I-BISIMPLE SEMIGROUPS

BY

R. J. WARNE

Let S be a semigroup and let E_S denote the set of idempotents of S. As usual E_S is partially ordered in the following fashion: if $e, f \in E_S$, $e \leqq f$ if and only if $ef = fe = e$. Let I denote the set of all integers and let I^0 denote the set of nonnegative integers. A bisimple semigroup S is called an I-bisimple semigroup if and only if E_S is order isomorphic to I under the reverse of the usual order. We show that S is an I-bisimple semigroup if and only if $S \cong G \times I \times I$, where G is a group, under the multiplication

$$\begin{aligned}(g, a, b)(h, c, d) &= (gf_{b-c,c}^{-1} h\alpha^{b-c} f_{b-c,d},\ a,\ b+d-c) \quad \text{if } b \geqq c,\\ &= (f_{c-b,a}^{-1} g\alpha^{c-b} f_{c-b,b} h,\ a+c-b,\ d) \quad \text{if } c \geqq b,\end{aligned}$$

where α is an endomorphism of G, α^0 denoting the identity automorphism of G, and for $m \in I^0$, $n \in I$,

$f_{0,n} = e$, the identity of G while if $m > 0$,

$f_{m,n} = u_{n+1}\alpha^{m-1} u_{n+2}\alpha^{m-2} \cdots u_{n+(m-1)}\alpha u_{n+m}$, where $\{u_n : n \in I\}$ is a collection of elements of G with $u_n = e$, the identity of G, if $n > 0$.

If we let $G = \{e\}$, the one element group, in the above multiplication we obtain $S = I \times I$ under the multiplication $(a, b)(c, d) = (a + c - r, b + d - r)$.

We will denote S under this multiplication by C^*, and we will call C^* the extended bicyclic semigroup. C^* is the union of the chain I of bicyclic semigroups C.

If S is an I-bisimple semigroup, we will write $S = (G, C^*, \alpha, u_i)$ where G is the structure group of S, α is the structure endomorphism of G, and $\{u_i\}$ is the sequence of "distinguished elements" of G.

An I-bisimple semigroup is a bisimple inverse semigroup without identity as contrasted to a bisimple ω-semigroup (a bisimple semigroup T such that E_T is order isomorphic to I^0 under the reverse of the usual order [7], [12]) which is a bisimple inverse semigroup with identity. If $S = (G, C^*, \alpha, u_i)$, the inverse of (g, m, n) is (g^{-1}, n, m) and $E_S = ((e, n, n) : n \in I)$. If $\mathscr{H}$ is Green's relation, $S/\mathscr{H} \cong C^*$, the extended bicyclic semigroup.

Necessary and sufficient conditions for two I-bisimple semigroups to be isomorphic are established, and an explicit determination of the homomorphisms of one I-bisimple semigroup onto another is given.

A complete description of the maximal group homomorphic image of an I-bisimple semigroup is given. Since we are also able to give an explicit description of the defining homomorphism, this result should have applications to the matrix

Received by the editors November 30, 1965.

representation of I-bisimple semigroups over fields. To perform the construction, we first determine the maximal cancellative homomorphic image of an ω-right cancellative semigroup (a right cancellative semigroup with identity whose ideal structure is order isomorphic to I^0 under the reverse of the usual order). We then utilize this result in conjunction with the description of the maximal group homomorphic image of a bisimple inverse semigroup with identity [8] to describe the maximal group homomorphic image of a bisimple ω-semigroup. Finally, this description, structural properties of I-bisimple semigroups, and a determination of the homomorphisms of a bisimple ω-semigroup into a group are used to give the desired construction.

It is shown that if ρ is a congruence relation on an I-bisimple semigroup $S=(G, C^*, \alpha, u_i)$, ρ is a group congruence (S/ρ is a group) or ρ is an idempotent separating congruence (each ρ-class contains at most one idempotent). The group congruences are in a one-to-one correspondence with the normal subgroups of the maximal group homomorphic image while the idempotent separating congruences are determined in terms of the α-invariant subgroups of G.

The ideal extensions of an I-bisimple semigroup are studied in [15], [16].

Unless otherwise specified, we will use the terminology, definitions, and notation of [2].

1. **The structure theory.** In this section we determine the structure of I-bisimple semigroups, and we also give an example of an I-bisimple semigroup with nontrivial distinguished elements.

$\mathscr{R}$, $\mathscr{L}$, $\mathscr{H}$, and $\mathscr{D}$ will denote Green's relations [2]. R_a will denote the $\mathscr{R}$-class containing a.

THEOREM 1.1 (REILLY [7]; SEE ALSO WARNE [12]). *S is a bisimple ω-semigroup if and only if $S\cong G\times I^0\times I^0$, where G is a group, under the multiplication*

$$(g, a, b)(h, c, d) = (g\alpha^{c-r}h\alpha^{b-r}, a+c-r, b+d-r)$$

where $r=\min(b, c)$ and α is an endomorphism of G, α^0 denoting the identity automorphism.

THEOREM 1.2 (WARNE [10], [11]). *Let S be a regular bisimple semigroup and let $e\in E_S$. Then, eSe is a regular bisimple semigroup with identity e. If E_S is linearly ordered, $S=U(eSe : e\in E_S)$ with $eSe\subset fSf$ if and only if $e\leqq f$ and each eSe is a bisimple inverse semigroup with identity e with $E_{eSe}=\{f\in E_S \mid e\geqq f\}$.*

THEOREM 1.3. *S is an I-bisimple semigroup if and only if $S\cong G\times I\times I$, where G is a group, under the multiplication*

$$(1.1)\qquad \begin{aligned}(g, a, b)(h, c, d) &= (gf_{b-c,c}^{-1}h\alpha^{b-c}f_{b-c,d}, a, b+d-c) \quad \text{if } b\geqq c,\\ &= (f_{c-b,a}^{-1}g\alpha^{c-b}f_{c-b,b}h, a+c-b, d) \quad \text{if } c\geqq b,\end{aligned}$$

where α is an endomorphism of G, α^0 denoting the identity automorphism of G, and for $m \in I^0$, $n \in I$,

$f_{0,n}=e$, the identity of G, and for $m>0$,

$f_{m,n}=u_{n+1}\alpha^{m-1}u_{n+2}\alpha^{m-2}\cdots u_{n+(m-1)}\alpha u_{n+m}$, where $\{u_n : n \in I\}$ is a collection of elements of G with $u_n=e$ if $n>0$.

Proof. Let us first consider $S=G\times I\times I$ under the multiplication (1.1). Since $f_{m,n}\alpha^c=f_{m+c,n}f_{c,m+n}^{-1}$ for $m \in I^0$, $n \in I$, and $c \in I^0$, the associative law may be verified by a routine calculation. Since $(g, m, n)\mathscr{R}(\mathscr{L})(h, p, q)$ if and only if $m=p$ $(n=q)$, S is bisimple. Since $E_S=\{(e, n, n) : n \in I\}$, S is an I-bisimple semigroup.

Let S^* be an I-bisimple semigroup and let $E_{S^*}=(e_i : i \in I)$ with $e_i<e_j$ if and only if $i>j$. If we let $S_i^*=e_jSe_i$, $S^*=\bigcup (S_i^* : i \in I$ and $i\leqq 0)$ with $S_i^*\subseteq S_i^*$ if and only if $i\geqq j$ by Theorem 1.2. Each S_i^* is a bisimple ω-semigroup with

$$E_{S_i^*} = (e_{i+n} : n \in I^0)$$

by Theorem 1.1 and Theorem 1.2. Thus, we may set $S_i^*=G_i\times I^0\times I^0$, where G_i is a group under the multiplication

$$(g, m, n)_i(h, p, q)_i = (g\beta_i^{p-r}h\beta_i^{n-r}, m+p-r, n+q-r)_i \text{ where } r = \min(n, p) \tag{1.2}$$

and β_i is an endomorphism of G_i by Theorem 1.1. Let us write $S_i^*=(G_i, \beta_i)_i$. We note that $e_{i+n}=(e, n, n)_i$ for $n \in I^0$. Let $\beta_0=\alpha_0$ and $G=G_0$. Thus, we may write $(G_0, \beta_0)_0=[G, \alpha_0]_0$. Suppose that $S_{i+1}^*=[G, \alpha_{i+1}]_{i+1}$ while $S_i^*=(G_i, \beta_i)_i$. Since $[g, 0, 0]_{i+1} \in R_{e_{i+1}} \cap L_{e_{i+1}}$, $[g, 0, 0]_{i+1}=(gf_i, 1, 1)_i$ where f_i is a one-to-one mapping of G onto G_i. Utilizing (1.2) we see that f_i is an isomorphism. For $g \in G$, let $g\alpha_i=gf_i\beta_if_i^{-1}$. Clearly, α_i is an endomorphism of G. If $g' \in G_i$, then $g'=gf_i$ for some $g \in G$. Thus, $g'\beta_if_i^{-1}=g'f_i^{-1}\alpha_i$ or $\beta_if_i^{-1}=f_i^{-1}\alpha_i$. Hence, by a straightforward calculation, $(g, m, n)_i\phi_i=[gf_i^{-1}, m, n]_i$ is an isomorphism of $(G_i, \beta_i)_i$ onto $[G, \alpha_i]_i$. Thus, we may set $(g, m, n)_i=[gf_i^{-1}, m, n]_i$. Hence, $[g, 0, 0]_{i+1}=[g, 1, 1]_i$. Thus, we may let $S^*=U(S_i' : i \in I, i\leqq 0)$ where $S_i'=[G, \alpha_i]_i$ and $[g, 0, 0]_{i+1}=[g, 1, 1]_i$. Since $[e, 0, 1]_{i+1} \in R_{e_{i+1}} \cap L_{e_{i+2}}$, $[e, 0, 1]_{i+1}=[z_i, 1, 2]_i$ for some $z_i \in G$. We may deduce from (1.2) that

$$\begin{aligned}[g, m, n]_{i+1} &= [e, m, 0]_{i+1}[g, 0, 0]_{i+1}[e, 0, n]_{i+1} \\ &= [z_i^{-1}\alpha_i^{m-1}\cdots z_i^{-1}\alpha_iz_i^{-1}gz_i\cdot z_i\alpha_i\cdots z_i\alpha_i^{n-1}, m+1, n+1]_i\end{aligned} \tag{1.3}$$

where if $m=0$ $(n=0)$, the left (right) multiplier of g becomes e. We note that

$$\begin{aligned}[z_i^{-1}g\alpha_{i+1}z_i, 2, 2]_i &= [g\alpha_{i+1}, 1, 1]_{i+1} = [e, 1, 1]_{i+1}[g, 0, 0]_{i+1} \\ &= [e, 2, 2]_i[g, 1, 1]_i = [g\alpha_i, 2, 2]_i.\end{aligned}$$

Hence, $g\alpha_i=z_i^{-1}g\alpha_{i+1}z_i$. Thus, by a straightforward calculation,

$$g\alpha_i = (z_{-1}\cdots z_i)^{-1}g\alpha_0(z_{-1}\cdots z_i).$$

For convenience, let

$$u_{i+1} = z_{-1}\cdots z_i \tag{1.4}$$

for $i \geqq -1$. For $n>0$, we set $u_n=e$, the identity of G. Thus,

$$g\alpha_i = u_{i+1}^{-1} g\alpha_0 u_{i+1}. \tag{1.5}$$

We now set $\alpha_0=\alpha$. We will show that $S^* \cong S$ where $S=G\times I\times I$ under the multiplication (1.1). Utilizing (1.1),

$$S_i = (e, i, i)S(e, i, i) = \{(g, m, n) : g \in G, m, n \in I, m \geqq i, n \geqq i\}.$$

As above, S_i is a bisimple ω-semigroup. Let $a_i=(e, i, i+1)$. Then each element of S_i may be uniquely expressed in the form $x=a_i^{-m}ga_i^n \in H_{(e,i+m,i+n)}$ where $g=a_i^m x a_i^{-n} \in H_{(e,i,i)}$. Thus, $(a_i^{-m}ga_i^n)\varphi_i=\langle g, m, n\rangle_i$ where $g=(g, i, i)$ and $a_i g=g\gamma_i a_i$ defines an isomorphism of S_i onto $\langle G, \gamma_i\rangle_i$ (the proof of the last statement is essentially given in [7, p. 164] and will thus be omitted). Since $a_i g=g\gamma_i a_i$,

$$\begin{aligned}(e, i, i+1)(g, i, i) &= (f_{1,i}^{-1}gf_{1,i}, i, i+1) = (u_{i+1}^{-1}gu_{i+1}, i, i+1)\\ &= (g\gamma_i, i, i)(e, i, i+1) = (g\gamma_i, i, i+1).\end{aligned}$$

Hence, by (1.5),

$$g\gamma_i = u_{i+1}^{-1}g\alpha u_{i+1} = g\alpha_i, \quad \text{i.e. } \gamma_i = \alpha_i. \tag{1.6}$$

We note that $a_{i+1}^{-0}ga_{i+1}^0=a_i^{-1}ga_i^1$. Thus, $\langle g, 0, 0\rangle_{i+1}=\langle g, 1, 1\rangle_i$.

We also have $a_{i+1}=a_i^{-1}za_i^2$ where $z=a_ia_{i+1}a_i^{-2}$. Therefore,

$$\begin{aligned}z &= (e, i, i+1)(e, i+1, i+2)(e, i+1, i)(e, i+1, i)\\ &= (u_{i+2}^{-1}u_{i+1}, i, i).\end{aligned}$$

Hence,

$$\langle e, 0, 1\rangle_{i+1} = \langle u_{i+2}^{-1}u_{i+1}, 1, 2\rangle_i = \langle z_i, 1, 2\rangle_i.$$

The last equality is valid by virtue of (1.4). Hence,

$$\langle g, m, n\rangle_{i+1} = \langle z_i^{-1}\alpha_i^{m-1}\cdots z_i^{-1}\alpha_i z_i^{-1}gz_i\cdot z_i\alpha_i\cdots z_i\alpha_i^{n-1}, m+1, n+1\rangle_i \tag{1.7}$$

where if $m=0$ $(n=0)$, the left- (right-) hand multiplier of g is e.

By virtue of (1.6) $[g, m, n]_i\theta_i=\langle g, m, n\rangle_i$ defines an isomorphism of S_i' onto S_i.

Let $x\theta=x\theta_i$ if $x \in S_i'$.

If $x \in S_{i+1}\subseteq S_i$, $x\theta_{i+1}=x\theta_i$ by virtue of (1.3) and (1.7). Thus, it follows easily that θ is an isomorphism of S onto S^*. Q.E.D.

COROLLARY 1.1. *An I-bisimple semigroup S contains an $\mathscr{H}$-class consisting of a single element if and only if $S\cong I\times I$ under the multiplication*

$$(a, b)(c, d) = (a+c-r, b+d-r) \tag{1.8}$$

where $r=\min(b, c)$.

Proof. Let S be an I-bisimple semigroup with the above property. Thus,

$$S \cong G\times I\times I,$$

where G is a group, under the multiplication (1.1). Since $(g, a, b)\mathscr{H}(h, c, d)$ if and only if $a=c$ and $b=d$, $G=\{e\}$ and $S\cong I\times I$ under (1.8). The converse follows from Theorem 1.1.

COROLLARY 1.2. *An I-bisimple semigroup S contains an $\mathscr{H}$-class consisting of a single element if and only if $S\cong I\times I$ under the multiplication*

$$(1.9) \qquad (a, b)(c, d) = (a+c, \max(b+c, d)).$$

Proof. Let S be an I-bisimple semigroup with the above property. Thus, $S\cong I\times I$ under (1.8) by Corollary 1.1. However, $(a, b)\varphi=(b-a, b)$ defines an isomorphism of $S=I\times I$ under (1.8) onto $S=I\times I$ under (1.9). By virtue of the above isomorphism the converse is a consequence of Corollary 1.1.

Thus, the only I-bisimple semigroup containing an $\mathscr{H}$-class consisting of a single element is the extended bicyclic semigroup.

COROLLARY 1.3. *Let S be an I-bisimple semigroup. Thus $\mathscr{H}$ is a congruence on S and $S/\mathscr{H}\cong C^*$.*

Proof. Let $S=(G, C^*, \alpha, u_i)$. By Theorem 1.1, $(g, a, b)\mathscr{H}(h, c, d)$ in S if and only if $a=c$ and $b=d$. Thus, it is easily seen that $\mathscr{H}$ is a congruence on S. Clearly, $S/\mathscr{H}$ contains a trivial $\mathscr{H}$-class. Thus the result is a consequence of Corollary 1.1 or Corollary 1.2.

EXAMPLE. An example of an I-bisimple semigroup with nontrivial distinguished elements.

First suppose that $S\cong G\times I\times I$ under the multiplication

$$(1.10) \qquad (g, m, n)(h, p, q) = (g\alpha^{p-r}h\alpha^{n-r}, m+p-r, n+q-r)$$

where $r=\min(n, p)$ and α is an endomorphism of G. Thus, $S_i=(e, i, i)S(e, i, i) =\{(g, m, n) : g\in G, m, n\in I, m\geqq i, n\geqq i\}$ and $S=\bigcup\{S_i : i\in I \text{ and } i\leqq 0\}$. Let $a_i=(e, i, i+1)$. Thus, as in the proof of Theorem 1.1, $(a_i^{-m}ga_i^n)\varphi_i=(g, m, n)_i$ where $g=(g, i, i)$ and $a_ig=g\alpha_ia_i$ is an isomorphism of S_i onto $(G, \alpha_i)_i$. Hence, since $a_ig=g\alpha_ia_i$,

$$(e, i, i+1)(g, i, i) = (g\alpha_i, i, i)(e, i, i+1),$$

$$(g\alpha, i, i+1) = (g\alpha_i, i, i+1).$$

Thus, $\alpha=\alpha_i$ and $S_i\cong(G, \alpha)_i$. We note that $a_{i+1}^{-0}ga_{i+1}^0=a_i^{-1}ga_i$. Thus

$$(1.11) \qquad (g, 0, 0)_{i+1} = (g, 1, 1)_i.$$

Furthermore, $a_{i+1}=a_i^{-1}za_i^2$ where $z=a_ia_{i+1}a_i^{-2}$. Thus,

$$z = (e, i, i+1)(e, i+1, i+2)(e, i+1, i)(e, i+1, i) = (e, i, i).$$

Hence,

$$(1.12) \qquad (e, 0, 1)_{i+1} = (e, 1, 2)_i.$$

Thus, utilizing (1.10), (1.11), and (1.12), we obtain

$$(1.13) \qquad (g, m, n)_{i+1} = (g, m+1, n+1)_i.$$

By a suitable choice of representative elements, any I-bisimple semigroup with $u_i = e$ must be reduced to an "inverse limit" of the above type. We will give an example of an I-bisimple semigroup where this reduction is not possible.

Let G be the group of integers under addition. Let $S = G \times I \times I$ under the multiplication

$$(1.14) \qquad \begin{aligned} (g, a, b)(h, c, d) &= (g+h2^{b-c}+f_{b-c,d}-f_{b-c,c}, a, b+d-c) \quad \text{if } b \geqq c, \\ &= (g2^{c-b}+h+f_{c-b,b}-f_{c-b,a}, c+a-b, d) \quad \text{if } c \geqq b, \end{aligned}$$

where

$f_{0,n} = 0$ for $n \in I$ while if $m \in I^0$ and $m > 0$,

$f_{m,n} = a_{n+1}2^{m-1} + a_{n+2}2^{m-2} + \cdots + a_{m+n-1}2 + a_{m+n}$, where $a_n = 0$ for $n > 0$ and for $n \leqq 0$

$$\begin{aligned} a_n &= 1 \quad \text{if } n \text{ is odd}, \\ &= 0 \quad \text{if } n \text{ is even}. \end{aligned}$$

By Theorem 1.1, S is an I-bisimple semigroup. As in the proof of Theorem 1.1, $S_i = (e, i, i)S(e, i, i) = \{(g, m, n) : g \in G, m, n \in I, m \geqq i, n \geqq i\}$ and

$$S = \bigcup (S_i : i \in I, i \leqq 0).$$

Let $(y_i : i \in I, i \leqq 0)$ be any sequence of elements of G and let $a_i = (y_i, i, i+1)$. Thus, as usual, $(a_i^{-m} g a_i^n)\varphi_i = (g, m, n)_i$ where $a_i g = g\gamma_i a_i$ and $g = (g, i, i)$ defines an isomorphism of S_i onto $(G, \gamma_i)_i$.

We again note that

$$(1.15) \qquad a_{i+1} = a_i^{-1} z a_i^2$$

where $z = a_i a_{i+1} a_i^{-2}$. Thus, by (1.14),

$$\begin{aligned} z &= (y_i, i, i+1)(y_{i+1}, i+1, i+2)(-y_i, i+1, i)(-y_i, i+1, i) \\ &= (y_{i+1} - 2y_i + a_{i+1} - a_{i+2}, i, i)_i. \end{aligned}$$

Hence, by (1.15),

$$(0, 0, 1)_{i+1} = (y_{i+1} - 2y_i + a_{i+1} - a_{i+2}, 1, 2).$$

Thus, by (1.12), if S is an I-bisimple semigroup "without factor terms", there must exist a sequence $\{y_i : i \in I, i \leqq 0\}$ of elements of G such that $y_{i+1} - 2y_i + a_{i+1} - a_{i+2} = 0$ for all $i \in I$ where $i \leqq 0$. Hence, $y_0 = 2y_{-1}$ while if $i \leqq -2$

$$\begin{aligned} y_{i+1} &= 2y_i - 1 \quad \text{if } i \text{ is even}, \\ &= 2y_i + 1 \quad \text{if } i \text{ is odd}. \end{aligned}$$

To simplify the notation, let $b_n = y_{-n}$. Thus, $b_0 = 2b_1$ while if $n \geqq 1$

$$\begin{aligned} b_n &= 2b_{n+1} - 1 \quad \text{if } n \text{ is odd,} \\ &= 2b_{n+1} + 1 \quad \text{if } n \text{ is even.} \end{aligned}$$

Let $b_0 = 2x_0$ where x_0 is chosen arbitrarily. Hence, $b_1 = x_0$. Thus, for $n \geqq 1$,

$$\begin{aligned} b_{n+1} &= \frac{x_0 + 1 - 2 + 2^2 + \cdots + (-1)^{n-1}2^{n-1}}{2^n}, \\ &= \frac{\frac{1}{3}(1-(-2)^n) + x_0}{2^n}. \end{aligned}$$

Hence, for $n \geqq 1$,

$$b_{2n+1} = \frac{\frac{1}{3}(1-2^{2n}) + x_0}{2^{2n}}.$$

Thus, $\lim_{n\to\infty} b_{2n+1} = -\frac{1}{3}$ which is impossible since each b_n is an integer.

REMARK. The above example is related to an example communicated to the author by Professor A. H. Clifford.

2. **The homomorphism theory.** In this section, we give necessary and sufficient conditions for two *I*-bisimple semigroups to be isomorphic, and we determine the homomorphisms of an *I*-bisimple semigroup onto an *I*-bisimple semigroup. The following theorem is obtained from [9, Theorem 2.3, Theorem 1.2, and Theorem 1.1]. A proof will be given elsewhere [14].

THEOREM 2.1 *Let $S = (G, C, \alpha)$ and $S^* = (G^*, C, \beta)$ be bisimple ω-semigroups. Let f be a homomorphism of G onto G^* and $z \in G^*$ such that $\alpha f = f\beta C_z$ where $xC_z = zxz^{-1}$ for $x \in G^*$. For each $(g, m, n) \in S$ define*

$$(2.1) \qquad (g, m, n)\theta = (z^{-1}\beta^{m-1}\cdots z^{-1}\beta z^{-1}(gf)z\cdot z\beta\cdots z\beta^{n-1}, m, n)$$

if $m>0$, $n>0$. If $m=0$ ($n=0$), the left- (right-) multiplier of gf is e^*, the identity of G^*.

Then, θ is a homomorphism of S onto S^* and conversely every such homomorphism is obtained in this fashion. θ is an isomorphism if and only if f is an isomorphism.

The condition for two bisimple ω-semigroups to be isomorphic (homomorphic) was given by Reilly [7] (Munn and Reilly [4]), although the isomorphism (homomorphism) was not exhibited.

If G is a group and $y \in G$, we will denote the inner automorphism of G determined by y by C_y, i.e. $xC_y = yxy^{-1}$ ($x \in G$).

THEOREM 2.2. *Let $S = (G, C^*, \alpha, u_i)$ and $S^* = (G^*, C^*, \beta, v_i)$ be I-bisimple semigroups. Then, S is isomorphic to S^* if and only if there exists a sequence*

$$\{z_i : i \in I, i \leqq 0\}$$

of elements of G^*, *a sequence* $\{f_i : i \in I, i \leqq 0\}$ *of isomorphisms of* G *onto* G^*, *and* $a \in I$ *such that for all* $i \in I$, $i \leqq 0$

$$z_{i+1}v_{i+a+2}^{-1}v_{i+a+1} = ((u_{i+2}^{-1}u_{i+1})f_iC_{z_i^{-1}})(z_i\beta C_{v_{i+a+1}^{-1}}), \tag{2.2}$$

$$f_i = f_{i+1}C_{z_i}, \tag{2.3}$$

$$\alpha C_{u_{i+1}^{-1}}f_i = f_i\beta C_{z_iv_{i+a+1}^{-1}}. \tag{2.4}$$

Proof. As in the proof of Theorem 1.1, $S = U(S_i : i \in I, i \leqq 0)$ where $S_i = (G, \alpha_i)_i$,

$$\alpha_i = \alpha C_{u_{i+1}^{-1}}, \tag{2.5}$$

$$(g, m, n)_{i+1} = (s_i^{-1}\alpha_i^{m-1}\cdots s_i^{-1}\alpha_is_i^{-1}gs_i\cdot s_i\alpha_i\cdots s_i\alpha_i^{n-1}, m+1, n+1)_i, \tag{2.6}$$

where if $m=0$ ($n=0$) the left (right) multiplier of g is e, the identity of G, and

$$s_i = u_{i+2}^{-1}u_{i+1}. \tag{2.7}$$

Similarly, $S^* = U(S_i^* : i \in I)$ where $S_i^* = [G^*, \beta_i]_i$ (also see p. 371)

$$\beta_i = \beta C_{v_{i+1}^{-1}}, \tag{2.8}$$

$$[g, m, n]_{i+1} = [t_i^{-1}\beta_i^{m-1}\cdots t_i^{-1}\beta_it_i^{-1}gt_i\cdot t_i\beta_i\cdots t_i\beta_i^{n-1}, m+1, n+1]_i, \tag{2.9}$$

where if $m=0$ ($n=0$), the left (right) multiplier of g is e^*, the identity of G^*, and

$$t_i = v_{i+2}^{-1}v_{i+1}. \tag{2.10}$$

First suppose that θ is an isomorphism of S onto S^*. Suppose that $(e, 0, 0)_0\theta = [e, 0, 0]_a$. Thus, θ induces an isomorphism θ_0 of $S_0 = (e, 0, 0)_0S(e, 0, 0)_0$ onto $[e, 0, 0]_aS^*[e, 0, 0]_a = S_a^*$. Hence, θ induces an isomorphism θ_i of S_i onto S_{i+a}^* for each $i \in I$ with $i \leqq 0$. Thus, by virtue of Theorem 2.1, for each i there exists an isomorphism f_i of G onto G^* and $z_i \in G^*$ such that

$$\alpha_if_i = f_i\beta_{i+a}C_{z_i}, \tag{2.11}$$

and

$$(g, m, n)_i\theta_i = [z_i^{-1}\beta_{i+a}^{m-1}\cdots z_i^{-1}\beta_{i+a}z_i^{-1}gf_iz_i\cdot z_i\beta_{i+a}\cdots z_i\beta_{i+a}^{n-1}, m, n]_{i+a}, \tag{2.12}$$

where if $m=0$ ($n=0$) the left (right) multiplier of gf is e^*.

Combining (2.5), (2.8), and (2.11), we obtain (2.4).

If $x \in S_{i+1} \subseteq S_i$, $x\theta = x\theta_{i+1} = x\theta_i$. Thus, since $(e, 0, 1)_{i+1} = [s_i, 1, 2]_i$ by (2.6),

$$[z_{i+1}, 0, 1]_{i+a+1} = [z_i^{-1}(s_if_i)z_i(z_i\beta_{i+a}), 1, 2]_{i+a} \tag{2.13}$$

by virtue of (2.12). However, by (2.9),

$$[z_{i+1}, 0, 1]_{i+a+1} = [z_{i+1}t_{i+a}, 1, 2]_{i+a}. \tag{2.14}$$

Thus, combining (2.13), (2.14), (2.10), (2.7), and (2.8), we obtain (2.2).

Furthermore, by (2.6), $(g, 0, 0)_{i+1}=(g, 1, 1)_i$. Hence, $(g, 0, 0)_{i+1}\theta_{i+1}=(g, 1, 1)_i\theta_i$. Thus, by (2.12),

$$[gf_{i+1}, 0, 0]_{i+a+1} = [z_i^{-1}gf_iz_i, 1, 1]_{i+a}. \tag{2.15}$$

However, by (2.9), we have

$$[gf_{i+1}, 0, 0]_{i+a+1} = [gf_{i+1}, 1, 1]_{i+a}. \tag{2.16}$$

Thus, combining (2.15) and (2.16) ,we obtain (2.3).

Let us now assume the conditions of the theorem are valid. By (2.4), (2.5), (2.8), and Theorem 2.1, (2.12) defines an isomorphism of S_i onto S^*_{i+a}.

By (2.12), (2.9), (2.10), (2.2), (2.7), (2.8), and (2.12),

$$\begin{aligned}(e, 0, 1)_{i+1}\theta_{i+1} &= [z_{i+1}, 0, 1]_{i+a+1} = [z_{i+1}t_{i+a}, 1, 2]_{i+a}\\ &= [z_{i+1}v_{i+a+2}^{-1}v_{i+a+1}, 1, 2]_{i+a}\\ &= [((u_{i+2}^{-1}u_{i+1})f_iC_{z_i^{-1}})z_i\beta C_{v_{i+a+1}^{-1}}, 1, 2]_{i+a}\\ &= [(s_if_iC_{z_i^{-1}})(z_i\beta_{i+a}), 1, 2]_{i+a}\\ &= [z_i^{-1}(s_if_i)z_i(z_i\beta_{i+a}), 1, 2]_{i+a} = (s_i, 1, 2)_i\theta_i.\end{aligned}$$

Thus,

$$(e, 0, n)_{i+1}\theta_{i+1} = (s_i\cdot s_i\alpha_i\cdots s_i\alpha_i^{n-1}, 1, n+1)_i\theta_i \text{ if } n \geqq 1. \tag{2.17}$$

By taking inverses, we obtain

$$(e, n, 0)_{i+1}\theta_{i+1} = (s_i^{-1}\alpha_i^{n-1}\cdots s_i^{-1}\alpha_is_i^{-1}, n+1, 1)_i\theta_i. \tag{2.18}$$

By (2.12), (2.9), (2.3), and (2.12),

$$\begin{aligned}(g, 0, 0)_{i+1}\theta_{i+1} &= [gf_{i+1}, 0, 0]_{i+a+1} = [gf_{i+1}, 1, 1]_{i+a}\\ &= [z_i^{-1}gf_iz_i, 1, 1]_{i+a} = (g, 1, 1)_i\theta_i.\end{aligned} \tag{2.19}$$

Thus, combining (2.17), (2.18), and (2.19), we obtain

$$\begin{aligned}(g, m, n)_{i+1}\theta_{i+1} &= (e, m, 0)_{i+1}\theta_{i+1}(g, 0, 0)_{i+1}\theta_{i+1}(e, 0, n)_{i+1}\theta_{i+1}\\ &= (s_i^{-1}\alpha_i^{m-1}\cdots s_i^{-1}\alpha_is_i^{-1}gs_i\cdot s_i\alpha_i\cdots s_i\alpha_i^{n-1}, m+1, n+1)_i\theta_i.\end{aligned} \tag{2.20}$$

Let us define

$$x\theta = x\theta_i \quad \text{if } x \in S_i. \tag{2.21}$$

Hence, by (2.6) and (2.20), θ defines an isomorphism of S onto S^*. Let N denote the natural numbers.

THEOREM 2.3. *Let $S=(G, C^*, \alpha, u_i)$ and $S^*=(G^*, C^*, \beta, v_i)$ be I-bisimple semigroups. Let $\{z_i : i \in I, i \leqq 0\}$ be a sequence of elements of G^*, and let $\{f_i : i \in I, i \leqq 0\}$*

be a sequence of homomorphisms of G onto G^*, and let a be an element of I such that for all $i \in I \backslash N$

$$z_{i+1}v_{i+a+2}^{-1}v_{i+a+1} = ((u_{i+2}^{-1}u_{i+1})f_iC_{z_i^{-1}})(z_i\beta C_{v_{i+a+1}^{-1}}), \tag{2.22}$$

$$f_i = f_{i+1}C_{z_i}, \tag{2.23}$$

$$\alpha C_{u_{i+1}^{-1}}f_i = f_i\beta C_{z_iv_{i+a+1}^{-1}}. \tag{2.24}$$

For each element $(g, m, n)_i \in S_i (i \in I\backslash N)$, define

$$(g, m, n)_i\theta = [z_i^{-1}\beta_{i+a}^{m-1}\cdots z_i^{-1}\beta_{i+a}z_i^{-1}gf_iz_i\cdot z_i\beta_{i+a}\cdots z_i\beta_{i+a}^{n-1}, m, n]_{i+a}$$

where the square brackets denote an element of S^* and where if $m=0$ $(n=0)$ the left (right) multiplier of gf_i is e^*, the identity of G^*.

Then, θ is a homomorphism of S onto S^* and conversely every such homomorphism is obtained in this fashion.

Proof. Let θ be a homomorphism of S onto S^*. Let us suppose that $(e, 0, 0)_i\theta = (e, 0, 0)_{a_i}$. Hence θ induces a homomorphism θ_i of S_i onto $S_{a_i}^*$. By Theorem 2.1, θ_i is given by

$$(g, m, n)_i\theta_i = [z_i^{-1}\beta_{a_i}^{m-1}\cdots z_i^{-1}\beta_{a_i}z_i^{-1}gf_iz_i\cdot z_i\beta_{a_i}\cdots z_i\beta_{a_i}^{n-1}m, n]_{a_i}, \tag{2.25}$$

where if $m=0$ $(n=0)$ the left (right) multiplier of gf_i is e^*, the identity of G^* and where, $z_i \in G^*$ and f_i is a homomorphism of G onto G^* such that

$$\alpha_if_i = f_i\beta_{a_i}C_{z_i}. \tag{2.26}$$

As usual, α_i and β_{a_i} are given by (2.5) and (2.8).

Since $(e, 0, 0)_{i+1} = (e, 1, 1)_i$ by (2.6), $(e, 0, 0)_{i+1}\theta_{i+1} = (e, 1, 1)_i\theta_i$. Thus, by (2.25),

$$[e, 0, 0]_{a_{i+1}} = [z_i^{-1}gf_iz_i, 1, 1]_{a_i}. \tag{2.27}$$

Clearly, $a_{i+1} \geqq a_i$, i.e., $a_{i+1} = a_i + b_i$ for some $b_i \in I^0$. Hence, by (2.9),

$$[e, 0, 0]_{a_{i+1}} = [e, 0, 0]_{a_i+b_i} = [e, b_i, b_i]_{a_i}.$$

Thus, by (2.27), $b_i = 1$ and $a_{i+1} = a_i + 1$. Hence, if we let $a_0 = a$, $a_i = a + i$ for all $i \in I$, $i \leqq 0$. Hence the remainder of the proof parallels that of Theorem 2.2.

3. **The maximal group homomorphic image.** In this section we describe the maximal cancellative homomorphic image of an ω-right cancellative semigroup(1), the maximal group homomorphic image of a bisimple ω-semigroup and finally the maximal group homomorphic image of an I-bisimple semigroup.

We first review constructions of Clifford [1] and of the author [8].

Let S be a semigroup with identity 1. The set of elements of S having a right inverse with respect to 1 is called the right unit subsemigroup of S. Let S be a

(1) The terminology of "ω-right cancellative semigroup" and "bisimple ω-semigroup" leads to some confusion. Thus, in [7], we employ the term "ω-bisimple semigroup".

bisimple inverse semigroup with identity and let P denote the right unit subsemigroup of S. The principal left ideals of P form a semilattice (with respect to inclusion). From each $\mathscr{L}$-class of P pick a fixed representative element and let $a \vee b$ $(a, b \in P)$ denote the representative element of the $\mathscr{L}$-class containing c where $Pa \cap Pb = Pc$. Define $(a^*b)b = a \vee b$. Then, $S \cong P \times P$ under the following definition of equality and multiplication:

(3.1) $(a, b) = (c, d)$ if $a = uc$ and $b = ud$ where u is a unit of P (an element of P which has a two-sided inverse with respect to 1, the identity of P).

(3.2) $(a, b)(c, d) = ((c^*b)a, (b^*c)d)$.

This construction is due to Clifford [1].

Let us now review the construction of the author [8] for describing the maximal group homomorphic image of a bisimple inverse semigroup with identity.

Let us define the following relation on P:

(3.3) If $a, b \in P$, $a\eta b$ if and only if there exists $h \in P$ such that $ha = hb$.

Then η is the minimal cancellative congruence on P or $\bar{P} = P/\eta$ is the maximal cancellative homomorphic image of P. Let $p \to \bar{p}$ denote the canonical homomorphism of P onto $\bar{P}$. Let $\bar{a}, \bar{b} \in \bar{P}$. We consider the set F of all pairs of elements of $\bar{P}$ writing them as fractions $\bar{b}/\bar{a}$. The relation $=$ between these fractions shall be defined thus:

$$\bar{b}/\bar{a} = \bar{d}/\bar{c} \tag{3.4}$$

shall mean that elements $\bar{x}$ and $\bar{y}$ exist in $\bar{P}$ such that $\bar{x}\bar{a} = \bar{y}\bar{c}$ and $\bar{x}\bar{b} = \bar{y}\bar{d}$.

The definition of the product is

$$\bar{b}/\bar{a} \cdot \bar{d}/\bar{c} = \bar{k}\bar{d}/\bar{h}\bar{a} \quad \text{where } \bar{h}\bar{b} = \bar{k}\bar{c}. \tag{3.5}$$

F is a group and the isomorphism of $\bar{P}$ into F is given by $\bar{a} \to \bar{a}/\bar{1}$.

THEOREM 3.1 (WARNE [8]). *With S, P, $\bar{P}$, and F as above the mapping $\phi: (a, b) \to \bar{b}/\bar{a}$ is a homomorphism of S onto F, and F is thereby the maximal group homomorphic image of S.*

In [14], we called a right cancellative semigroup P with identity an ω-right cancellative semigroup if and only if its ideal structure (the set of principal left ideals of P ordered by inclusion) is order isomorphic to I^0 under the reverse of the usual order. The structure of ω-right cancellative semigroups was given by Rees.

THEOREM 3.2 (REES [6]). *S is an ω-right cancellative semigroup if and only if $S \cong G \times I^0$, where G is a group, under the multiplication*

$$(g, m)(h, n) = (g(h\alpha^m), m+n) \tag{3.6}$$

where α is an endomorphism of G, α^0 being interpreted as the identity transformation.

If P is an ω-right cancellative semigroup, we will write $P = (G, I^0, \alpha)$ where G is the structure group and α is the structure endomorphism of P.

THEOREM 3.3. *Let $P=(G, I^0, \alpha)$ be an ω-right cancellative semigroup and let e be the identity of G. If $N=\{g \in G \mid g\alpha^n = e$ for some $n \in I^0\}$, N is a normal subgroup of G. If $(xN)\theta=(x\alpha)N$, $x \in G$, θ is an endomorphism of G/N. Let $g \to \bar{g}$ denote the natural homomorphism of G onto G/N. The maximal cancellative homomorphic image $\bar{P}$ of P is $(G/N, I^0, \theta)$ and the canonical homomorphism of P onto $\bar{P}$ is given by $(g, m)\eta=(\bar{g}, m)$.*

Proof. Let η be the minimum cancellative congruence relation on P. By (3.6) and (3.3), $(g, k)\eta(h, j)$ if and only if $k=j$ and there exists $s \in I^0$ such that $g\alpha^s=h\alpha^s$. If we define $A\rho B$ ($A, B \in G$) if and only if $A\alpha^c=B\alpha^c$ for some $c \in I^0$, ρ is a congruence relation on G. Let N denote the congruence class containing the identity, i.e., $N=\{A \in G \mid A\alpha^c=e$ for some $c \in I^0\}$. It is easy to see that the mapping $(AN)\theta=(A\alpha)N$ is an endomorphism of G/N. Let $\bar{P}$ be the maximal cancellative homomorphic image of P under the natural homomorphism

$$(g, j) \to \overline{(g, j)}.$$

If we define

$$\overline{(g, j)}\delta = (\bar{g}, j)$$

it is easily seen that δ is an isomorphism of $\bar{P}$ onto $(G/N, I^0, \theta)$.

REMARK 3.1. By the proof of [12, Theorem 3.1], S is a bisimple ω-semigroup (G, C, α) if and only if its right unit subsemigroup P is the ω-right cancellative semigroup (G, I^0, α).

We now completely describe the maximal group homomorphic image of a bisimple ω-semigroup (including the defining homomorphism). If σ is an equivalence relation on a set X, we let x_σ denote the equivalence class containing the element x of X.

THEOREM 3.4. *Let $S=(G, C, \alpha)$ be a bisimple ω-semigroup and let e denote the identity of G. If $N=\{g \in G \mid g\alpha^n=e$ for some $n \in I^0\}$, N is a normal subgroup of G. If $(xN)\theta=(x\alpha)N$, $x \in G$, θ is an endomorphism of G/N. Let $g \to \bar{g}$ denote the natural homomorphism of G onto G/N. Let us define a relation σ on $G/N \times (I^0)^2$ by the rule*

$$((\bar{g}, a, b), (\bar{h}, c, d)) \in \sigma \tag{3.7}$$

if and only if there exists $x, y \in I^0$ such that $x+a=y+c$, $x+b=y+d$, and $\bar{g}\theta^x=\bar{h}\theta^y$. Then, σ is an equivalence relation on $G/N \times (I^0)^2$. Furthermore, the rule

$$(\bar{g}, a, b)_\sigma(\bar{h}, c, d)_\sigma = (\bar{g}\theta^c\bar{h}\theta^b, a+c, b+d)_\sigma \tag{3.8}$$

defines a binary operation on $G/N \times (I^0)^2/\sigma = V$ whereby V becomes a group which is the maximal group homomorphic image of S.

The canonical homomorphism of S onto V is given by $(g, a, b)\gamma=(\bar{g}, a, b)_\sigma$.

Proof. By Theorem 3.3 and Remark 3.1, $\bar{P}=(G/N, I^0, \theta)$. We will utilize (3.4) and (3.5) to determine the group of fractions F of $\bar{P}$. Utilizing (3.4), it is easily seen that

$$(\bar{B}, b)/(\bar{A}, a) = (\bar{D}, d)/(\bar{C}, c) \tag{3.9}$$

if and only if there exists $x, y \in I^0$ such that $x+a=y+c$, $x+b=y+d$, and

$$(\bar{A}^{-1}\bar{B})\theta^x = (\bar{C}^{-1}\bar{D})\theta^y.$$

By (3.5),

$$(\bar{B}, b)/(\bar{A}, a)\cdot(\bar{D}, d)/(\bar{C}, c) = (\bar{K}(\bar{D}\theta^k), k+d)/(\bar{H}(\bar{A}\theta^h), h+a)$$

where $\bar{H}(\bar{B}\theta^h)=\bar{K}(\bar{C}\theta^k)$ and $h+b=k+c$. Thus, applying (3.9) with $x=c$ and $y=h$, we obtain

$$(3.9)' \qquad (\bar{B}, b)/(\bar{A}, a)\cdot(\bar{D}, d)/(\bar{C}, c) = ((\bar{C}^{-1}\bar{D})\theta^b, b+d)/((\bar{B}^{-1}\bar{A})\theta^c, a+c).$$

It is easy to see that σ is an equivalence relation and that V is a groupoid.

Hence by (3.9) and (3.9)′

$$(3.10) \qquad ((\bar{B}, b)/(\bar{A}, a))\varphi = (\bar{A}^{-1}\bar{B}, a, b)_\sigma$$

defines an isomorphism of F onto V.

Let S^* be the semigroup constructed from $P=(G, I^0, \alpha)$ (see Remark 3.1) by means of the Clifford construction. Thus, utilizing Theorem 3.2, (3.1), and [12, p. 572, Equation 3.4], it is easily seen that $(g, a, b)\lambda=((A, a), (B, b))$ where $g=A^{-1}B$, defines an isomorphism of $S=(G, C, \alpha)$ onto S^*.

By Theorem 3.1, F is the maximal group homomorphic image of S^* under the homomorphism

$$((A, a), (B, b))\phi = (\bar{B}, b)/(\bar{A}, a).$$

Hence, V is the maximal group homomorphic image of S under the homomorphism $(g, a, b)\gamma=(g, a, b)\lambda\phi\varphi=(\bar{g}, a, b)$.

Note. Another construction of the maximal group homomorphic image of S is given in [4].

The following result is obtained from [9, Theorem 2.3 and Theorem 1.1] and its proof will be given elsewhere [14].

Theorem 3.5 *Let $S=(G, C, \alpha)$ be a bisimple ω-semigroup and let G^* be a group. Let f be a homomorphism of G into G^* such that $fC_z=\alpha f$ where $xC_z=zxz^{-1}$ for $x \in G^*$. Then, $(g, m, n)\phi=z^{-m}gfz^n$ is a homomorphism of S into G^* and, conversely, every such homomorphism is obtained in this fashion.*

Theorem 3.6. *Let $S=(G, C^*, \alpha, u_i)$ be an I-bisimple semigroup and let e be the identity of G. If $N=\{g \in G \mid g\alpha^n=e \text{ for some } n \in I^0\}$, N is a normal subgroup of G. If $(xN)\theta=(x\alpha)N$, θ is an endomorphism of G/N. Let $g \to \bar{g}$ be the natural homomorphism of G onto G/N. Let us define a relation σ on $G/N\times(I^0)^2$ by the rule $((\bar{g}, a, b), (\bar{h}, c, d)) \in \sigma$ if and only if there exists $x, y \in I^0$ such that $x+a=y+c$, $x+b=y+d$, and $\bar{g}\theta^x=\bar{h}\theta^y$. Then, σ is an equivalence relation on $G/N\times(I^0)^2$. Furthermore, the rule $(\bar{g}, a, b)_\sigma(\bar{h}, c, d)_\sigma=(\bar{g}\theta^c\bar{h}\theta^b, a+c, b+d)_\sigma$ defines a binary operation on $G/N\times(I^0)^2/\sigma=H$ whereby H becomes a group which is the maximal group homomorphic image of S. The homomorphism of S onto H is given by*

$$(3.11) \qquad (g, m, n)_i\phi = (x_i^{-1}\theta^{m-1}\cdots x_i^{-1}\theta x_i^{-1}\bar{g}\delta_i x_i\cdot x_i\theta\cdots x_i\theta^{n-1}, m, n)_\sigma$$

where if $m=0$ $(n=0)$ *the left* (*right*) *multiplier of* $\bar{g}\delta_i$ *is* $\bar{e}$ *and where*

$$
\begin{aligned}
x_0 &= \bar{e},\\
x_{-1} &= \bar{u}_0^{-1} \quad \textit{while for } i \leqq -2,\\
x_i &= \bar{u}_0^{-1}(\bar{u}_{-1}^{-1}\theta)\cdots\bar{u}_{i+1}^{-1}\theta^{-(i+1)}\bar{u}_{i+2}\theta^{-(i+1)}\bar{u}_{i+3}\theta^{-(i+2)}\cdots\bar{u}_0\theta,\\
\bar{g}\delta_0 &= \bar{g} \quad \textit{while if } i \leqq -1,\\
\bar{g}\delta_i &= \bar{u}_0^{-1}\cdot\bar{u}_{-1}^{-1}\theta\cdots\bar{u}_{i+1}^{-1}\theta^{-(i+1)}\bar{g}\theta^{-i}\bar{u}_{i+1}\theta^{-(i+1)}\cdots\bar{u}_{-1}\theta\bar{u}_0.
\end{aligned}
$$

Proof. We first use Theorem 3.5 to determine a homomorphism ϕ_i of S_i into H for each $i\in I$ with $i\leqq 0$. Let x_i snd δ_i be defined as in the statement of the theorem. In the notation of Theorem 3.5, let $G^*=H$, $z_i=(x_i,0,1)_\sigma$, and $gf_i=(\bar{g}\delta_i,0,0)_\sigma$. Clearly, f_i is a homomorphism of G into H. Let us first verify the condition of Theorem 3.5.

$$
\begin{aligned}
z_i g f_i z_i^{-1} &= (x_i,0,1)_\sigma(\bar{g}\delta_i,0,0)_\sigma(x_i^{-1},1,0)_\sigma\\
&= (x_i\bar{g}\delta_i\theta,0,1)_\sigma(x_i^{-1},1,0)_\sigma\\
&= ((x_i\bar{g}\delta_i\theta)\theta x_i^{-1}\theta,1,1)_\sigma\\
&= (x_i\bar{g}\delta_i\theta x_i^{-1},0,0)_\sigma\\
&= (\bar{u}_0^{-1}\cdot\bar{u}_{-1}^{-1}\theta\cdots\bar{u}_{i+1}^{-1}\ \theta^{-(i+1)}\{\bar{u}_{i+2}\theta^{-(i+1)}\bar{u}_{i+3}\theta^{-(i+2)}\cdots\bar{u}_0\theta\\
&\qquad\cdot\bar{u}_0^{-1}\theta\bar{u}_{-1}^{-1}\theta^2\cdots\bar{u}_{i+2}^{-1}\theta^{-(i+1)}\}\bar{u}_{i+1}^{-1}\theta^{-i}\bar{g}\theta^{-i+1}\\
&\qquad\cdot\bar{u}_{i+1}\theta^{-i}\{\bar{u}_{i+2}\theta^{-(i+1)}\cdots\bar{u}_{-1}\theta^2\bar{u}_0\theta\\
&\qquad\cdot\bar{u}_0^{-1}\theta\cdots\bar{u}_{i+2}^{-1}\theta^{-(i+1)}\}\bar{u}_{i+1}\theta^{-(i+1)}\cdots\bar{u}_{-1}\theta\bar{u}_0,0,0)_\sigma\\
&= (\bar{u}_0^{-1}\cdot\bar{u}_{-1}^{-1}\theta\cdots\bar{u}_{i+1}^{-1}\theta^{-(i+1)}\bar{u}_{i+1}^{-1}\theta^{-i}\bar{g}\theta^{-i+1}\\
&\qquad\cdot\bar{u}_{i+1}\theta^{-i}\bar{u}_{i+1}\theta^{-(i+1)}\cdots\bar{u}_{-1}\theta\bar{u}_0,0,0)_\sigma.
\end{aligned}
\tag{3.12}
$$

By (2.5), $g\alpha_i=u_{i+1}^{-1}g\alpha u_{i+1}$. Thus,

$$\overline{g\alpha_i} = \bar{u}_{i+1}^{-1}\overline{g\alpha}\bar{u}_{i+1} = \bar{u}_{i+1}^{-1}\bar{g}\theta\bar{u}_{i+1}. \tag{3.13}$$

The last equality follows from the statement of the theorem. Thus, using (3.13),

$$
\begin{aligned}
g\alpha_i f_i &= (\overline{g\alpha_i}\delta_i,0,0)_\sigma = ((\bar{u}_{i+1}^{-1}\bar{g}\theta\bar{u}_{i+1})\delta_i,0,0)_\sigma\\
&= (\bar{u}_0^{-1}\cdot\bar{u}_1^{-1}\theta\cdots\bar{u}_{i+1}^{-1}\theta^{-(i+1)}(\bar{u}_{i+1}^{-1}\bar{g}\theta\bar{u}_{i+1})\theta^{-i}\bar{u}_{i+1}\theta^{-(i+1)}\cdots\bar{u}_{-1}\theta\bar{u}_0,0,0)_\sigma\\
&= (\bar{u}_0^{-1}\cdot\bar{u}_{-1}^{-1}\theta\cdots\bar{u}_{i+1}^{-1}\theta^{-(i+1)}\bar{u}_{i+1}^{-1}\theta^{-i}\bar{g}\theta^{-i+1}\bar{u}_{i+1}\theta^{-i}\\
&\qquad\cdot\bar{u}_{i+1}\theta^{-(i+1)}\cdots\bar{u}_{-1}\theta\bar{u}_0,0,0)_\sigma.
\end{aligned}
\tag{3.14}
$$

Thus, comparing (3.12) and (3.14), we see that $z_igf_iz_i^{-1}=g\alpha_if_i$ as desired. Hence, by Theorem 3.5,

$$
\begin{aligned}
(g,m,n)_i\phi_i &= (x_i^{-1},1,0)_\sigma^m(\bar{g}\delta_i,0,0)_\sigma(x_i,0,1)_\sigma^n\\
&= (x_i^{-1}\theta^{m-1}\cdots x_i^{-1}\theta x_i^{-1}\bar{g}\delta_i x_i\cdot x_i\theta\cdots x_i\theta^{n-1},m,n)_\sigma,
\end{aligned}
$$

where if $m=0$ $(n=0)$ the left (right) multiplier of $\bar{g}\delta_i$ is $\bar{e}$, defines a homomorphism of S_i into H.

We note that $(g, m, n)_0\phi_0=(\bar{g}, m, n)_\sigma$. Hence, by Theorem 3.4, ϕ_0 is a homomorphism of S_0 onto H.

Let us define $x\phi=x\phi_i$ if $x\in S_i$. We will show that ϕ is a homomorphism of S onto H. We note that

$$\begin{aligned}(g, 1, 1)_i\phi_i &= (x_i^{-1}\bar{g}\delta_i x_i, 1, 1)_\sigma \\ &= (\bar{u}_0^{-1}\theta\cdots\bar{u}_{i+2}^{-1}\theta^{-(i+1)}\{\bar{u}_{i+1}\theta^{-(i+1)}\cdots\bar{u}_{-1}\theta\bar{u}_0\bar{u}_0^{-1}\cdot\bar{u}_{-1}^{-1}\theta\cdots\bar{u}_{i+1}^{-1}\theta^{-(i+1)}\} \\ &\quad\cdot\bar{g}\theta^{-i}\{\bar{u}_{i+1}\theta^{-(i+1)}\cdots\bar{u}_{-1}\theta\bar{u}_0\cdot\bar{u}_0^{-1}\cdot\bar{u}_{-1}^{-1}\theta\cdots\bar{u}_{i+1}^{-1}\theta^{-(i+1)}\} \\ &\qquad\cdot\bar{u}_{i+2}\theta^{-(i+1)}\cdots\bar{u}_0\theta, 1, 1)_\sigma \\ &= (\bar{u}_0^{-1}\theta\cdots\bar{u}_{i+2}^{-1}\theta^{-(i+1)}\bar{g}\theta^{-i}\bar{u}_{i+2}\theta^{-(i+1)}\cdots\bar{u}_0\theta, 1, 1)_\sigma \\ &= (\bar{g}\delta_{i+1}\theta, 1, 1)_\sigma = (\bar{g}\delta_{i+1}, 0, 0)_\sigma = (g, 0, 0)_{i+1}\phi_{i+1}.\end{aligned}$$

Let $s_i=u_{i+2}^{-1}u_{i+1}$. Thus,

$$\begin{aligned}(s_i, 1, 2)_i\phi_i &= (x_i^{-1}\bar{s}_i\delta_i x_i\cdot x_i\theta, 1, 2)_\sigma \\ &= (\bar{u}_0^{-1}\theta\cdots\bar{u}_{i+2}^{-1}\theta^{-(i+1)}\{\bar{u}_{i+1}\theta^{-(i+1)}\cdots\bar{u}_{-1}\theta\bar{u}_0\cdot\bar{u}_0^{-1}\bar{u}_{-1}^{-1}\theta\cdots\bar{u}_{i+1}^{-1}\theta^{-(i+1)}\} \\ &\quad\cdot\bar{s}_i\theta^{-i}\{\bar{u}_{i+1}\theta^{-(i+1)}\cdots\bar{u}_{-1}\theta\bar{u}_0\cdot\bar{u}_0^{-1}\bar{u}_{-1}^{-1}\theta\cdots\bar{u}_{i+1}^{-1}\theta^{-(i+1)}\} \\ &\quad\cdot\{\bar{u}_{i+2}\theta^{-(i+1)}\cdots\bar{u}_0\theta\cdot\bar{u}_0^{-1}\theta\cdots\bar{u}_{i+2}^{-1}\theta^{-(i+1)}\} \\ &\qquad\cdot\bar{u}_{i+1}^{-1}\theta^{-i}\bar{u}_{i+2}\theta^{-i}\cdots\bar{u}_0\theta^2, 1, 2)_\sigma \\ &= (\bar{u}_0^{-1}\theta\cdots\bar{u}_{i+2}^{-1}\theta^{-(i+1)}\{\bar{u}_{i+2}^{-1}\theta^{-i}\bar{u}_{i+1}\theta^{-i}\bar{u}_{i+1}^{-1}\theta^{-i}\bar{u}_{i+2}\theta^{-i}\}\cdots\bar{u}_0\theta^2, 1, 2)_\sigma \\ &= (\bar{u}_0^{-1}\theta\cdots\bar{u}_{i+2}^{-1}\theta^{-(i+1)}\bar{u}_{i+3}\theta^{-(i+1)}\cdots\bar{u}_0\theta^2, 1, 2)_\sigma \\ &= ((\bar{u}_0^{-1}\cdots\bar{u}_{i+2}^{-1}\theta^{-(i+2)}\bar{u}_{i+3}\theta^{-(i+2)}\cdots\bar{u}_0\theta)\theta, 1, 2)_\sigma \\ &= (x_{i+1}\theta, 1, 2)_\sigma = (x_{i+1}, 0, 1)_\sigma = (e, 0, 1)_{i+1}\phi_{i+1}\end{aligned}$$

Hence,

$$\begin{aligned}(g, m, n)_{i+1}\phi_{i+1} &= ((e, m, 0)_{i+1}(g, 0, 0)_{i+1}(e, 0, n)_{i+1})\phi_{i+1} \\ &= (s_i^{-1}\alpha_i^{m-1}\cdots s_i^{-1}\alpha_i s_i^{-1}gs_i\cdot s_i\alpha_i\cdots s_i\alpha_i^{n-1}, m+1, n+1)_i\phi_i\end{aligned}$$

where if $m=0$ ($n=0$) the left- (right-) hand multiplier of g is e. Hence, if $x\in S_{i+1}\subseteq S_i$, i.e.,

$$x = (g, m, n)_{i+1} = (s_i^{-1}\alpha_i^{m-1}\cdots s_i^{-1}\alpha_i s_i^{-1}gs_i\cdot s_i\alpha_i\cdots s_i\alpha_i^{n-1}, m+1, n+1)_i,$$

then $x\phi_{i+1}=x\phi_i$. Thus, ϕ is a homomorphism of S onto H.

We now will show that H is the maximal group homomorphic image of S under the homomorphism ϕ.

Let G^* be an arbitrary group and let ρ be a homomorphism of S onto G^*. We denote ρ/S_i by ρ_i. Thus, ρ_i is a homomorphism of S_i into G^*. Since H is the maximal group homomorphic image of S_0 under the homomorphism ϕ_0 by virtue of Theorem 3.4, there exists a homomorphism γ of H onto the subgroup $S_0\rho_0$ of G^* such that $(g, m, n)_0\phi_0\gamma=(g, m, n)_0\rho_0$ for all $(g, m, n)_0\in S_0$.

Next, suppose that $(g, m, n)_{i+1}\phi_{i+1}\gamma=(g, m, n)_{i+1}\rho_{i+1}$ where γ is a homomorphism of H onto $S_{i+1}\rho_{i+1}$.

By virtue of Theorem 3.5, there exists $v_i \in G^*$ and a homomorphism η_i of G into G^* such that $v_i g\eta_i v_i^{-1} = g\alpha_i\eta_i$ for all $g \in G$. Furthermore $(g, a, b)_i\rho_i = v_i^{-a} g\eta_i v_i^b$ for $(g, a, b)_i \in S_i$.

Since $(g, 0, 0)_{i+1} = (g, 1, 1)_i$, $(g, 0, 0)_{i+1}\rho_{i+1} = (g, 1, 1)_i\rho_i$. Thus, $g\eta_{i+1} = v_i^{-1} g\eta_i v_i$. Hence, $g\eta_i = v_i g\eta_{i+1} v_i^{-1}$.

Since $(e, 0, 1)_{i+1} = (s_i, 1, 2)_i$, $(e, 0, 1)_{i+1}\rho_{i+1} = (s_i, 1, 2)_i\rho_i$. Thus,

$$\begin{aligned} v_{i+1} &= v_i^{-1}(s_i\eta_i)v_iv_i \\ &= v_i^{-1}(v_i(s_i\eta_{i+1})v_i^{-1})v_iv_i \\ &= s_i\eta_{i+1}v_i. \end{aligned}$$

Hence, $v_i = s_i^{-1}\eta_{i+1}v_{i+1}$. Thus,

$$\begin{aligned} g\eta_i &= (s_i^{-1}\eta_{i+1}v_{i+1})g\eta_{i+1}(v_{i+1}^{-1}s_i\eta_{i+1}) \\ &= s_i^{-1}\eta_{i+1}(v_{i+1}g\eta_{i+1}v_{i+1}^{-1})s_i\eta_{i+1} \\ &= s_i^{-1}\eta_{i+1}(g\alpha_{i+1}\eta_{i+1})s_i\eta_{i+1} \\ &= (s_i^{-1}g\alpha_{i+1}s_i)\eta_{i+1}. \end{aligned}$$

We recall that by p. 369 $s_i^{-1}g\alpha_{i+1}s_i = g\alpha_i = u_{i+1}^{-1}g\alpha u_{i+1}$. Thus,

$$\overline{s_i^{-1}g\alpha_{i+1}s_i} = \bar{u}_{i+1}^{-1}\overline{g\alpha}\bar{u}_{i+1} = \bar{u}_{i+1}^{-1}\bar{g}\theta\bar{u}_{i+1}.$$

Hence,

$$\begin{aligned} (s_i^{-1}g\alpha_{i+1}s_i, 0, 0)_{i+1}\phi_{i+1} &= ((\overline{s_i^{-1}g\alpha_{i+1}s_i})\delta_{i+1}, 0, 0)_\sigma \\ &= (\bar{u}_0^{-1}\bar{u}_{-1}^{-1}\theta\cdots\bar{u}_{i+2}^{-1}\theta^{-(i+2)}(\overline{s_i^{-1}g\alpha_{i+1}s_i})\theta^{-(i+1)} \\ &\qquad \cdot\bar{u}_{i+2}\theta^{-(i+2)}\cdots\bar{u}_{-1}\theta\bar{u}_0, 0, 0)_\sigma \\ &= (\bar{u}_0^{-1}\bar{u}_{-1}^{-1}\theta\cdots\bar{u}_{i+2}^{-1}\theta^{-(i+2)}(\bar{u}_{i+1}^{-1}\bar{g}\theta\bar{u}_{i+1})\theta^{-(i+1)} \\ &\qquad \cdot\bar{u}_{i+2}\theta^{-(i+2)}\cdots\bar{u}_{-1}\theta\bar{u}_0, 0, 0)_\sigma \\ &= (\bar{u}_0^{-1}\bar{u}_{-1}^{-1}\theta\cdots\bar{u}_{i+1}^{-1}\theta^{-(i+1)}\bar{g}\theta^{-i}\bar{u}_{i+1}\theta^{-(i+1)}\cdots\bar{u}_{-1}\theta\bar{u}_0, 0, 0)_\sigma \\ &= (\bar{g}\delta_i, 0, 0)_\sigma = (g, 0, 0)_i\phi_i. \end{aligned}$$

Thus,

$$\begin{aligned} (g, 0, 0)_i\rho_i &= g\eta_i = (s_i^{-1}g\alpha_{i+1}s_i)\eta_{i+1} \\ &= (s_i^{-1}g\alpha_{i+1}s_i, 0, 0)_{i+1}\rho_{i+1} \\ &= (s_i^{-1}g\alpha_{i+1}s_i, 0, 0)_{i+1}\phi_{i+1}\gamma = (g, 0, 0)_i\phi_i\gamma. \end{aligned}$$

We next note that

$$\begin{aligned} (s_i^{-1}, 0, 1)_{i+1}\phi_{i+1} &= (\bar{s}_i^{-1}\delta_{i+1}x_{i+1}, 0, 1)_\sigma \\ &= (\bar{u}_0^{-1}\cdot\bar{u}_{-1}^{-1}\theta\cdots\bar{u}_{i+2}^{-1}\theta^{-(i+2)}(\bar{u}_{i+1}^{-1}\bar{u}_{i+2})\theta^{-(i+1)} \\ &\qquad \cdot\bar{u}_{i+2}\theta^{-(i+2)}\cdots\bar{u}_{-1}\theta\bar{u}_0x_{i+1}, 0, 1)_\sigma \\ &= (\bar{u}_0^{-1}\bar{u}_{-1}\theta\cdots\bar{u}_{i+1}^{-1}\theta^{-(i+1)}\bar{u}_{i+2}\theta^{-(i+1)}\{\bar{u}_{i+2}\theta^{-(i+2)} \\ &\qquad \cdots\bar{u}_{-1}\theta\bar{u}_0\bar{u}_0^{-1}\cdot\bar{u}_{-1}^{-1}\theta\cdots\bar{u}_{i+2}^{-1}\theta^{-(i+2)}\}\bar{u}_{i+3}\theta^{-(i+2)}\cdots\bar{u}_0\theta, 0, 1)_\sigma \\ &= (\bar{u}_0^{-1}\bar{u}_{-1}\theta\cdots\bar{u}_{i+1}^{-1}\theta^{-(i+1)}\bar{u}_{i+2}\theta^{-(i+1)}\bar{u}_{i+3}\theta^{-(i+2)}\cdots\bar{u}_0\theta, 0, 1)_\sigma \\ &= (x_i, 0, 1)_\sigma = (e, 0, 1)_i\phi_i. \end{aligned}$$

Thus,

$$(e, 0, 1)_i\rho_i = s_i^{-1}\eta_{i+1}v_{i+1} = (s_i^{-1}, 0, 1)_{i+1}\rho_{i+1}$$
$$= (s_i^{-1}, 0, 1)_{i+1}\phi_{i+1}\gamma = (e, 0, 1)_i\phi_i\gamma.$$

Hence, $(g, m, n)_i\phi_i\gamma=(g, m, n)_i\rho_i$ for all $(g, m, n) \in S_i$. Thus, if $x \in S$, $x\phi\gamma=x\rho$.

Clearly, $H\gamma=G^*$. Thus, H is the maximal group homomorphic image of S under the homomorphism ϕ.

4. **The congruences.** In this section, we will determine the congruence relations on an I-bisimple semigroup $S=(G, C^*, \alpha, u_i)$. We first show that every congruence relation ρ on S is either an idempotent separating congruence (each ρ-class of S contains at most one idempotent) or a group congruence (S/ρ is a group). The idempotent separating congruences are uniquely determined by the α-invariant subgroups of G. Clearly, the group congruences are uniquely determined by the normal subgroups of the maximal group homomorphic image of S (see §3).

We first show that every congruence on $S=(G, C^*, \alpha, u_i)$ is either an idempotent separating congruence or a group congruence.

Let S be an inverse semigroup and let ρ be a congruence relation on S. Let $\{N_\alpha : \alpha \in J\}$ denote the collection of idempotent ρ-classes of S and let $N_\alpha \cap E_S=E_\alpha$. Thus $E_S=U(E_\alpha : \alpha \in J)$ (each N_α contains an idempotent [5]) and $E_\alpha \cap E_\beta=\square$ if $\alpha\neq\beta$.

Furthermore,

(4.1) $E_\alpha E_\beta \subseteq E_\gamma$ for some $\gamma \in J$.

(4.2) If $a \in S$ and $\alpha \in J$, there exists a $\gamma \in J$ such that $a^{-1}E_\alpha a \subseteq E_\gamma$.

THEOREM 4.1. *If S is a bisimple ω-semigroup each congruence on S is either an idempotent separating congruence or a group congruence.*

Proof. Let $S=(G, C, \alpha)$ and let ρ be a congruence relation on S. Let E_0 denote the class containing $(e, 0, 0)$. If $E_0\neq E_S$, let $(e, k+1, k+1)$ be the first element of E_S not contained in E_0. Thus, $E_0=\{(e, j, j) : 0\leqq j\leqq k\}$. Suppose that $k>0$. Hence, by (4.2), $(e, 1, 1+k)(e, \mathrm{k}+1, k+1)(e, k+1, 1)=(e, 1, 1) \in E_0$ and

$$(e, 1, 1+k)(e, 2k+1, 2k+1)(e, k+1, 1) = (e, k+1, k+1) \in E_0$$

since $(e, k+1, 0)(e, 0, 0)(e, 0, k+1)=(e, k+1, k+1)$ and

$$(e, k+1, 0)(e, k, k)(e, 0, k+1) = (e, 2k+1, 2k+1)$$

are contained in the same class by (4.2). Hence, we have a contradiction. Thus, $k=0$, and $E_0=\{(e, 0, 0)\}$. Let us next consider E_γ, say. Let (e, n, n) denote the first element of E_γ and suppose that $(e, n+1, n+1) \in E_\gamma$. Thus,

$$(e, 0, n)(e, n, n)(e, n, 0) = (e, 0, 0) \in E_0$$

and $(e, 0, n)(e, n+1, n+1)(e, n, 0)=(e, 1, 1) \in E_0$ and we again have a contradiction, i.e., $(e, n+1, n+1) \notin E_\gamma$. If $(e, n+s, n+s) \in E_\gamma$ with $s>1$,

$$(e, n+s, n+s)(e, n+1, n+1) = (e, n+s, n+s) \in E_\gamma$$

and hence $(e, n, n)(e, n+1, n+1)=(e, n+1, n+1) \in E_\gamma$, a contradiction. Thus, each E_γ consists of a single point, i.e., ρ is idempotent separating. If $E_S=E_0$, S/ρ is an inverse semigroup [5] with a single idempotent, i.e., S/ρ is a group.

Theorem 4.1 has been established by Munn and Reilly [4] by different methods.

THEOREM 4.2. *If S is an I-bisimple semigroup, each congruence relation on S is either a group congruence or an idempotent separating congruence.*

Proof. Let ρ be a congruence relation on S. Clearly, $\rho \mid S_i \times S_i$ where $S_i=e_iSe_i$ is a congruence relation ρ_i on S_i. Thus, since S_i is a bisimple ω-semigroup, ρ_i is an idempotent separating congruence or a group congruence by Theorem 4.1. Let us suppose that ρ_0 is an idempotent separating congruence. Assume that ρ_{i+1} is idempotent separating. Let e and f be distinct idempotents of $S_{i+1} \subseteq S_i$. If ρ_i is a group congruence, $e\rho_i f$. Thus, $e\rho f$ and hence $e\rho_{i+1}f$, a contradiction. Therefore, ρ_i is idempotent separating. Hence, since $S=U(S_i : i \in I, i \leqq 0)$ by Theorem 1.2, ρ is an idempotent separating congruence by induction. Similarly, if ρ_0 is a group congruence, ρ is a group congruence.

We next will determine the idempotent separating congruences of an I-bisimple semigroup S.

We will make use of the determination of the idempotent separating congruences for an arbitrary inverse semigroup.

If ρ is a congruence relation on an inverse semigroup S, the kernel of ρ is the inverse image of $E_{S/\rho}$ under the canonical homomorphism.

THEOREM 4.3 (PRESTON [5]). *Let $\{N_e : e \in E_S\}$ be a collection of disjoint subgroups of the inverse semigroup S and let $N=U(N_e : e \in E_S)$. Furthermore, suppose that*

(4.3) $N_eN_f \subseteq N_{ef}$,

(4.4) $a\, N_f a^{-1} \subseteq N_g$ *where* $a \in S$ *and* $g=afa^{-1}$.

Define the relation ρ_N over S by $a\rho_N b$ if and only if for some $e \in E_S$, $aa^{-1}=e=bb^{-1}$ and $ab^{-1} \in N_e$. Then ρ_N is an idempotent separating congruence over S with kernel N.

Conversely, every idempotent separating congruence ρ over S has a kernel N of the above type such that ρ_N is ρ.

THEOREM 4.4. *Let $S=(G, C^*, \alpha, u_i)$ be an I-bisimple semigroup. There exists a 1-1 correspondence between the idempotent separating congruences on S and the α-invariant subgroups of G. If ρ^V is the congruence corresponding to the α-invariant subgroup V, $\rho^V_{(g,a,b)}=((vg, a, b) : v \in V)$, i.e., $(g, a, b)\rho^V(h, c, d)$ if and only if $a=c$, $b=d$, and $Vg=Vh$. If V_1, V_2 are α-invariant subgroups of G, $V_1 \subseteq V_2$ if and only if $\rho^{V_1} \subseteq \rho^{V_2}$.*

Proof. Let V be an α-invariant subgroup of G and let $N_{(e,a,a)}=\{(v, a, a) : v \in V\}$ and let $N=U(N_{(e,a,a)} : a \in I)$. It follows by routine calculation that $N_{(e,a,a)}$ is a subgroup of S isomorphic to V, $N_{(e,a,a)}N_{(e,b,b)} \subseteq N_{(e,a,a)(e,b,b)}$ and

$$(g, a, b)N_{(e,c,c)}(g^{-1}, b, a) \subseteq N_{(e,t,t)}$$

where $(e, t, t)=(g, a, b)(e, c, c)(g^{-1}, b, a)$. Thus, ρ_N is an idempotent separating congruence of S by Theorem 4.3. We will denote ρ_N by ρ^V.

Conversely, suppose that ρ is an idempotent separating congruence of S. Thus, by Theorem 4.3, $\rho=\rho_N$ where N is given in the statement of Theorem 4.3 and $N_{(e,0,0)}=((v, 0, 0) : v \in V)$ where V is a subgroup of G. Since

$$(h, 0, 1)(e, 0, 0)(h^{-1}, 1, 0) = (e, 0, 0), \quad (h, 0, 1)N_{(e,0,0)}(h^{-1}, 1, 0) \subseteq N_{(e,0,0)}$$

by (4.4). Thus, if $v \in V$, $(h, 0, 1)(v, 0, 0)(h^{-1}, 1, 0)=(h(v\alpha)h^{-1}, 0, 0)$ and V is an α-invariant subgroup of G. Let $N_{(e,b,b)}$ denote the subgroup of N containing (e, b, b). Thus, $N_{(e,b,b)}=\{(w, b, b) : w \in W\}$ where W is a subgroup of G. Since $(e, 0, b)(e, b, b)(e, b, 0)=(e, 0, 0)$, if $w \in W$,

$$(e, 0, b)(w, b, b)(e, b, 0) = (w, 0, 0) \in N_{(e,0,0)}$$

by (4.4). Hence $W \subseteq V$, and similarly, $V \subseteq W$. Thus, $\rho=\rho^V$ and we have the desired correspondence. If $(g, c, d) \in S$, we next show that $\rho^V_{(g,c,d)}=\{(vg, c, d) : v \in V\}$. If $(h, a, b) \in \rho^V_{(g,c,d)}$, $a=c$ and $b=d$ and $(h, c, d)(g^{-1}, d, c)=(hg^{-1}, c, c) \in N_{(e,c,c)}$ by Theorem 4.3. Thus, $\rho^V_{(g,c,d)} \subseteq \{(v, g, c, d) : v \in V\}$. Using Theorem 4.3, the desired equality follows by a routine calculation. If V_1 and V_2 are α-invariant subgroups of G, clearly $V_1 \subseteq V_2$ implies that $\rho^{V_1} \subseteq \rho^{V_2}$. If $\rho^{V_1} \subseteq \rho^{V_2}$, $v \in V_1$ implies that $(v, 0, 0)\rho^{V_2}(e, 0, 0)$, i.e., $v \in V_2$.

COROLLARY 4.1. *If S is an I-bisimple semigroup, $\mathscr{H}$ is the maximal idempotent separating congruence of S.*

Proof. By Corollary 1.3, $\mathscr{H}$ is a congruence on S. By [3, p. 389, Theorem 2], every idempotent separating congruence of S is contained in $\mathscr{H}$.

BIBLIOGRAPHY

1. A. H. Clifford, *A class of d-simple semigroups*, Amer. J. Math. **75** (1953), 547–556.

2. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Math. Surveys No. 7, Amer. Math. Soc., Providence, R. I., 1961.

3. W. D. Munn, *A certain sublattice of the lattice of congruences on a regular semigroup*, Proc. Cambridge Philos. Soc. **60** (1964), 385–391.

4. W. D. Munn and N. R. Reilly, *Congruences on a bisimple ω-semigroup*, Proc. Glasgow Math. Assoc. **7** (1966), 184–192.

5. G. B. Preston, *Inverse semigroups*, J. London Math. Soc. **29** (1954), 346–403.

6. D. Rees, *On the ideal structure of a semi-group satisfying a cancellation law*, Quart. J. Math. Oxford Ser. **19** (1948), 101–108.

7. N. R. Reilly, *Bisimple ω-semigroups*, Proc. Glasgow Math. Assoc. **7** (1966), 160–169.

8. R. J. Warne, *Matrix representation of d-simple semigroups*, Trans. Amer. Math. Soc. **106** (1963), 427–435.

9. ———, *Homomorphisms of d-simple inverse semigroups with identity*, Pacific J. Math. **14** (1964), 1111–1222.

10. ———, *A characterization of certain regular d-classes in semigroups*, Illinois J. Math. (1965), 304–306.

11. R. J. Warne, *Regular D-classes whose idempotent obey certain conditions*, Duke Math. J. **33** (1966), 187–196.

12. ———, *A class of bisimple inverse semigroups*, Pacific J. Math. **18** (1966), 563–577.

13. ———, *On certain bisimple inverse semigroups*, Bull. Amer. Math. Soc. **72** (1966), 679–682.

14. ———, *Homomorphisms of L-bisimple semigroups*, (to appear).

15. ———, *Extensions of I-bisimple semigroups*, Canad. J. Math. **19** (1967), 419–426.

16. ———, Addendum to [15], Canad. J. Math., (to appear).

17. ———, *Bisimple inverse semigroups mod groups*, Duke Math. J. **34** (1967).

WEST VIRGINIA UNIVERSITY,
MORGANTOWN, WEST VIRGINIA