APPROXIMATION BY BOUNDED ANALYTIC FUNCTIONS: UNIFORM CONVERGENCE AS IMPLIED BY MEAN CONVERGENCE[1]

BY

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In three recent notes [1], [2], [3] I have discussed uniform convergence by polynomials (in the complex variable) to a given function as a consequence of convergence in the mean of those polynomials to the given function, and also convergence in the mean of one order as a consequence of convergence in the mean of a lower order. The present note contains analogs of those results, but now for approximation by bounded analytic functions. As a first illustration of the new results, we have

THEOREM 1. *Let* Γ *be an analytic Jordan curve contained in the simply-connected region D of the z-plane, and suppose we have for some function* $f(z)$ *continuous on* Γ *and functions* $f_n(z)$ *analytic in D*

$$\int_\Gamma |f(z)-f_n(z)|^q\,|dz| \leqq A/n^{q\alpha}, \qquad q > 0, \tag{1}$$

$$|f_n(z)| \leqq AR^n, \qquad z \text{ in } D. \tag{2}$$

Then for $\alpha+1/p-1/q>0$ *and* $0<q<p\leqq\infty$ *we have for the pth power norm on* Γ

$$\|f(z)-f_n(z)\|_p \leqq A/n^{\alpha+(1/p)-(1/q)}. \tag{3}$$

Here and below the constants A are independent of n and z, and may change from one inequality to another.

For $p=\infty$ we consider the first member of (3) as the Tchebycheff (uniform) norm of $[f(z)-f_n(z)]$ on Γ, with a similar interpretation in later formulas. As is usual in the study of convergence by bounded analytic functions, we note (see for instance [4, §2.2]) that there exist for each n and N polynomials $P_{n,N}(z)$ of respective degrees N such that we have

$$|f_n(z)-p_{n,N}(z)| \leqq AR^n/R_1^N, \qquad z \text{ on } \Gamma, \quad R_1 > 1. \tag{4}$$

If we choose the integer λ so large that $R_1^\lambda > R$, there follow

$$|f_n(z)-p_{n,\lambda n}(z)| \leqq A(R/R_1^\lambda)^n, \qquad z \text{ on } \Gamma, \tag{5}$$

$$\int_\Gamma |f_n(z)-p_{n,\lambda n}(z)|^q\,|dz| \leqq A/n^{q\alpha}. \tag{6}$$

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Standard algebraic inequalities depending on q yield by (1) and (6)

$$\int_\Gamma |f(z)-p_{n,\lambda n}(z)|^q\,|dz| \leqq A_0/n^{q\alpha}. \tag{7}$$

The polynomials $p_{n,\lambda n}(z)$ are defined only for the degrees $\lambda n=\lambda, 2\lambda, 3\lambda, \ldots$, but to obtain polynomials $P_m(z)$ for all degrees we may set $P_m(z)=p_{n,\lambda n}(z)$ for $\lambda n \leqq m < \lambda(n+1)$, whence for $m=1, 2, 3, \ldots$

$$\int_\Gamma |f(z)-P_m(z)|^q\,|dz| \leqq \frac{A_0}{n^{q\alpha}} \leqq \frac{A_1}{[\lambda(n+1)]^{q\alpha}} \leqq \frac{A_1}{m^{q\alpha}},$$

provided $A_1 \geqq A_0\lambda^{q\alpha}(n+1)^{q\alpha}/n^{q\alpha}$ for all n. Consequently $f(z)$ has various known properties on Γ. Thus by [3, Theorem 11] we have (since $\alpha+1/p>1/q$)

$$\|f(z)-p_{n,\lambda n}(z)\|_p \leqq A/n^{\alpha+(1/p)-(1/q)}. \tag{8}$$

Inequality (8) together with (5) now yields (3).

The reader may notice the validity of

COROLLARY 1. *In Theorem* 1 *the second member of* (1) *may be replaced by* $A\varepsilon_n^q$, *where* ε_n (>0) *is monotonic nonincreasing as n increases, is such that* $r^n=o(\varepsilon_n)$ *for every* r (<1), *with the property* $\varepsilon_n=O(\varepsilon_{\lambda n})$ *whenever integral* $\lambda>1$, *and where the expression* $(2^{m-1}\leqq n<2^m)$,

$$\frac{2^{mr}\varepsilon_n+2^{(m+1)r}\varepsilon_{2^m}+2^{(m+2)r}\varepsilon_{2^{m+1}}+\cdots}{n^r\varepsilon_n}, \qquad p \geqq 1, \tag{9}$$

$$\frac{(2^m)^{pr}\varepsilon_n^p+(2^{m+1})^{pr}\varepsilon_{2^m}^p+(2^{m+2})^{pr}\varepsilon_{2^{m+1}}^p+\cdots}{n^{pr}\varepsilon_n^p}, \qquad p<1, \tag{10}$$

where $r=1/q-1/p$, *has a meaning and is bounded as* $n\to\infty$; *the second member of* (3) *is to be replaced by* $An^{1/q-1/p}\varepsilon_n$, *assumed to approach zero.*

In the proof, the second members of (6), (7), and (8) are to be replaced by $A\varepsilon_n^q$, $A_0\varepsilon_n^q$, $An^{1/q-1/p}\varepsilon_n$ respectively.

Both Theorem 1 and Corollary 1 can be modified in hypothesis and conclusion so that the first member of (1) is a double integral taken over the interior of Γ, as we now indicate.

First we state a result [3, Theorem 14] on degree of convergence by polynomials:

THEOREM 2. *Let E be the closed interior of an analytic Jordan curve, and let a function* $f(z)$ *continuous on E and polynomials* $p_n(z)$ *of respective degrees n be given such that we have for the qth power norm on E*

$$\|f(z)-p_n(z)\|'_q \leqq \varepsilon_n, \qquad q>0 \tag{11}$$

and where ε_n *has the first three properties of Corollary 1. Let us suppose the expression* (9) *or* (10) *with r replaced by* $s=2/q-2/p$ *exists and is bounded as* $n\to\infty$, *where* $2^{m-1}\leqq n<2^m$. *Then we have for* $0<q<p\leqq\infty$,

$$\|f(z)-p_n(z)\|'_p \leqq An^{2/q-2/p}\varepsilon_n, \tag{12}$$

where the second member is supposed to approach zero. In particular we may choose $\varepsilon_n = n^{-\alpha}$, $\alpha > 2/q - 2/p$.

Second, we indicate the analog of Theorem 2 for approximation by bounded analytic functions, which is thus an extension of Theorem 2, in the spirit of Theorem 1 and its Corollary as an extension of [1, Theorem 2].

THEOREM 3. *Let E be the closed interior of an analytic Jordan curve contained in the simply-connected region D, and suppose some function $f(z)$ analytic interior to E, continuous on E, and functions $f_n(z)$ analytic throughout D satisfy*

$$\iint_E |f(z) - f_n(z)|^q \, dS \leqq A/n^{q\alpha}, \qquad q > 0, \tag{13}$$

$$|f_n(z)| \leqq AR^n, \qquad z \text{ in } D. \tag{14}$$

Then for $\alpha > 2/q - 2/p$ and $0 < q < p \leqq \infty$ we have

$$\|f(z) - f_n(z)\|_p' \leqq A/n^{\alpha + 2/p - 2/q}. \tag{15}$$

Theorem 3 follows by the methods of proof of [1, Theorem 4] and the present Theorem 2. Like Theorem 1, Theorem 3 can be generalized in a suitable corollary:

COROLLARY 1. *In Theorem* 3 *the second member of* (13) *may be replaced by $A\varepsilon_n^q$ where ε_n (>0) is arbitrary monotonic nonincreasing, and is such that $r^n = o(\varepsilon_n)$ for every r (<1), with the property $\varepsilon_n = O(\varepsilon_{\lambda n})$ whenever integral $\lambda > 1$, and where the expression* (9) *or* (10) *with r replaced by s has a meaning and is bounded as $n \to \infty$, with $2^{m-1} \leqq n < 2^m$. The second member of* (15) *is to be replaced by $An^{2/q - 2/p}\varepsilon_n$, and is assumed to approach zero.*

The preceding results, primarily relating to approximation by bounded analytic functions, have an analog for approximation on a curve rather than in a region:

THEOREM 4. *Let Γ be an analytic Jordan curve contained in a region D not necessarily simply-connected, and suppose we have for some function $f(z)$ continuous on Γ and functions $f_n(z)$ analytic in D*

$$\int_\Gamma |f(z) - f_n(z)|^q \, |dz| \leqq A\varepsilon_n^q, \qquad q > 0, \tag{16}$$

$$|f_n(z)| \leqq AR^n, \qquad z \text{ in } D, \tag{17}$$

where ε_n (>0) is monotonic nonincreasing, is such that $r^n = o(\varepsilon_n)$ for every r (<1), and with the property $\varepsilon_n = O(\varepsilon_{\lambda n})$ whenever integral $\lambda > 1$, and where the expression (9) *or* (10) *has a meaning and is bounded as $n \to \infty$, with $2^{m-1} \leqq n < 2^m$. Then if $n^{1/q - 1/p}\varepsilon_n \to 0$ and $0 < q < p \leqq \infty$, we have for the pth power norm on Γ*

$$\|f(z) - f_n(z)\|_p \leqq An^{1/q - 1/p}\varepsilon_n. \tag{18}$$

In particular we may choose $\varepsilon_n = n^{-\alpha}$, $\alpha > 1/q - 1/p$.

In the proof of Theorem 4, we assume the origin to lie interior to Γ, approximate the $f_n(z)$ on Γ by polynomials in z and $1/z$, and use the method of [3]. Details are left to the reader.

Theorem 4 applies to approximation on the unit circumference Γ to a real or complex function $f(z)$ by real or complex bounded analytic functions $f_n(z)$, or with the substitution $z=e^{i\theta}$, approximation on the real line $-\infty<\theta<\infty$ to a function with period 2π by bounded analytic functions with period 2π in a strip containing the line. In particular if $f_n(z)$ is a polynomial in z and $1/z$ of degree n satisfying (16), then (17) follows if D is an annulus containing Γ in its interior with boundary components having 0 as center, and $f_n(e^{i\theta})$ is a trigonometric polynomial of order n. Compare here [2, Theorems 6–9].

Theorem 4 suggests approximation by bounded analytic functions in a multiply connected region, as measured by a line integral over the boundary:

THEOREM 5. *Let E be a closed bounded region whose boundary Γ consists of a finite number of mutually disjoint analytic Jordan curves, and which lies in a region D. Suppose for some function $f(z)$ analytic interior to E and continuous on E and for functions $f_n(z)$ analytic in D we have* (16) *and* (17), *where ε_n satisfies the conditions of Theorem* 4. *Then if $n^{1/q-1/p}\varepsilon_n\to 0$ and $0<q<p\leqq\infty$ we have* (18) *for the pth power norm on Γ. In particular we may choose $\varepsilon_n=n^{-\alpha}$, $\alpha>1/q-1/p$.*

To prove Theorem 5, we merely apply Theorem 4 to each component of Γ and of $f(z)$.

Our primary topic in the foregoing theorems is degree of uniform convergence of the $f_n(z)$ to $f(z)$, so it is natural to assume those functions continuous in the closed regions considered. Some comments on uniform convergence in subregions as a consequence of mean convergence on the boundary or over a region are made in [5, §5.8].

We proceed to study the analog of Theorem 5, using as norm a double integral, whose proof is more involved than that of Theorem 5:

THEOREM 6. *Let E be a closed bounded region whose boundary Γ consists of a finite number of mutually disjoint analytic Jordan curves, and which lies in a region D. Suppose for some function $f(z)$ analytic interior to E, continuous on E, and for functions $f_n(z)$ analytic in D we have*

$$\iint_E |f(z)-f_n(z)|^q\, dS \leqq A\varepsilon_n^q, \qquad q>0, \tag{19}$$

and (14), *where ε_n satisfies the conditions of Corollary* 1 *to Theorem* 3. *Then if $n^{2/q-2/p}\varepsilon_n\to 0$ and $0<q<p\leqq\infty$ we have*

$$\|f(z)-f_n(z)\|'_p \leqq An^{2/q-2/p}\varepsilon_n, \tag{20}$$

where we assume the second member approaches zero. In particular we may choose $\varepsilon_n=n^{-\alpha}$, $\alpha>2/q-2/p$.

Let the components of Γ be $\Gamma_1, \Gamma_2, \ldots, \Gamma_\nu$ where Γ_1 bounds a closed finite region E_1 containing E, and Γ_j $(j>1)$ bounds a closed infinite region E_j containing E. Let Γ_j' be a variable analytic Jordan curve interior to E $(j=1, 2, \ldots, \nu)$ which together with Γ_j bounds a closed annular region G_j, where the G_j are mutually disjoint. Since the curve Γ_j' lies in E, there follows from (19) by [5, §5.3, Lemma 2]

$$|f(z)-f_n(z)| \leqq A\varepsilon_n, \quad z \text{ on } \Gamma_j', \tag{21}$$

where A varies with Γ_j'.

If z is an arbitrary point interior to E, the Γ_j' can be chosen so that z lies exterior to the G_j, and indeed z lies interior to the region bounded by all ν of the Γ_j'. For this point z, the value of $f(z)$ is represented by the Cauchy integral of $f(z)$ over $\sum \Gamma_j'$, so we may write $f(z) = \sum f^{(j)}(z)$ for z interior to E, and similarly $f_n(z) \equiv \sum f_n^{(j)}(z)$ for z interior to E, where the ν components $f^{(j)}(z)$ and $f_n^{(j)}(z)$ of $f(z)$ and $f_n(z)$ are represented by the Cauchy integrals of $f(z)$ and $f_n(z)$ over the respective Γ_j' but are independent of the Γ_j' having the required properties. By inequality (21) we have for z on any closed subset of E disjoint from G_j

$$|f^{(j)}(z)-f_n^{(j)}(z)| \leqq A\varepsilon_n \qquad (j = 1, 2, \ldots, \nu). \tag{22}$$

The functions $f^{(j)}(z)$ and $f_n^{(j)}(z)$ are defined throughout the interior of E_j and inequality (22) is valid also for z on $E_j - G_j$ minus a neighborhood of Γ_j'.

It is natural to attempt to use (22) to obtain an inequality on the functions $f^{(j)}(z)-f_n^{(j)}(z)$ on each E_k, but this procedure is complicated by the fact that $\nu-1$ of these regions are infinite and the surface integral norm cannot be used directly.

We may choose points $\alpha_1=\infty, \alpha_2, \ldots, \alpha_\nu$ fixed in the respective regions $D_1, D_2, \ldots, D_\nu$ exterior to E bounded by $\Gamma_1, \Gamma_2, \ldots, \Gamma_\nu$, and choose in each D_j and in D an analytic Jordan curve Γ_j'' separating α_j from E but so that the region D_0 bounded by $\sum \Gamma_j''$ contains no point not in D. The components of $f_n(z)$ already defined can be represented by Cauchy integrals of $f_n(z)$ over the curves Γ_j'', and we have by (14)

$$|f_n^{(j)}(z)| \leqq AR^n, \qquad z \text{ in } D_j^0, \tag{23}$$

where D_j^0 is a suitable closed region containing E_j in its interior and separated by Γ_j'' from α_j.

We fasten our attention now on $E_1, f^{(1)}(z)$, and $f_n^{(1)}(z)$. Inequality (22) yields

$$|f^{(j)}(z)-f_n^{(j)}(z)| \leqq A\varepsilon_n, \qquad z \text{ on } G_1, \quad j > 1,$$

$$\iint_{G_1} \sum_{j>1} |f^{(j)}(z)-f_n^{(j)}(z)|^q \, dS \leqq A\varepsilon_n^q,$$

and by (19) with the integral over G_1 there follows

$$\iint_{G_1} |f^{(1)}(z)-f_n^{(1)}(z)|^q \, dS \leqq A\varepsilon_n^q. \tag{24}$$

The point set G_1 is to some extent variable, so we deduce also by (22) and by the finiteness of the area of E_1,

$$\iint_{E_1-G_1} |f^{(1)}(z)-f_n^{(1)}(z)|^q \, dS \leqq A\varepsilon_n^q, \tag{25}$$

where the new E_1-G_1 contains in its interior the partial boundary Γ_1' of the G_1 in (24). Then by (24) and (25) we have

$$\iint_{E_1} |f^{(1)}(z)-f_n^{(1)}(z)|^q \, dS \leqq A\varepsilon_n^q. \tag{26}$$

By (23) and (26) we are in a position to apply Corollary 1 to Theorem 3, which establishes

$$\iint_{E} |f^{(1)}(z)-f_n^{(1)}(z)|^p \, dS \leqq An^{2p/q-2}\varepsilon_n^p; \tag{27}$$

the integral may be taken over E_1 or E. This proof does not apply directly to (27) with 1 replaced by j (>1) because the area of E_j is then infinite.

However, for $j>1$ we make a linear transformation $w=\phi(z)$ that carries α_j to infinity, which then transforms E_j into a finite region of the w-plane. By the method of proof of (24) we establish

$$\iint_{G_j} |f^{(j)}(z)-f_n^{(j)}(z)|^q \, dS \leqq A\varepsilon_n^q, \qquad dS = dS_z.$$

With the transformation $w=\phi(z)$, $z=\psi(w)$, we may set $dS_w=|\phi'(z)|^2 \, dS_z$, where $|\phi'(z)|$ is bounded and bounded from zero except near $z=\alpha_j$ and $z=\infty$ and their images, whence for the integral over the image of G_j

$$\iint |f^{(j)}[\psi(w)]-f_n^{(j)}[\psi(w)]|^q \, dS_w \leqq A\varepsilon_n^q. \tag{28}$$

By (22) we may write (28) for the integral over the image of a new E_j-G_j containing the partial boundary Γ_j' of the previously used G_j (by the boundedness of the area of the image of E_j). There follows for the integral over the image of E_j this same inequality (28).

By virtue of (23) interpreted in the w-plane, we can now apply Corollary 1 to Theorem 3, which proves for the integral over the image of E_j or E

$$\iint |f^{(j)}[\psi(w)]-f_n^{(j)}[\psi(w)]|^p \, dS_w \leqq An^{2p/q-2}\varepsilon_n^p.$$

We use this integral over the image of E, on which $\psi'(w)$ is bounded and bounded from zero, so there follows $(j>1)$

$$\iint_{E} |f^{(j)}(z)-f_n^{(j)}(z)|^p \, dS_z \leqq An^{2p/q-2}\varepsilon_n^p,$$

and (27) yields (20), which completes the proof of Theorem 6.

We add now some general comments on the theorems already proved. If the $f_n(z)$ of Theorem 1 are polynomials of respective degrees n satisfying (1), inequality (2) is a consequence of (1). For inequality (1) implies the boundedness ($n \to \infty$) of

$$\int_\Gamma |f_n(z)|^q \, |dz|, \tag{29}$$

and (2) follows where D is an arbitrary finite region bounded by a level locus Γ_R, by [5, §5.2, Lemma]. Here we denote generically by Γ_ρ ($\rho > 1$) the locus $|\phi(z)| = \rho$ in the complement K of E, where $w = \phi(z)$ maps K onto $|w| > 1$, $\phi(\infty) = \infty$. A more general remark can be made:

REMARK. *Let E be a closed limited point set whose complement is simply connected and whose boundary Γ has positive linear measure. If the rational functions $f_n(z)$ of respective degrees n satisfy* (1), *and if the poles of the $f_n(z)$ have no limit point on E, then for a suitably chosen region D containing E, inequality* (2) *is satisfied.*

An inequality

$$\int_\Gamma |f_n(z)|^q \, |dz| \leqq L^q, \qquad q > 0, \tag{30}$$

follows by the method of treatment of (29). If the $f_n(z)$ have no poles on or interior to Γ_B, $B > 1$, then [5, §9.8, Lemma III] we have for z on and within Γ_Z

$$|f_n(z)| \leqq AL[(BZ-1)/(B-Z)]^n, \qquad 1 < Z < B, \tag{31}$$

so we may choose D as the closed interior of Γ_Z by identifying (31) with (2).

The Remark just established deserves a number of additional comments.

1°. It is immaterial whether the hypothesis of the Remark is chosen as (1) or as the replacement of (1) as in Corollary 1 to Theorem 1. In either case we obtain (30) at once.

2°. Let the hypothesis (1) of the Remark be replaced by the inequality

$$|f(z) - f_n(z)| \leqq A\varepsilon_n, \qquad z \text{ on } \Gamma. \tag{32}$$

The uniform boundedness of the rational functions $f_n(z)$ follows on Γ, and an appropriate lemma [5, §9.7, Lemma I] yields (31) for z on or within Γ_Z if all poles of the $f_n(z)$ lie exterior to Γ_B, $1 < Z < B$. Thus D can be chosen as the interior of Γ_Z. This comment is of interest in connection with approximation also in the real domain, as in [6].

3°. The hypothesis of the Remark may be replaced by an inequality for the double integral:

$$\iint_E |f(z) - f_n(z)|^q \, dS \leqq A\varepsilon_n^q, \tag{33}$$

say under the hypothesis of Theorem 2, where the rational functions $f_n(z)$ of respective degrees n have no limit point of poles on E. We obtain the boundedness of the integrals

$$\iint_E |f_n(z)|^q \, dS,$$

hence [5, §5.3, Lemma II] there follows on an arbitrary closed region E' interior to E the uniform boundedness of the $f_n(z)$. Let the poles of the $f_n(z)$ have no limit point on or exterior to E_ρ, $\rho > 1$. Then for E' sufficiently large in E, the locus $(E')_\rho$ can be chosen as near E_ρ as desired (but interior to E_ρ), so in particular we can choose E' so that $(E')_\rho$ contains in its interior some E_B, $B > 1$, which contains E in its interior. If $|f_n(z)| \leqq L$ for z on E', we have for z on $(E')_Z$ (chosen to contain E and be contained in E_B)

$$|f_n(z)| \leqq AL[(\rho Z - 1)/(\rho - Z)]^n, \qquad 1 < Z < \rho,$$

by [5, §9.7, Lemma I]. The region D can be chosen as the interior of $(E')_Z$.

4°. The Remark can be extended so as to apply even if the complement of E is not simply connected, provided the boundary of E consists of a finite number of mutually disjoint analytic Jordan curves. We assume that $f_n(z)$ is a sequence of rational functions of respective degrees n whose poles have no limit point on E; it follows for instance that inequality (16) implies (17). Compare here Theorem 4 and [2, Theorems 6, 7, and 8].

5°. The reasoning involved in the Remark may apply even if the approximating functions $f_n(z)$ are no longer rational functions, provided each $f_n(z)$ is meromorphic with not more than n poles in each of one or more suitable regions. For instance we might consider approximation on a Jordan curve E containing in its interior a closed simply connected region E_0, where the functions $f_n(z)$ are respectively meromorphic with no more than n poles in the complement E_1 of E_0, continuous and bounded on the boundary of E_0.

Throughout this paper we have assumed for simplicity that the Jordan curves involved are analytic. That assumption can be somewhat weakened, as by the use of curves of type B in [1], and of type B' in [2].

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