1. Introduction. Suppose \( A_n = \{a_1, a_2, \ldots, a_{2n}\} \) is a set of \( 2n \) points lying in the open interval \((0, 1)\) such that \( a_i < a_{i+1}, i=1, \ldots, 2n-1 \) and that \( W \) is a decomposition of \( A_n \) into two element sets. Suppose also that \( f \) is a mapping of the half open interval \([0, 1)\) into the plane such that (1) \( f(t) = f(t') \) for \( t < t' \) if and only if \( \{t, t'\} \in W \), (2) \( \text{Im} f \) can be expressed as the sum of a finite number of straight line intervals such that no point of \( f(A_n) \) is an endpoint of one of the intervals and, (3) \( f(t) \to f(0) \) as \( t \to 1 \). The decomposition \( W \) is said to determine the double point structure of \( f \) and \( W \) is said to have property \( P \) provided it is true that if \( U \) and \( V \) are subsets of \( W \) such that \( U = W - V \), then there exist \( \{u_1, u_2\} \in U \) and \( \{v_1, v_2\} \in V \) such that \( u_1 < v_1 < u_2 < v_2 \) or \( v_1 < u_1 < v_2 < u_2 \). If \( W \) has property \( P \) and the double point structure of \( f \) is determined by \( W \) then \( f \) is said to have property \( P \) or be prime. It is now possible to state two of the main results.

**Theorem 2 (The Invariance of Boundary Theorem).** If \( A_n \) (as above) is a set of \( 2n \) points lying in \((0, 1)\), \( W \) is a decomposition of \( A_n \) into two element sets and \( f \) and \( g \) are prime mappings whose double point structure is determined by \( W \), then there is a natural one-to-one correspondence between the complementary domains of \( \text{Im} f \) and those of \( \text{Im} g \) according to the equation \( f^{-1}(\text{Bd} U) = g^{-1}(\text{Bd} V) \), where \( U \) and \( V \) are corresponding complementary domains of \( \text{Im} f \) and \( \text{Im} g \), respectively.

**Theorem 3.** Given \( A_n, W, f \) and \( g \) as in Theorem 2, and assuming that the unbounded complementary domains of \( \text{Im} f \) and \( \text{Im} g \) correspond, then there is a homeomorphism \( h \) from \( E^2 \) onto \( E^2 \) such that \( hf = g \).

The main use of Theorem 3 (to the author), and, certainly the context in which it arose, are now described. Given \( f \) as in paragraph one, the set \( \text{Im} f \) can be considered [1] as the projection of a polygonal knot in regular position, where the set \( f(A_n) \) is the set of double points of the projection. Suppose \( g \) is a one-to-one mapping of \([0, 1)\) into \( E^3 \) so that (1) \( \pi g = f \), where \( \pi(x, y, z) = (x, y, 0) \), (2) \( \text{Im} g \) is the sum of a finite number of straight line intervals, and (3) \( g(t) \to g(0) \) as \( t \to 1 \). D. E. Penney [6] has been studying the idea of associating with \( g \) (or \( \text{Im} g \)) a "word" \( f(a_1)^{e_1}f(a_2)^{e_2}\cdots f(a_{2n})^{e_{2n}} \), where if \( f(a_i) = f(a_j) \) and the \( z \) coordinate of \( g(a_i) \) is larger than the \( z \) coordinate of \( g(a_j) \), then \( e_i = 1 \) (or is suppressed) and \( e_j = -1 \). The technique is illustrated in Figure 1.

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In [6] Penney describes a set of "admissible" operations with "words", one example of which is the cancellation of the $aa^{-1}$ in the "word" associated with Figure 1. His Theorem 1 (applied here) says that there is some knot isomorphic to the one of Figure 1 and whose word is $b^{-1}cd^{-1}be^{-1}dc^{-1}e$. A prime word is one where $\pi g$ is a prime mapping. Penney's Theorem 3 says that if $F$ and $G$ are knots with words $W_1$ and $W_2$, respectively, $W_2$ is prime and can be obtained by a finite number of "admissible" operations on $W_1$, then $F$ and $G$ are isomorphic. Theorem 3 of this paper is one of the preliminaries to the Theorem 3 of Penney's paper.

For other references in the field of topology see Gauss [2], Nagy [5] and Treybig [9]. For references in the field of topological analysis see Marx [3] and Titus [7] and [8].

2. Definitions. In addition to the definitions stated in the introduction it is desirable to state a few others. Given $A_n$ a subset of $(0, 1)$ as above, let $N(A_n)$ denote the set of all mappings $f$ of $[0, 1)$ into the plane such that there is a decomposition $W$ of $A_n$ into two element sets and such that $f$ and $W$ are related as above. Let $G(A_n)$ denote $\{[a_1, a_2], \ldots, [a_{2n-1}, a_{2n}], [0, a_1]+[a_{2n}, 1]\}$. The notation $[0, a_1]+[a_{2n}, 0)$ will be shortened to $[a_{2n}, a_1]$. As in the case of other intervals, the points $a_{2n}$ and $a_1$ will be called the endpoints of this set. Given a collection $H$ of point sets let $H^*$ denote the sum (or union) of the sets in $H$. Given a decomposition $W$ of $A_n$ as above, then $W$ will be said to have property $Q$ provided it is true that if $\{a_i, a_j\} \in W$ then there exists $\{a_i, a_j\} \in W$ such that $i < r < j$ or $r < i < s < j$. If $W$ determines the double point structure of $f \in N(A_n)$ and $W$ has property $Q$, then $f$ will be said to have property $Q$. It is easy to see that property $P$ implies property $Q$ for $n > 1$.

3. A lemma.

**Lemma 1.** If $f \in N(A_n)$ and each of $AB$ and $CD$ is an arc lying in $\text{Im} f$ such that $\{A, B, C, D\} \subset f(A_n)$, then (1) there exists $H, K$ such that $H \subset G(A_n), K \subset G(A_n), f(H^*) = AB$ and $f(K^*) = CD$, and (2) if $AB$ and $CD$ intersect, then the first (last) point of $CD$ on $AB$ in the order from $A$ to $B$ is in $f(A_n)$.

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Proof. (1) Let \( H = \{ G \in G(A_n) : f(G \cdot G \cdot A_n) \) intersects \( AB \} \). If \( f(G \cdot G \cdot A_n) \) intersects \( AB \) then it must be a subset of arc \( AB \) or it would contain one of the endpoints \( A \) or \( B \). Therefore, since \( AB \) is closed, \( H \) contains a finite number of closed sets and \( f(H^*) \) is dense in \( AB \), it follows that \( f(H^*) = AB \). Define \( K \) for \( CD \) analogously.

(2) Suppose the first point \( X \) of \( CD \) on \( AB \) in the order from \( A \) to \( B \) is not in \( f(A_n) \). There exists \( G \in G(A_n) \) so that \( X \in f(G \cdot G \cdot A_n) \) so \( G \in H \cdot K \). Thus there is an open interval containing \( X \) which is a subset of both \( AB \) and \( CD \). This yields a contradiction.

4. The theorems.

**Theorem 1.** If \( A_n \) (as above) is a set of \( 2n \) points lying in \((0,1)\), \( W \) is a decomposition of \( A_n \) into two element sets and \( f \) and \( g \) are prime mappings whose double point structure is determined by \( W \), then for each complementary domain \( U \) of \( \text{Im} f \) there is a unique complementary domain \( V \) of \( \text{Im} g \) such that \( f^{-1}(\text{Bd} U) = g^{-1}(\text{Bd} V) \).

Proof. By Theorem 7 of [9] the collection \( H_1 \) of all elements \( K \) of \( G(A_n) \) such that \( f(K \cdot K \cdot A_n) \) intersects \( \text{Bd} U \) has the property that \( f(H_1^*) = \text{Bd} U \), and is the only subcollection of \( G(A_n) \) with this property. Suppose \( W = \{ a_r, a_s \} \in W \) \((r<s)\) and that each of \( I_1 \) and \( I_2 \) is an element of \( H_1 \) such that \( a_r \) is an endpoint of \( I_1 \) and \( a_s \) is an endpoint of \( I_2 \). (see Theorem 9 of [9]). By Theorem 5 of [9] there is a complementary domain \( V \) of \( \text{Im} g \) such that \( g(I_1) + g(I_2) \subset \text{Bd} V \). Let \( H_2 \) denote the unique subcollection of \( G(A_n) \) such that \( g(H_2^*) = \text{Bd} V \). The idea is to show that \( H_2 = H_1 \), so suppose that \( H_1 \neq H_2 \).

If \( n = 1 \) then it follows that \( H_1 = H_2 \), so the previous assumption means that \( n > 1 \), and that \( W \) has property \( Q \). By Theorem 9 of [9], (1) \( \text{Bd} U \) (\( \text{Bd} V \)) is a simple closed curve, and (2) if \( a_r \in A_n \cdot K \in K \in H_1 \) \((H_2)\) there is exactly one other element \( L \) of \( H_1 \) \((H_2)\) containing an element \( a_s \) of \( A_n \) such that \( f(a_r) = f(a_s) \), and furthermore \( K \) and \( L \) do not intersect. With the aid of Lemma 1 \( H_1 \) \((H_2)\) can be expressed as the sum of two subcollections \( \{ I_2 \} \) and \( \{ J_1, J_2, \ldots, J_{m_2} \} \) \((\{ I_2 \} \) and \( \{ K_1, \ldots, K_{m_2} \})\) such that (1) \( f(I_2) \) and \( f(\sum J_p) \) \((g(I_2) \) and \( g(\sum K_p))\) are two arcs which meet only in their endpoints, and (2) \( f(J_p) \) intersects \( f(I_2) \) \((g(K_p) \) intersects \( g(I_2)\)\) if and only if \( |p-q| \leq 1, 1 \leq p, q \leq m_1 \) \((m_2)\), (3) \( K_1 = J_1 = I_1 \). There is an integer \( n_1 \) such that \( J_p = K_p \) for \( 1 \leq p \leq n_1 \), but \( J_{n_1+1} \neq K_{n_1+1} \). Furthermore, there is an integer \( n_2 > n_1 \) such that \( f(J_{n_2}) \) intersects \( f(\sum K_p) \), but if \( n_1 < q < n_2 \) then \( f(J_q) \) does not intersect \( f(\sum K_p) \).

As above \( H_2 \) \((H_1)\) is the sum of two subcollections \( A_1 \) \((A_2) \) and \( B_1 \) \((B_2) \) such that (1) \( \text{Bd} U \) \((\text{Bd} V)\) is the sum of two arcs \( f(A_1^*) \) \((f(A_2^*) \) and \( f(A_2^*) \) \((g(B_1^*) \) and \( g(B_2^*) \)) having only their endpoints in common, (2) \( A_1 = \{ J_{n_1+1}, \ldots, J_{n_2} \} \), and (3) \( f(B_1^*) \) \((g(B_2^*))\) has the same end points as \( f(A_1^*) \) \((g(A_2^*))\) but they have no other point in common.

If \( f(B_2^*) \) intersects \( f(A_1^*) \) in a point distinct from one of its endpoints, then \( f(B_2^*) \) contains an arc \( D_2 \) and \( f(B_1^*) \) contains an arc \( D_1 \) such that (1) \( D_1 \) and \( D_2 \) do not intersect, (2) \( D_1 \) and \( D_2 \) lie except for their endpoints in the complement
of \( U \) and the endpoints of \( D_1 \) separate those of \( D_2 \) on \( Bd \ U \). By Theorem 11, p. 147 of [4], \( D_1 \) and \( D_2 \) must intersect, which is a contradiction. Therefore, the situation is that (1) \( f(A^*_1) \cdot f(A^*_2) = P + Q = f(B^*_1) \cdot f(B^*_2) \), (2) \( f(H^*_1) \cdot f(H^*_2) = P + Q \), where \( P = f(a_i), \ Q = f(a_p), \ \{a_i, a_j\} \in W, \ \{a_p, a_q\} \in W, \) and \( a_i \) is an endpoint of \( J_{n_1} = K_{n_1} \). It may be supposed without loss of generality that \( i < j, p, q \) and that \( a_q \) is an endpoint of \( J_{n_2} = K_{n_2} \). Let \( K \) denote the collection of all elements of \( G(A_n) \) which have none of \( a_i, a_j, a_p, a_q \) for an endpoint. The idea now is to try to obtain a subset \( L \) of \( K \) such that \( f(L^*) \) is connected and intersects both of \( f(A^*_1), f(A^*_2) \) and \( f(B^*_1), f(B^*_2) \). Suppose that \( J_{n_1} = K_{n_1} = [a_i, a_j] \), and note that one of \( [a_i, a_j] \) and \( [a_p, a_{j + 1}] \) is an element of \( A_2 \) and the other is an element of \( B_2 \).

Case 1. \( i < p, q < j \). Let \( L = \{[a_1, a_2], \ldots, [a_{i - 1}, a_i], [a_{j + 1}, a_{j + 2}], \ldots, [a_{p + 1}, a_p], [a_{j + 1}, a_{j + 2}], \ldots, [a_{2n}, a_1] \} \).

Case 2. \( i < q, j < p \) and \( [a_p, a_{p + 1}] \) is an element of \( A_2 \) or of \( B_2 \). Let

\[ L = [a_1, a_2], \ldots, [a_{i - 2}, a_{i - 1}], [a_{i - 1}, a_i] \]

Case 4. \( i < j, p < q \) and \( [a_q, a_{q + 1}] \) is not an element of \( A_2 + B_2 \).

(a) \( j < p \). In this case each of the sets

\[ L_1 = \{[a_{i + 1}, a_{i + 2}], \ldots, [a_{j - 2}, a_{j - 1}]\}, \]

\[ L_2 = \{[a_{i + 1}, a_{i + 2}], \ldots, [a_{p - 2}, a_{p - 1}]\}, \]

has the property that \( f(L^*_i) \) is connected (\( i = 1, 2, 3 \)) and intersects \( f(A^*_1) \) or \( f(B^*_1) \). Since \( W \) has property \( Q \) there is an element \( \{a_v, a_w\} \) of \( W \) such that \( i < v < q \) and \( w < i \) or \( w > q \). Let \( L = L_1 + \{[a_{q + 1}, a_{q + 2}], \ldots, [a_{2n}, a_1], [a_1, a_2], \ldots, [a_{i - 2}, a_{i - 1}]\} \) where \( a_v \) is an endpoint of an element of \( L \).

(b) \( p < j \). Let

\[ L_1 = \{[a_{i + 1}, a_{i + 2}], \ldots, [a_{p - 2}, a_{p - 1}]\}, \]

\[ L_2 = \{[a_{p + 1}, a_{p + 2}], \ldots, [a_{j - 2}, a_{j - 1}]\} \]

If each of \( f(U^*_1), f(U^*_2) \) and \( f(U^*_3) \) intersects one of \( f(A^*_1) \) and \( f(B^*_1) \), then proceed as in 4(a). If not, then \( f(U^*_1) \) must be the one that fails to meet \( f(A^*_1) \) or \( f(B^*_1) \). Since \( f \) is prime there exists \( \{a_u, a_v\} \in W \) such that \( i < v < q \) and \( u < i \) or \( q < w \). If \( p < v < j \) or \( j < v < q \) proceed as in 4(a). Now suppose \( i < v < p \). By condition 1 of Theorem 19 of [9] there is an element \( \{a_v, a_u\} \) of \( W \) such that \( u \neq j, p < u < q \), and \( t < q \) or \( t < i \) proceed as in 4(a). If \( i < t < p \) let

\[ L = L_1 + L_m + \{[a_{q + 1}, a_{q + 2}], \ldots, [a_{2n}, a_1], [a_1, a_2], \ldots, [a_{i - 2}, a_{i - 1}]\} \]
where \( a_n \) is an endpoint of an element of \( L_m \). This concludes cases 1–4 and shows that in any event the desired collection \( L \) is obtained.

Suppose \( f(L^*) \) intersects \( f(A_n^*) \) and \( f(A_n^*) \). There is an arc \( AB \) from point \( A \) in \( f(A_n^*).f(A_n) \) to a point \( B \) in \( f(A_n^*).f(A_n) \). There is a subarc \( CD \) of \( AB \) such that (1) \( C \) and \( D \) are in \( f(A_n) \), and (2) \( C \) belongs to one of \( f(A_n^*) \) and \( f(A_n^*) \) and \( D \) belongs to the other and arc \( CD \) misses \( f(B_n^*) \) (or use \( f(A_n^*) \) and \( f(A_n^*) \) and require that \( CD \) miss \( f(A_n^*) \), or use \( f(A_n^*) \) and \( f(B_n^*) \) and require that \( CD \) miss \( f(B_n^*) \), or use \( f(A_n^*) \) and \( f(B_n^*) \) and require that \( CD \) miss \( f(A_n^*) \) and \( 3 \) \( CD \) contains no proper subarc with the same property. Let \( M = \{ L' \in L : f(L') \subset CD \} \). (It follows by Lemma 1 that \( CD \) is the sum of such sets.)

Suppose for example that \( C \in f(A_n^*) \) and \( D \in f(A_n^*) \). The arc \( CD \) and the arc \( f(B_n^*) \) have endpoints that separate each other on \( Bd U \). Therefore \( CD \) and \( f(B_n^*) \) intersect, a contradiction.

If \( C \in f(A_n^*) \) and \( D \in f(A_n^*) \) let \( R \) denote the first point of \( f(B_n^*) \) in the order \( QCP \) on \( f(A_n^*) \). Let \( N \) denote \( \{ L' \in L : f(L') \subset \text{arc CR} \} \). In this case \( g(M^*) + g(N^*) \) and \( g(A_n^*) \) are arcs whose endpoints separate each other on \( Bd V \). This involves a contradiction.

These two cases suffice to show how to handle the other two, so this concludes the proof of Theorem 1.

Proof of Theorem 2. For each complementary domain \( U \) of \( Im f \), let \( U' \) denote the unique complementary domain of \( Im g \) guaranteed by Theorem 1. But starting with \( Im g \) and letting a \( U' \) correspond to \( U'' \) one must obtain the relation

\[ U \to U' \to U'' = U. \]

Proof of Theorem 3. Suppose the complementary domain \( U \) of \( Im f \) corresponds to the complementary domain \( U' \) of \( Im g \) under the correspondence guaranteed by Theorem 2. Let \( B(U) \) denote the subcollection of \( G(A_n) \) such that \( f(B(U)^*) = Bd U. \) Of course, \( g(B(U)^*) = Bd U' \). Also for \( B_1, B_2 \in B(U) \) \( f(B_1) \) intersects \( f(B_2) \) if and only if \( g(B_1) \) intersects \( g(B_2) \). Remember also that for \( \{ a_i, a_j \} \in W, f(a_i) = f(a_j) \) and \( g(a_i) = g(a_j) \).

By the Schoenflies Theorem there exist homeomorphisms \( h_1 \) and \( h_2 \) mapping \( U \) and \( U' \), respectively, onto \( T = \{ z : |z| \leq 1 \} \) (for the case of the unbounded components use \( T = \{ 0 \} \)). Define \( h \) mapping \( U \) onto \( U' \) by \( h(h_1^{-1}(0)) = h_2^{-1}(0) \) and otherwise, for \( P \in W - \{ 0 \} \) let \( t \) be a number in \( [0, 1] \) such that \( P = rh_1(f(t)) \) for some \( r \) satisfying \( 0 < r \leq 1 \) and define \( h(h_1^{-1}(P)) \) to be \( h_2^{-1}(rh_2(g(t))) \). Let \( h \) be defined on all other \( U' \)'s analogously. It is a simple matter to check that \( h \) is a homeomorphism from \( E^2 \) onto \( E^2 \) such that \( hf = g \). This concludes the proof of Theorem 3.

The collection \( N(A_n) \) could also be defined for mappings into the two spheres, where the crossing of straight line intervals is replaced by the crossing of arcs (see [4, p. 182]). In this case Theorem 3 holds unrestricted in the sense that boundedness requirements simply disappear.
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