A FIXED-POINT THEOREM FOR INWARD AND OUTWARD MAPS

BY

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The Schauder-Tychonoff theorem states that a continuous function from a compact convex subset of a locally convex topological vector space into itself must have a fixed point ([1, Chapter V, 10.5], or [2]). Using this theorem, we obtain here a stronger result, stating that a map from such a set into the surrounding vector space has a fixed point if the directions in which the points are moved satisfy a certain “inwardness” condition.

It follows immediately that a symmetrical “outwardness” condition also implies the existence of a fixed point. We find also that under the latter condition the image of the map necessarily includes the original set!

1. Definitions. Let $X$ be a topological vector space, and $K$ a compact convex subset of $X$.

We shall call a map $F: K \to X$ “inner” if $F(K) \subseteq K$.

Given $x \in K$, let us define the “inward set” of $x$ with respect to $K$ as the set of points of the form $(1-\alpha)x + \alpha y$, for $y \in K$, $\alpha \geq 0$. It can be thought of as the union of all rays originating at $x$ and drawn so as to pass through some other point $y$ of $K$. For $z \in K$, $\neq x$, a necessary and sufficient condition for $z$ to lie in the inward set of $x$ is that the line segment connecting $x$ and $z$ meet $K$ in some point other than $x$.

A map $F: K \to X$ will be called “inward” if for all $x \in K$, $F(x)$ belongs to the inward set of $x$. The class of inward maps clearly includes the class of inner maps.

Similarly, the “outward set” of $x$ with respect to $K$ will mean the set of points $(1-\alpha)x + \alpha y$ for $y \in K$ and (N.B.) $\alpha \leq 0$, and $F$ will be called outward if $F(x)$ always belongs to the outward set of $x$.

The “weakly inward” and “weakly outward” sets of a point $x$ will be defined as the closures of the inward and outward sets of $x$, respectively. “Weakly inward” and “weakly outward” maps will mean maps taking every $x$ to a member of the appropriate set.

We note that if $F$ is a (weakly) inward map, then the map $x \mapsto 2x - F(x)$ is (weakly) outward, and conversely. Also, $x$ is a fixed point of one map if and only if it is a fixed point of the other. Hence fixed-point results for (weakly) inward maps

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are equivalent to such results for (weakly) outward maps. We shall derive our results by considering maps of the former type.

2. The strictly-convex-normed case. Suppose that the $X$ considered above is a strictly convex normed linear space. Then to every point $y \in X$ there corresponds a unique point $N_K(y) \in K$ whose distance to $y$ is minimal. The function $N_K$ so defined is a continuous retraction of $X$ onto $K$.

Given $x \in K$, let us define the “normal-outward set of $x$” to be the set of points $y \neq x$ such that $N_K(y) = x$.

Given any $x \in K$, the weakly-inward set of $x$ and the normal-outward set of $x$ are disjoint. To show this, we must find, given $y$ in the normal-outward set of $x$, a neighborhood of $y$ containing no members of the inward set of $x$. We claim that the open ball of radius $\| x - y \|$ about $y$ has this property. For, given $z$ in this ball, all points of the segment joining $z$ and $x$, other than $x$ itself, belong to this ball, and hence are nearer to $y$ than $x$ is. Since $x$ is the point of $K$ nearest to $y$, no point of this segment can belong to $K$, hence $z$ is not in the inward set of $x$.

Let us call a map $F: K \to X$ “nowhere normal-outward” if $F(x)$ belongs to the normal-outward set of $x$ for no $x$. It is clear from the above that the class of maps so defined includes the weakly inward maps.

**Theorem 2.1.** Let $X$ be a strictly convex normed linear space, $K$ a compact convex subset of $X$, and $F$ a nowhere normal-outward map from $K$ into $X$. Then $F$ has a fixed point.

**Proof.** $N_K F$ is a continuous map of $K$ into itself. Hence by the Schauder-Tychnoff theorem, there exists $x$ in $K$ such that $N_K F(x) = x$. Looking at the definition of the “normal-outward set of $x$”, we see that $F(x)$ must belong to that set unless $F(x) = x$. Hence $F(x) = x$. □

3. The case $X = R^\infty$, and a Fibering Lemma. Let $R^\infty$ designate the space of all sequences $x = (x_1, x_2, \ldots)$ of real numbers, with the product topology. Let $p_i: R^\infty \to R$ be the $i$th-coordinate map.

**Lemma 3.1.** Let $K$ be a compact convex subset of $R^\infty$, and $F$ a weakly inward map from $K$ into $R^\infty$. Then $F$ has a fixed point.

**Proof.** Suppose $F$ has no fixed point. Then the sets $U_i = \{ x \in K \mid p_i F(x) \neq p_i(x) \}$ cover $K$, hence some finite number of them—say $U_1, \ldots, U_n$—cover $K$. Thus the function $\sup_{1 \leq i \leq n} |p_i F(x) - p_i(x)|$ is nowhere zero, hence is bounded away from zero. We can clearly assume it is everywhere $\geq 1$.

By the assumption that $F$ is weakly inward, we can, for every $x \in K$, find $y \in K$ and $\alpha \geq 0$, such that the first $n$ coordinates of $u(x) = (1 - \alpha) x + \alpha y$ differ from those of $F(x)$ by less than $1/2^n$. Now it is clear that the $y$ and $\alpha$ chosen for a given $x$ will work for all $x'$ in a neighborhood of $x$ in $K$. Hence by compactness of $K$, we can handle all points of $K$ by choosing $y$ and $\alpha$ from some finite set; $u$ can thus be
chosen as a function which, though not necessarily continuous, will take on all its values in a compact set. Hence for each \( i > 0 \), we can find a real number \( B_i \) such that \( \forall x \in K, |p_i(x)| < B_i, |p_i(F(x))| < B_i, \) and \( |p_i(F(x) - u(x))| < B_i \). Multiplication of each coordinate by an independent constant is a linear homeomorphism on \( \mathbb{R}^\infty \), hence preserves all the structure we are considering. Consequently, we may assume \( B_i \leq 2^{-i} \) for all \( i > n \). (We have already put conditions on the first \( n \) coordinates.)

Now let \( H \) designate the space of all \( L_2 \) (square-summable) sequences of real numbers under the \( L_2 \) norm—a vector subspace of \( \mathbb{R}^\infty \), but with a stronger topology. Let \( B \) be the set of all \( x = (x_1, x_2, \ldots) \) such that \( \forall i |x_i| \leq B_i \). \( B \) lies in \( H \), and it is easily shown that the \( \mathbb{R}^n \) and \( H \)-topologies agree on \( B \). Hence \( K \) is a compact convex subset of \( H \) and \( F \) is continuous in the topology of \( H \).

We note that for all \( x \in K \), \( u(x) \) is at a distance less than 1 from \( F(x) \), since \( |p_i(u(x) - F(x))| < 1/2^i \) (for \( i \) both \( \leq n \) and \( > n \)). On the other hand, \( F(x) \) is at a distance at least 1 from \( x \) (see first paragraph). Hence \( u(x) \), a point on a ray drawn from \( x \), through some other point of \( K \), is nearer to \( F(x) \) than \( x \) is. Some point on the line segment between \( x \) and \( u(x) \) will both be closer to \( F(x) \) than \( x \) is and be in \( K \). Hence \( x \) is not the point of \( K \) nearest \( F(x) \). So \( F \) is a nowhere normal-outward map without a fixed point, contradicting Theorem 2.1.

We shall obtain our most general form of the fixed-point theorem from the above by the Fibering Lemma and the corollary below. (This is a strengthened form of the argument used in the Dunford-Schwartz lemma [1, Chapter V, 10.4]—the analogous step in the proof of the Schauder-Tychonoff theorem.) Note that our lemma merely requires \( K \) to be Lindelöf (every open covering has a countable subcovering), though in the case we are interested in, it is compact.

**Lemma 3.2 (Fibering Lemma).** Let \( X \) be a topological vector space whose topology is induced by linear functionals, let \( K \) be a Lindelöf subset of \( X \), and let \( F: K \to X \) be a continuous map. Then, given any countable family \( G_0 \) of continuous linear functionals on \( X \), there is a continuous linear map \( p: X \to R'^\infty \), and a continuous map \( F': p(K) \to R'^\infty \) such that:

1. \( F' p = p F \),
2. For each \( f \in G_0 \) there exists an \( f': R^\infty \to R \) such that \( f = f' p \).

**Proof.** We shall first show that any continuous real-valued function \( g \) on \( K \) is (in a sense to be made clear) “continuously determined” by a countable family of continuous linear functionals of \( X \).

Given \( g \) we can, by the Lindelöf assumption, find for each \( \varepsilon > 0 \) a countable covering of \( K \) by open sets \( (U_a)_{a \in A_\varepsilon} \), such that for each \( a \) in the index-set \( A_\varepsilon \), and \( x, x' \in U_a \), \( |g(x) - g(x')| < \varepsilon \). By the assumption on the topology of \( X \), each \( U_a \) can be assumed of the form

\[
\{x \in K \mid f_{a1}(x) \in (a_{a1}, b_{a1}), \ldots, f_{an}(x) \in (a_{an}, b_{an})\}
\]

where the \( f_{ai} \) are continuous linear functionals on \( X \), and the \( a_{ai} < b_{ai} \) are real numbers.
Let $A$ be the union of index-sets $A_1 \cup A_{1/2} \cup A_{1/4} \cup \cdots$. Then it follows from our construction that $g$ is a continuous function of the family of maps $(f_{il})_{i=1,\ldots,n_0}$. I.e., $g$ can be written $g_0f$, where $f$ is the product map of the $f_{il}$ sending $X$ into $R^{(G(i)_{i=1,\ldots,n_0} \times A_{i/2})}$, and $g_0$ is a continuous function on $f(K)$, as we desired.

We are given a set of functionals $G_0$; for $j > 0$ let us, inductively, make $G_j$ a countable family of linear functionals which, for every $f \in G_{j-1}$, "continuously determines" $fF$. Let $G = \bigcup_{j=0}^\infty G_j$, which we can reindex $(f_l)_{l=1,2,\ldots}$, since it is countable. (If $G$ is finite, we let $f_l = 0$ for large $i$.) Let $p : X \to R^n$ be the product of this family of maps. It is clear from our construction that for all $i, f_iF$ is continuously determined by $p$, thus $pF$ is continuously determined by $p$, i.e., we can write $pF = F'p$, where $F'$ is continuous. On the other hand, for every $f \in G_0, f$ will equal some $f_l$; letting $F'$ be the $i$th projection map we have $f = f'p$.

**Corollary 3.3.** Lemma 3.2 still holds if the hypothesis "the topology of $X$ is determined by linear functionals," is replaced by "linear functionals distinguish points of $X$, and $K$ has compact closure."

**Proof.** Let $X'$ designate $X$ with the topology induced by the continuous linear functionals. Since the closure of $K$ is compact in $X$, the closure of $K$, and hence $K$ itself, has the same topology in $X'$ as in $X$. Hence continuous maps on $K$ remain continuous in the $X'$ topology; and we get our results by applying Lemma 3.3 in $X'$.  

4. **The general fixed-point theorems.**

**Theorem 4.1.** Let $X$ be a topological vector space such that continuous linear functionals distinguish points. Let $K$ be a compact convex subset of $X$, and $F : K \to X$ a weakly inward map. Then $F$ has a fixed point.

**Proof.** Given any continuous linear functional $f$ on $X$ let $S_f$ be the set of $x \in K$ such that $fF(x) = f(x)$. We claim that any finite intersection of the sets $S_f$ is non-empty.

Indeed, given any finite set, $G_0$, of such functionals, we apply Corollary 3.3, getting maps $p : X \to R^n$ and $F' : p(K) \to R^n$. The set $p(K)$ is compact and convex. Further, $F'$ is weakly inward, for it is easy to see that the property of lying in the weakly inward set of a point is preserved under continuous linear maps of the vector space.

So by Lemma 3.1, $F'$ has a fixed point $p(z), z \in X$. We claim that $z \in \bigcap_{G_0} S_f$. For given any $f \in G_0$, we compute

$$fF(z) = f'pF(z) = f'F'p(z) = f'p(z) = f(z).$$

It follows from the compactness of $K$ that the intersection of the sets $S_f$ over all linear functionals $f$ is nonempty. Clearly, a point in this intersection is a fixed point of $F$.  

Given a map $F$ on a compact convex set $K$ in a topological space $X$, let us call a coset $C$ of a closed subspace of $X$ a nonempty section of $K$ if $C \cap K \neq \emptyset$; and let us call it an invariant section if $F(C \cap K) \subset C$. The proof in Dunford-Schwartz [1] of the general case of the Schauder-Tychonoff theorem makes use of the fact that the inverse image of a fixed point under a fibering is a nonempty invariant section, and that if $C$ is a nonempty invariant section and $F$ is an inner map, then $(F \mid K \cap C): (K \cap C) \to C$ is again an inner map. (Zorn's lemma is then used.) The latter result also holds for inward maps, but not for weakly inward maps! In fact, for $F$ weakly inward, a nonempty invariant section need not contain a fixed point. For example, let $K$ be the disc $x^2 + (y-1)^2 \leq 1$ in the $xy$-plane, and let $F$ send the point $(x, y)$ to the end of the clockwise segment of length $\sqrt{2y/2}$ tangent to $K$ at the point $((1-(y-1)^2)^{1/2}, y)$. Then $y=0$ and $y=2$ are both invariant sections, but only $y=0$ has a fixed point. (See Figure 1.)
It was this observation that forced us to look, not at sets based on an arbitrary choice, but sets such as the $S_i$'s which cannot exclude any potential fixed points.

**Lemma 4.2.** Suppose a compact convex subset $K$ of a topological vector space $X$ contains the point 0. Then the outward set of any point $x$ is closed under multiplication by constants $c > 1$. Hence so is the weakly outward set.

*Proof.* See Figure 2, where $z = (1 - \alpha)x + \alpha y$ ($\alpha < 0$) is an arbitrary point of the outward set of $x$. The reader can easily supply the numerical argument, getting $cz$ in the form $(1 - \alpha')x + \alpha'y$. \(\square\)

**Theorem 4.3.** Let $X$ be a topological vector space such that continuous linear functionals distinguish points. Let $K$ be a compact convex subset of $X$, and $F: K \to X$ a weakly outward map. Then:

1. $F$ has a fixed point,
2. $F(K) \supseteq K$.

*Proof.* (1) is clear from Theorem 4.1 and our original discussion of the relationship between “inwardness” and “outwardness” conditions.

To show (2), let us suppose the contrary. Clearly, we can assume that 0 is a point of $K - F(K)$. The complement $U$ of $F(K)$ is a neighborhood of 0, so we can choose $c > 1$ such that $cU \supseteq K$. Then $cF(K)$ is disjoint from $K$, and so the map $cF$ can have no fixed points. But, by Lemma 4.2 it is clear that $cF$ is weakly outward. Contradiction.

**References**