| W_m - V \| < \epsilon/4 \). Suppose \( W_m \in \mathcal{A}_p \). Then

\[
\| X_m (p)(X_m^{(p)})^* \|^2 = (1/2p) \left| \sum_{s=0}^{2p-1} \alpha_s \bar{\alpha}_{ts} \right|^2 > 2^{5n}\epsilon^2.
\]

The outer summation is over pairs \( (r, t) \) such that \( pE_{rt} \in \mathcal{E}_n^D \), since \( pE_{rs} \in \mathcal{E}_n^D \). Fix \( p \) from here on.

Consider \( \sum_{s=0}^{2p-1} P'(s)^{(p)+3n} \), which has its matrix units in \( \mathcal{E}_n^D \). Then \( \sum P'(s) \) is in \( R(\mathcal{E}^P_{j+1}) \) for any \( j \geq 1 \), and if \( V \in \gamma_j \), \( V(\sum P'(s))V^* = T \in R(\mathcal{E}^P_{j+1}) \). So there exists a sequence \( T \), \( T \in \mathcal{A}, \| T - T' \| \to 0 \), and \( T' \in \mathcal{A}_q (q \in \Gamma) \) implies \( T_{j+1}^{(q)} = \sum \eta_{ts}^q E_{ts} \) with \( \eta_{ts}^q \) in \( \mathcal{E}_n^P \). Choose \( T \), such that \( \| V(\sum P'(s))V^* - T \| < \epsilon/2 \). Since \( \sum P'(s) \) is a projection, of norm at most one,

\[
\left\| W_m \left( \sum P'(s) \right) W_m^* - V \left( \sum P'(s) \right) V^* \right\| < \epsilon/2,
\]
and thus it follows that
\[
\left[ W_m \left( \sum_s P'(s) \right) W_m^* - T_v \right] < \epsilon.
\]

On the other hand, we can apply Lemmas 3.9 and 3.11 with \( W_m \) replacing \( W \). Take \( q \) to be such that \( q \in \Gamma, q \geq p + 3n, \) and \( T_v \in \mathfrak{M}_q \). Since \( Q^{(q)} = \sum \lambda_{cd} E_{cd} \) \((E_{cd} \in \mathcal{N}_{n-1} \text{ or } \mathcal{N}_n)\) and \( T_{v}^{(q)} = \sum \eta_{ih} E_{ih} \) \((E_{ih} \in \mathcal{C}_{j-1}^P, \) where \( j-1 < n-1 \)), we have \((T_{v}^{(q)}, Q^{(q)}) = 0\). Therefore
\[
\begin{align*}
\left[ P_{e_{rr}} W_m^{(q)} \sum P'(s)^{q} W_m^{*} p E_{tt} - p E_{rr} T_v^{(q)} p E_{tt} \right]^2 \\
= \left[ A(r, t)^{q} + Q(r, t)^{q} - p E_{rr} T_v^{(q)} p E_{tt} \right]^2 \\
\geq \left[ Q(r, t)^{q} \right]^2 \geq \left| \sum_s a_{rs} a_{ts} \right|^2 / 2^n + 5n.
\end{align*}
\]

Finally, we have:
\[
e^2 \geq \sum_{(r,t)} \left| p E_{rr} \left( W_m^{(q)} \sum_s P'(s) W_m^{*} - T_v^{(q)} \right) p E_{tt} \right|^2,
\]
\[
e^2 \geq \sum_{(r,t)} \left| \sum_s a_{rs} \right|^2 / 2^n + 5n > e^2,
\]

which is a contradiction.

Therefore \( \lim_{m} \| V_m \| = 0 \) and so \( \lim_{m} \| V_m - V \| = 0 \), where \( \| V_i \| \leq 1 \) and \( V_i \in \mathfrak{M}_z \) \((z \in \Gamma)\) implies \( V_i^{(z)} = \sum \beta_{rs} E_{rs} \) with \( E_{rs} \in \mathcal{C}_n^{D} \).

(ii) To show: Suppose \( j < k \leq n \) and suppose there exists \( W_m \in \mathfrak{M} \) such that \( \| W_m \| \leq 1, \lim_{m} \| W_m - V \| = 0 \), and \( W_m \in \mathfrak{M}_p \) implies \( W_m^{(p)} = \sum \delta_{rs} E_{rs} \) with \( E_{rs} \in \mathcal{C}_k^{P} \). Then there exists \( V_m \) with the same properties except that \( V_m^{(p)} = \sum \beta_{rs} E_{rs} \) with \( E_{rs} \in \mathcal{C}_{k-1}^{P} \).

We let the assumed \( W_m^{(p)} = V_m^{(p)} + X_m^{(p)} \), where the matrix units of the two summands are in \( \mathcal{C}_{k-1}^{P} \) and \( \mathcal{N}_k \) respectively. The argument proceeds much as in part (i), with \( \sum P_{h}(s) \) replacing \( \sum P'(s) \), so that Lemmas 3.10 and 3.11 apply. Since \( V(\sum P_{h}(s)) V^* = T \) in \( R(\mathcal{C}_{j-1}^{P}) \) and since \( j-1 < k-1 \), the desired orthogonality holds between \( Q \) (in \( \mathcal{N}_{k-1}^{P} \)) and \( T_v \) (the sequence of matrices converging to \( T \)). We are led to conclude that \( \lim_{m} \| X_m \| = 0 \), and that \( V \) is the metric limit of \( V_m \).

Since we can extend this as far as \( k = j + 1 \) by a finite induction process, the lemma is proved.

Theorem 3.13. For \( j = 1, 2, \ldots, n-1 \), if \( R(\mathcal{V}_j) \) is the ring generated by \( \mathcal{V}_j \) as defined in Lemma 3.12, then \( R(\mathcal{V}_j) = R(\mathcal{C}_{j-1}^{P}) \). Thus, \( N(R(\mathcal{C}_{j-1}^{P})) = R(\mathcal{C}_{j-1}^{P}) \).

Proof. By Lemma 3.12, \( R(\mathcal{V}_j) \subseteq R(\mathcal{C}_{j-1}^{P}) \).

For the reverse inclusion, take \( T \in R(\mathcal{C}_{j-1}^{P}) \). Let \( V_T^{(p)} = \sum \pm E_{rs} \) with \( E_{rs} \in \mathcal{C}_{j-1}^{P} \) and signs arbitrary. Then \( V_T^* T V_T^* \) is in \( R(\mathcal{C}_{j-1}^{P}) \) since all three operators are.
Next let $V_2^{[p]} = \sum \pm_i^p E_{rs}$ with $\tau E_{rs}$ in $\mathcal{N}_2$. Take a sequence $T_m \in \mathfrak{M}$, $T_m \to T$, and if $T \in \mathfrak{M}$, $T_m^{(1)} = \sum \beta_{cd} E_{cd}$ with $\sigma E_{cd}$ in $\mathfrak{M}_{p-1}$. If $z = \max \{p, q\}$, then

$$V_2^{[q]} T_m^{(1)} T_2^{[q]} = \left[ \sum \delta_{rs}^z E_{rs} \right] \left[ \sum \beta_{cd}^z E_{cd} \right] \left[ \sum \delta_{rs}^z E_{rs} \right],$$

where the matrix units of the first sum are in $\mathfrak{M}_z$, those of the second in $\mathfrak{M}_{p-1}$, and those of the third in $\mathfrak{M}_2$, by Lemma 3.6. Calculating by means of §3.1, we see that each matrix unit of this product is in $\mathfrak{M}_{p-1}$. Hence $V_2 T_m V_2^*$ is in $R(\mathfrak{M}_{p-1})$, and so its strong limit $V_2 T V_2^*$.

But all unitaries of the form $V_1$ or $V_2$ are sufficient to generate $R(\mathfrak{M}_2)$. Therefore $R(\mathfrak{M}_2) \subset R(\mathfrak{M}_2)$, and hence $R(\mathfrak{M}_2) = R(\mathfrak{M}_2)$.

**Theorem 3.14.** If $\mathfrak{N} = \{V : \mathcal{V}(\mathfrak{M}_{p-1}) \mathcal{V}^* = R(\mathfrak{M}_{p-1})$, $V$ unitary, $V \in \mathfrak{M}\}$, then $R(\mathfrak{N}) = R(\mathfrak{M}) = \mathfrak{U}$. Thus, $N(R(\mathfrak{M}_{p-1})) = \mathfrak{U}$.

**Proof.** Obviously $R(\mathfrak{N}) \subset R(\mathfrak{M})$.

For the reverse inclusion, let $T \in R(\mathfrak{M}_{p-1})$. Consider in turn four types of unitaries $V_i^{[p]} = \sum \pm_i^p E_{rs}$ ($i = 1, 2, 3, 4$ and signs arbitrary). For $i = 1$, the matrix units are to be in $\mathfrak{M}_{p-1}$; for $i = 2$, in $\mathfrak{M}_{p-1}$; for $i = 3$, in $\mathfrak{M}_{p-1}$; for $i = 4$, in $\mathfrak{M}_2$. By Lemma 3.6, these classes are preserved under the unitaries $U_t$. So calculations like those in the proof of Theorem 3.13 show that $V_i V_2$ is in $R(\mathfrak{M}_{p-1})$ for $i = 1, 2, 3, 4$.

But all unitaries of these types are sufficient to generate $R(\mathfrak{M}_{p-1})$, or $\mathfrak{U}$. Therefore $R(\mathfrak{M}_{p-1}) \subset R(\mathfrak{N})$, and $R(\mathfrak{N}) = \mathfrak{U}$.

**Remark.** Theorems 3.13 and 3.14, together with Theorem 3.5 and Lemma 3.7, show that for each $\mathfrak{R}_n$, $n = 1, 2, 3, \ldots$, we have $R_n \subset N(R_n) \subset \cdots \subset N^{n+1}(\mathfrak{R}_n) = \mathfrak{U}$. In order to prove that $\mathfrak{R}_n$ is $M$-semiregular $(n+1 = M)$, we need only show that $N(R_n)$, $N^2(R_n)$, \ldots, $N^n(R_n)$ are not factors. $(N^{n+1}(\mathfrak{R}_n) = N^M(\mathfrak{R}_n)$ is the factor $\mathfrak{U}$).

**Theorem 3.15.** For $k = 1, 2, \ldots, n$, $N^k(\mathfrak{R}_n)$ is not a factor.

**Proof.** If $k \neq n$, $N^k(\mathfrak{R}_n) = N^{k-1}(\mathfrak{P}_n) = R(\mathfrak{P}_{n-1}) = R(\mathfrak{M}_{p-1})$. Consider the projection $1 \mathcal{E}_{00} = 1 \mathcal{E}_{00}^{(p)} \in \mathfrak{R}_n \subset N^k(\mathfrak{R}_n)$, if $A$ is any operator in $N^k(\mathfrak{R}_n)$, there is a sequence $A_m \to A$ such that if $A_m \in \mathfrak{M}_{p-1}$, $A_m^{[p]} = \sum \alpha_{rs} E_{rs}$ with $\tau E_{rs}$ in $\mathfrak{M}_{p-1}$. Then

$$(1 \mathcal{E}_{00} A_{m} 1 \mathcal{E}_{00})^{[p]} = 1 \mathcal{E}_{00} A_{m}^{[p]} 1 \mathcal{E}_{00} = \sum \alpha_{rs} E_{rs} E_{rs}^{\tau} E_{rs}^{\tau} 1 \mathcal{E}_{00} = \sum \alpha_{rs} E_{rs}^{\tau} E_{rs}^{\tau} 1 \mathcal{E}_{00} = \mathcal{A}_{m}^{[p]}.$$ 

Thus $1 \mathcal{E}_{00} A_{m} 1 \mathcal{E}_{00} = A_m$, and taking strong limits, $1 \mathcal{E}_{00} A 1 \mathcal{E}_{00} = A$.

Therefore $1 \mathcal{E}_{00}$ commutes with $N^k(\mathfrak{R}_n)$, $1 \mathcal{E}_{00} \not\in \alpha I$, $1 \mathcal{E}_{00} \in N^k(\mathfrak{R}_n)$, and so $N^k(\mathfrak{R}_n)$ is not a factor.

**Bibliography**


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