SIMPLE SYSTEMS OF ROOTS IN L^*-ALGEBRAS

BY

V. K. BALACHANDRAN

1. **Introduction.** In this paper we study semisimple L^*-algebras having simple systems of roots. Such algebras—called regular L^*-algebras in the sequel—are shown to be precisely those semisimple L^*-algebras which are orthogonal sums of separable simple L^*-algebras. Since the latter class of algebras is known (Schue [7]), the structure of the regular L^*-algebras is completely determined. We also extend to regular L^*-algebras some of the classical results from the theory of finite-dimensional semisimple Lie algebras. In particular we obtain extensions of the isomorphism theorem and the existence theorem for the Weyl basis (Theorems 3, 4).

The notions of the real form and the compact form can be introduced for semisimple L^*-algebras in quite a natural way. It turns out that the compact form that we introduce is unique. Also in case the L^*-algebra is regular, the compact form has the same kind of relationship with the Weyl basis as in the (finite-dimensional) Lie theory (see Remark 7).

A principal tool employed extensively in this paper consists in constructing suitable finite-dimensional L^*-subalgebras (cf. Propositions 2, 3) and applying to them the theorems of the classical theory of Lie algebras. In particular it is by this method that we obtain one of the main results (Proposition 8) leading to the structure theorem stated at the outset.

2. **Preliminaries.** The requisite background material on L^*-algebras will be found in Schue [7], [8]. For the convenience of the reader the principal results of Schue's theory which are required in this paper are collected here.

Let L denote a semisimple L^*-algebra. If H is a Cartan subalgebra of L and $\Delta=\{\alpha\}$ the set of nonzero roots of L relative to H, then we have the Cartan decomposition

$$L = H \oplus V; \qquad V = \sum_{\alpha\in\Delta} V_\alpha, \tag{1}$$

where V_α is the α root subspace and $\oplus$ denotes the orthogonal (or the Hilbert space direct) sum.

The root subspaces V_α have the following properties: (i) each V_α is one-dimensional, (ii) $V_\alpha^* = V_{-\alpha}$, where V_α^* denotes the image of V_α under the conjugation (*) of L, and (iii) $[V_\alpha, V_\beta]=V_{\alpha+\beta}$ if $\alpha+\beta$ is a root $\neq 0$, $[V_\alpha, V_{-\alpha}]\subseteq H$ and $[V_\alpha, V_\beta]=\{0\}$ if $\alpha+\beta$ is not a root.

Each root $\alpha=\alpha(h)$ is a continuous linear functional on H so that there is a unique vector $h_\alpha\in H$ such that $\alpha(h)=\langle h, h_\alpha\rangle$ where $\langle\cdot\rangle$ denotes the inner product. Also

Received by the editors November 24, 1966.

h_α is selfadjoint (i.e., $h_\alpha^* = h_\alpha$) and $h_\alpha = [v_\alpha, v_\alpha^*]$ for any $v_\alpha \in V_\alpha$ with $\|v_\alpha\| = 1$. Further, the set $\{h_\alpha : \alpha \in \Delta\}$ is *total* in H; i.e., for any $h \in H$, $\langle h, h_\alpha \rangle = 0$ for all h_α implies $h = 0$ [1, Lemma 6].

We write $\langle \alpha, \beta \rangle = \langle h_\alpha, h_\beta \rangle$ and call $\alpha \perp \beta$ ($= \alpha$ orthogonal to β) if $\langle \alpha, \beta \rangle = 0$. Also we shall say that a set $\{\alpha\}$ is linearly independent if the set $\{h_\alpha\}$ is linearly independent.

If α and β are any two (distinct) roots, the α-series containing β is the set of roots $\beta + i\alpha$ where i is an integer, $p \leqq i \leqq q$, and we have

$$2\langle \alpha, \beta \rangle / \langle \alpha, \alpha \rangle = -(p+q). \tag{2}$$

Some Notations. (See also under Remark 1.) $\boldsymbol{C}$, $\boldsymbol{R}$, $\boldsymbol{Q}$ denote as usual the complex, the real and the rational fields respectively.

Let L be a semisimple L^*-algebra. For any subset S of L the set of all (finite) $\boldsymbol{C}$-linear combinations of elements of S will be denoted by $\mathrm{Sp}_{\boldsymbol{C}}\, S = \mathrm{Sp}, S$ and the set of all $\boldsymbol{Q}$-linear combinations of elements by $\mathrm{Sp}_{\boldsymbol{Q}}\, S$. The orthogonal complement of S in L is denoted by $S^\perp$.

If H is a Hilbert space we write dim H to mean the orthogonal dimension, i.e., cardinality of an orthonormal basis for H. For an arbitrary set S, $|S|$ will denote the cardinality of S.

Lemma 1. *Let a semisimple L^*-algebra L have an orthogonal decomposition $L = \sum \oplus L_j$ where L_j are semisimple. Let H be a Cartan subalgebra of L. If α is a nonzero root relative to H, the root subspace V_α belongs precisely to one L_j. Write $\Delta_j = \{\alpha : V_\alpha \subseteq L_j\}$. Then the closure of* Sp $\{h_\alpha : \alpha \in \Delta_j\}$ *in H is a Cartan subalgebra H_j of L_j, and the restrictions to H_j of the $\alpha \in \Delta_j$ are precisely the roots of L_j.*

The proof of this lemma is quite straightforward and is omitted.

Since every semisimple L^*-algebra L has an (unique) orthogonal decomposition $L = \sum \oplus \tilde{L}_j$ where $\tilde{L}_j$ are simple L^*-subalgebras [7, p. 71], we obtain the corollaries:

Corollary 1. *Every root α of L is a root of one of its simple components $\tilde{L}_j$.*

Corollary 2. *Suppose $\beta_1, \beta_2, \ldots, \beta_n$ are linearly independent roots of L. If there is a root γ of L such that*

$$\gamma = \sum c_i \beta_i, \quad c_i \neq 0, \tag{3}$$

then all β_i and γ are roots of the same simple component $\tilde{L}_j$.

Proof. By Corollary 1, γ is a root of some $\tilde{L}_j$ and the β_i are easily seen to be roots of this $\tilde{L}_j$.

3. **Semisimple L^*-subalgebras.** In this section we describe a method of constructing semisimple and simple L^*-subalgebras of a semisimple L^*-algebra. Particular cases of this construction have already been used by Schue in his paper [7].

DEFINITION 1. Let L be a semisimple L^*-algebra, H a Cartan subalgebra of L, and Δ the set of nonzero roots of L relative to H. A subset Δ_0 of Δ is called a root system (relative to H) if it satisfies the conditions: $\alpha \in \Delta_0$ implies $-\alpha \in \Delta_0$; and $\alpha, \beta \in \Delta_0$, $\alpha+\beta \in \Delta$ implies $\alpha+\beta \in \Delta_0$.

REMARK 1. For any subset S of Δ, if we write $(\mathrm{Sp})^\frown S = \mathrm{Sp}\, S \cap \Delta$ then $(\mathrm{Sp})^\frown S$ is clearly a root system. We may also observe that, as in the finite-dimensional Lie algebra case (cf. [5, p. 117]), we have the result:

$$(\mathrm{Sp})^\frown S = \mathrm{Sp}_Q\, S \cap \Delta. \tag{4}$$

DEFINITION 2. A subset S of Δ is called indecomposable if S is not expressible as a disjoint union of nonempty subsets S_1, S_2 with $S_1 \perp S_2$.

REMARK 2. If $\alpha, \beta \in \Delta$, then a finite sequence of roots $\alpha_1, \alpha_2, \dots, \alpha_n$ of Δ with $\alpha=\alpha_1$, $\beta=\alpha_n$ and $\langle \alpha_i, \alpha_{i+1}\rangle \neq 0$, is called a chain $C_{\alpha\beta}$ from α to β. Obviously a chain is an indecomposable set. Further, it is easy to see that a set S is indecomposable if and only if for any two roots α, β in S there is a $C_{\alpha\beta}$ in S (i.e., whose elements belong to S). Finally, we have the result that L is simple if and only if Δ is indecomposable (cf. [7, pp. 71, 74]).

PROPOSITION 1. *Each root system Δ_0 in L determines a semisimple L^*-subalgebra $L_0 = L_0(\Delta_0)$, with a Cartan subalgebra $H_0 = H \cap L_0$, whose roots relative to H_0 are precisely the roots in Δ_0. L_0 is simple if Δ_0 is indecomposable. If Δ_0 is finite, L_0 is finite-dimensional; if Δ_0 is infinite,* $\dim L_0 = |\Delta_0|$.

Proof. If we define

$$H_0 = \mathrm{Sp}\,\{h_\alpha : \alpha \in \Delta_0\}, \qquad V_0 = \sum_{\alpha\in\Delta_0} \oplus V_\alpha,$$

it is straightforward to verify that $H_0 \oplus V_0$ is the required subalgebra L_0. The assertions concerning the dimension of L_0 are immediate consequences of the fact that, in an infinite-dimensional semisimple L^*-algebra L,

$$\dim L = \dim H = |\Delta| \qquad [2, \text{Theorem } 1]. \tag{5}$$

COROLLARY. *If Δ_0 has a finite basis, L_0 is finite-dimensional and consequently Δ_0 is a finite set.*

PROPOSITION 2. *Suppose S is a subset of Δ, $\Delta_0 = (\mathrm{Sp})^\frown S$ and L_0 the L^*-subalgebra of L determined by Δ_0 in accordance with Proposition* 1: $L_0 = L_0(\Delta_0) = L_0(S)$. *Then L_0 is finite-dimensional if S is finite, and* $\dim L_0 = |S|$ *if S is infinite. Finally, L_0 is simple if S is indecomposable.*

Proof. In view of the Corollary to Proposition 1 it suffices to consider the case where S is infinite. Since $\Delta_0 = (\mathrm{Sp})^\frown S = \Delta \cap \mathrm{Sp}_Q\, S$ (by (4)), each root in Δ_0 is a rational linear combination of roots from S. Hence, since S is infinite, by (5), $\dim L_0 = |\Delta_0| = |S|$. The final assertion concerning the simplicity of L_0 is an immediate consequence of Proposition 1 and the (easily proved) fact that Δ_0 is indecomposable whenever S is indecomposable.

PROPOSITION 3. *Let L be a simple L^*-algebra and Δ the set of its nonzero roots relative to some Cartan subalgebra H. For any subset S of Δ there exists a simple L^*-subalgebra L_0 of L, with Cartan subalgebra H_0, such that*

(i) $S \subseteq \Delta_0$, *where Δ_0 is the set of roots of L_0 (relative to H_0),*

(ii) *L_0 is finite-dimensional if S is finite,*

(iii) *L_0 is infinite-dimensional and* $\dim L_0 = |S|$ *if S is infinite.*

Proof. Since Δ is indecomposable, for any two roots α, β there is a chain $C_{\alpha\beta}$ of roots. For each pair of roots α, β in S select a chain $C_{\alpha\beta}$ and denote by S' the union of all sets $C_{\alpha\beta}$ as α, β vary in S. It is easily seen that any two roots γ, δ in S' can be joined by a chain $C_{\gamma\delta}$ in S'. Hence S' is indecomposable and therefore by Proposition 2, L_0 with $\Delta_0 = (\mathrm{Sp})^\frown S'$ is simple.

Next we observe that S' is finite whenever S is finite. Also when S is infinite

$$|S'| = |\bigcup C_{\alpha\beta}| \leqq \aleph_0 |S| = |S|$$

so that $|S'| = |S|$. The assertions (ii) and (iii) concerning the dimension of L_0 now follow from Proposition 2.

4. Fundamental sets of roots.

DEFINITION 3. A subset $F = \{\rho_i\}$ of Δ is called a fundamental set of roots or briefly a f.s.r. of L (relative to H) if the following conditions are satisfied:

(i) The ρ_i are linearly independent.

(ii) Every root α in $(\mathrm{Sp})^\frown F = (\mathrm{Sp})^\frown_Q F$ is a linear combination $\sum c_i\rho_i$ of ρ_i with coefficients c_i all nonnegative or all nonpositive.

A f.s.r. F which is a basis for Δ is called a f.b.r. (fundamental basis of roots).

DEFINITION 4. A semisimple L^*-algebra which has a f.b.r. is called a *regular* L^*-algebra.

REMARK 3. Suppose α is a root of a semisimple L^*-algebra L. If $\Delta_0 = (\mathrm{Sp})^\frown \{\alpha\}$, the L^*-subalgebra $L_0 = L_0(\Delta_0)$ is finite-dimensional (by Proposition 2). By a well-known result in Lie algebras [5, p. 117] it follows that $\Delta_0 = \{\alpha, -\alpha\}$. Thus a set consisting of a single root is a f.s.r.

PROPOSITION 4. *Let $F = \{\rho_i\}$ be a f.s.r. If $\gamma \in (\mathrm{Sp})^\frown F$, then*

$$\gamma = \sum k_i\rho_i \tag{6}$$

where k_i are all integers of the same sign and $|k_i| \leqq 6$.

Proof. We may assume $\gamma = c_1\rho_1 + c_2\rho_2 + \cdots + c_n\rho_n$, $c_i \neq 0$. Write

$$S = \{\rho_1, \rho_2, \ldots, \rho_n\}.$$

If $L_0 = L_0(S)$ is the L^*-subalgebra determined by S (Proposition 2), L_0 is finite-dimensional, and also simple in view of Corollary 2 to Lemma 1. The hypothesis on F implies that every root in Δ_0 is a linear combination of ρ_i with all coefficients (rationals) of the same sign. But then the ρ_i are the simple roots of L_0 and $c_i = k_i$ are integers [5, pp. 120–121].

Finally, an examination of the coefficients k_i occurring in the representation of the highest root $\alpha = \sum k_i \rho_i$ for the simple Lie algebras of the different types (cf. [3, pp. 163–164]) show that $0 \leqq k_i \leqq 6$. Hence for all roots of L_0, and in particular for γ, we have $|k_i| \leqq 6$. This completes the proof.

The lemma which follows will be needed later (§7).

LEMMA 2. *Let L be a semisimple L^*-algebra. Every finite f.s.r. F (unless it is a basis for Δ) can be extended to a f.s.r. $\tilde{F}$ by the addition of a suitable root.*

Proof. Let $F = \{\rho_1, \rho_2, \ldots, \rho_{n-1}\}$. Let β be any root of L such that

$$S = \{\rho_1, \ldots, \rho_{n-1}, \beta\}$$

is linearly independent. Consider the finite-dimensional L^*-subalgebra $L_0 = L_0(S)$ determined by S. Introduce the lexicographic ordering in Δ (cf. §5) induced by the ordering $\rho_1, \ldots, \rho_{n-1}, \beta$ of the basis S. Denote by S' the subset of elements of Δ_0 which are not linear combinations of the ρ_i and set

$$\rho_n = \inf \gamma : \gamma \in S'.$$

Since $\beta \in S'$, $\beta \geqq \rho_n$.

We assert that the ρ_i $(i = 1, 2, \ldots, n)$ are simple relative to the ordering introduced. To prove this suppose $\rho_i = \delta + \delta'$; $\delta, \delta' > 0$. First consider the case $\rho_i \neq \rho_n$. Then $\delta, \delta' \notin S'$ (for the contrary would imply $\delta, \delta' \geqq \rho_n > \rho_i$). Hence by the hypothesis on F, δ and δ' are nonnegative integral linear combinations of ρ_i $(i = 1, \ldots, n-1)$. Therefore the relation $\rho_i = \delta + \delta'$ cannot hold unless one of δ, δ' is zero. This contradiction proves that ρ_i is simple in this case. Next consider the case $\rho_i = \rho_n = \delta + \delta'$. Since $\rho_n \in S'$, at least one of $\delta, \delta' \in S'$ (otherwise ρ_n would be in $(\mathrm{Sp})^\wedge F$). If $\delta \in S'$, then $\delta \geqq \rho_n$ and $\delta' = 0$ which is a contradiction. Hence ρ_n also is simple. Thus $\tilde{F} = \{\rho_1, \rho_2, \ldots, \rho_n\}$ is a simple system of roots for L_0 from which it follows (cf. Remark 4) that $\tilde{F}$ is a f.s.r.

(The above construction for the simple root ρ_n was suggested by a result in I. Satake [6, p. 289].)

5. Lexicographic ordering of roots.

DEFINITION 5. Let $S \subseteq \Delta$ be a basis for Δ (over $\boldsymbol{C}$). Then S is also a basis for Δ over $\boldsymbol{R}$ (even over $\boldsymbol{Q}$) so that Δ is a subset of the real vector space H_0 of all $\boldsymbol{R}$-linear combinations of elements of S. H_0 can be made into an ordered vector space in the following standard way. Let the basis S be well ordered as $S = \{\beta_i\}$ in an arbitrary fashion. Any vector x in H_0 is a real linear combination $\sum \lambda_i \beta_i$, with β_i in S, and we define $x > 0$ if the last nonzero coefficient λ_i is positive; also we define $x > y$ if $x - y > 0$. We shall call this ordering of H_0, and also that induced on its subset Δ, the lexicographic ordering determined by the well ordered basis S.

Denote by Δ^+ the subset of Δ consisting of the positive roots (relative to this ordering). Obviously $S \subseteq \Delta^+$. Though S is well ordered the same need not be true of the whole of Δ^+ in general. However, we have

THEOREM 1. *If L is a regular L^*-algebra, there exists a lexicographic ordering for Δ under which Δ^+ is well ordered.*

Proof. Let $S=F$ be a f.b.r. for Δ and consider a lexicographic ordering determined by F. Then every root α in Δ^+ (relative to this ordering) is a nonnegative integral linear combination $\alpha=\sum k_i\rho_i$ of roots ρ_i in F. It follows, from the fact that k_i are nonnegative integers, that Δ^+ is well ordered.

REMARK 4. Suppose Δ is ordered as above in Theorem 1. Then it is easy to see (cf. [5, p. 121]) that the roots in F are precisely the positive roots ρ which cannot be written as a sum $\rho=\alpha+\beta$ of positive roots α, β. In view of this property we shall henceforth call ρ a simple root and $F=\Pi$ a simple system of roots of (the regular L^*-algebra) L.

Suppose next there is an arbitrary lexicographic ordering for Δ making Δ^+ well ordered. Then an argument similar to the finite-dimensional case [5, p. 120] will show that the simple roots under this ordering form a f.b.r. of L.

6. **Simple systems of roots.** Simple root systems in L^*-algebras share with simple root systems of Lie algebras many of their useful properties. Some of these properties are stated below. They are obtained by constructing, using Proposition 2, suitable finite-dimensional L^*-subalgebras and applying to them the known results of the Lie theory [5, pp. 112–123].

In what follows $\Pi=\{\rho_i\}$ will denote a simple system of roots of a regular L^*-algebra L. As in the finite-dimensional case we call $A=(A_{ij})$, where

$$A_{ij} = 2\langle \rho_i, \rho_j\rangle/\langle \rho_i, \rho_i\rangle$$

the Cartan matrix associated with Π; $A_{ij}=0$, -1, -2, or -3, for $i\neq j$ (cf. [5, p. 121]).

PROPOSITION 5. (i) *If $\rho_1, \rho_2 \in \Pi$, $\rho_1\neq\rho_2$, then $\rho_1-\rho_2$ is not a root; and $\langle\rho_1, \rho_2\rangle \leqq 0$.*

(ii) *For any positive root $\alpha\notin\Pi$, there exists a simple root ρ such that $\alpha-\rho$ is a positive root; also α has a representation $\alpha=\rho_{i_1}+\rho_{i_2}+\cdots+\rho_{i_k}$ where the ρ_{i_j} are not necessarily distinct and are such that every partial sum $\rho_{i_1}+\cdots+\rho_{i_m}$ $(m\leqq k)$ is a root.*

(iii) *For any finite set of simple roots $\rho_1, \rho_2, \ldots, \rho_n$ the sequences $(k_1, k_2, \ldots, k_n)$ such that $\sum k_i\rho_i$ is a root can be determined from A by an algorithm, so that Π and A determine Δ.*

REMARK 5. Let $\Pi=\{\rho_i\}$ be a simple system of roots of a regular L^*-algebra L. Choose $e_{\rho_i}\in V_{\rho_i}$ with $\|e_{\rho_i}\|=1$. Then $e_{-\rho_i}=e^*_{\rho_i}$ and $h_{\rho_i}=[e_{\rho_i}, e_{-\rho_i}]$. Now write

$$e_i = e_{\rho_i}, \quad f_i = 2e_i^*/\langle\rho_i, \rho_i\rangle, \quad h_i = 2h_{\rho_i}/\langle\rho_i, \rho_i\rangle.$$

Then

$$[h_i, e_i] = 2e_i, \quad [h_i, f_i] = -2f_i, \quad [e_i, f_i] = h_i.$$

In other words the elements e_i, f_i, h_i have the canonical multiplication table for a three-dimensional split simple Lie algebra as given in [5, p. 121]. Also these elements clearly span a 3-dimensional L^*-subalgebra of L. Now select for each positive root α a representative $\alpha = \rho_{i_1} + \rho_{i_2} + \cdots + \rho_{i_k}$ as explained in (ii) of Proposition 5. Set

$$e_\alpha = [e_{i_k} e_{i_{k-1}} \cdots e_{i_1}], \qquad e_{-\alpha} = [f_{i_k} f_{i_{k-1}} \cdots f_{i_1}]$$

where we have used the abbreviation $[x_k x_{k-1} \cdots x_1]$ for $[x_k[x_{k-1}[\cdots[x_2, x_1]\cdots]]]$. The elements

$$h_i;\ e_\alpha,\ e_{-\alpha} \quad (\rho_i \in \Pi, \alpha \in \Delta^+)$$

form a linearly independent set T which is total in L. We may call T a *total basis* for L. Because of the linearity and continuity of the Lie bracket, the multiplication table for elements of T determines that for all of L. It can further be shown by the same method as in [5, p. 125, Theorem 2] that the multiplication table for T has rational coefficients which are determined by the Cartan matrix A.

PROPOSITION 7. *A regular L^*-algebra L is simple if and only if Π is indecomposable.*

Proof. The proof of this proposition proceeds on the same lines as in [5, p. 128] for the corresponding finite-dimensional theorem. If $\Pi = \Pi_1 \cup \Pi_2$ with $\Pi_1, \Pi_2 \neq \varnothing$ and $\Pi_1 \perp \Pi_2$, then write $H_1 =$ closure of $\mathrm{Sp}\{h_{\rho_j} : \rho_j \in \Pi_1\}$, $V_1 = \sum' \oplus V_\alpha$ where summation is taken over all roots α in $(\mathrm{Sp})^\wedge\, \Pi_1$, and $L_1 = H_1 \oplus V_1$. L_1 is clearly a closed subalgebra of L such that $0 \subset L_1 \subset L$. Now consider the normalizer $N(L_1)$ of L_1; $N(L_1)$ is a closed subalgebra of L. An argument similar to the one used in the finite-dimensional proof cited above shows that $N(L_1) = L$. Hence L_1 is a closed ideal of L and therefore L is not simple. Conversely, if L is not simple an entirely analogous argument as in the finite-dimensional proof shows that Π is decomposable.

7. **Structure of regular L^*-algebras.** We observe that by virtue of Remark 4 a regular L^*-algebra can be redefined as a semisimple L^*-algebra with a simple system Π of roots. The following lemma is an easy consequence of the definition of a simple system (Definition 3) and Lemma 1.

LEMMA 3. *An orthogonal sum $L = \sum \oplus L_j$ of semisimple L^*-algebras L_j is a regular L^*-algebra if and only if each component L_j is a regular L^*-algebra.*

PROPOSITION 8. *If a simple L^*-algebra is a regular L^*-algebra, it is separable.*

Proof. Let L be a simple L^*-algebra which is regular, so that L has a simple system Π of roots. For each ρ_0 in Π denote by $M(\rho_0)$ the set of all ρ in Π such that $\rho \neq \rho_0$, $\langle \rho, \rho_0 \rangle \neq 0$. Call $|M(\rho_0)|$ the multiplicity $m(\rho_0)$ of ρ_0.

We shall first prove: $m(\rho) \leqq 3$ for all ρ in Π. Suppose indeed for a $\rho = \rho_0$, $m(\rho_0) > 3$. Then we can pick out four roots ρ_i $(i = 1, 2, 3, 4)$ such that $\langle \rho_0, \rho_i \rangle \neq 0$. Write $\Delta_0 = (\mathrm{Sp})^\wedge \{\rho_0, \rho_1, \ldots, \rho_4\}$ and let $L_0 = L_0(\Delta_0)$ denote the corresponding L^*-subalgebra (cf. Proposition 2). L_0 will be simple and the Dynkin diagram for L_0 will have at the vertex ρ_0, four lines issuing from it. But this is impossible [5, p. 130]. Hence the result.

Next we assert that there is at most one ρ with $m(\rho) = 3$. To prove this, suppose we have $m(\rho_1) = m(\rho_2) = 3$, $\rho_1 \neq \rho_2$. Write $S = M(\rho_1) \cup M(\rho_2)$. By Proposition 3 there exists a finite dimensional simple L^*-algebra L_0 with $S \subseteq \Delta_0$. By successive applications of Lemma 2, S can be extended to a simple system Π_0 of L_0. Then the Dynkin diagram for Π_0 will have at each of the vertices ρ_1, ρ_2 three lines issuing from the vertex. But an examination of the diagrams for the different types of simple Lie algebras will show that this is not possible. This proves our assertion. Further an exactly similar argument as above will also show that if $m(\rho_0) = 3$, $M(\rho_0) = \{\rho_1, \rho_2, \rho_3\}$ then one of the ρ_i has $m(\rho_i) = 1$.

Again it is an easy consequence of the indecomposability of Π that $m(\rho) \geqq 2$ for all roots ρ except possibly two of them, say, ρ_1, ρ_2 for which $m(\rho_1) = m(\rho_2) = 1$. We conclude therefore that for the simple system Π, $m(\rho) = 2$ for all ρ in Π except possibly three of them ρ', ρ'', ρ''' for which we have: $m(\rho') = m(\rho'') = 1$ and $m(\rho''') = 3$.

We now proceed to prove that Π is countable. We may assume that Π is infinite. We have to consider various possibilities. Our method consists in picking out in each of the different cases that arise a certain countable (infinite) subset S of Π and showing, with the help of the properties established for $m(\rho)$ and the indecomposability of Π, that $\Pi = S$. By making use of the fact, that the Dynkin diagram (without weights) for any finite chain of Π corresponds to that of one of the finite-dimensional simple Lie algebras, it is not difficult to see that the cases considered below (for Π) are the only ones that can occur. In each case the corresponding subset S that can be selected is indicated by a diagram.

Case 1. $m(\rho) = 2$ for all ρ in Π.

$$S: \begin{array}{ccccccccccc} \cdots & \circ & \cdots & \circ & — & \circ & — & \circ & \cdots & \circ & \cdots \\ & \rho_{-n} & & \rho_{-1} & & \rho_0 & & \rho_1 & & \rho_n & \end{array}$$

Case 2. $m(\rho_1) = 1$, $m(\rho) = 2$ for all $\rho \neq \rho_1$.

$$S: \begin{array}{cccccc} \circ & — & \circ & \cdots & \circ & \cdots \\ \rho_1 & & \rho_2 & & \rho_n & \end{array}$$

Case 3. $m(\rho_0) = m(\rho_1) = 1$; $m(\rho_2) = 3$; $m(\rho) = 2$ for all $\rho \neq \rho_0, \rho_1, \rho_2$.

$$S: \begin{array}{cccccccc} & & \rho_2 & & & & & \\ \circ & — & \circ & — & \circ & \cdots & \circ & \cdots \\ \rho_1 & & | & & \rho_3 & & \rho_n & \\ & & \circ & & & & & \\ & & \rho_0 & & & & & \end{array}$$

To prove that $S = \Pi$ (in all the cases) it suffices to consider Case 1 since the proof for the other two cases is similar. If $S \neq \Pi$, we could select a ρ' outside S. By the indecomposability of Π, there is a chain $C_{\rho_0\rho'}$ in Π. Let ρ'' be the first root of the chain lying outside S. Then the preceding root ρ lies in S and $\langle\rho, \rho''\rangle \neq 0$. This clearly implies $m(\rho)>2$, which is a contradiction. Hence $S=\Pi$. The countability of Π follows immediately, whence L is separable (by (5) of §3).

THEOREM 2 (STRUCTURE THEOREM). *The regular L^*-algebras are precisely the L^*-algebras which are orthogonal sums of separable simple L^*-algebras.*

Proof. It is an immediate consequence of the definition of regularity and the results obtained by Schue in [7, p. 76] that every separable simple (or even semisimple) L^*-algebra is regular. Hence by Lemma 3 an orthogonal sum of separable simple L^*-algebras is also regular. Next suppose that L is a semisimple L^*-algebra which is regular. If $L=\sum \oplus \tilde{L}_j$ is the decomposition of L into its simple components $\tilde{L}_j$, then $\tilde{L}_j$ are regular by Lemma 3. Hence by Proposition 8 $\tilde{L}_j$ are separable and this completes the proof.

8. The isomorphism theorem.

DEFINITION 6. Let Π be a simple system of roots (relative to H) of a regular L^*-algebra L with a Cartan subalgebra H. Following Schue [7, p. 76] we call the set $\langle h_{\rho_1}, h_{\rho_2}\rangle$ as ρ_1, ρ_2 vary in Π the graph G of Π. If L' is also a regular L^*-algebra with Π' as a simple system of roots (relative to an H') and G' the corresponding graph, then G is said to be isomorphic with G' if there is a map $\rho \to \rho'$ of Π onto Π' such that $\langle h_{\rho_1}, h_{\rho_2}\rangle = \langle h_{\rho_1'}, h_{\rho_2'}\rangle$ for all ρ_1, ρ_2 in Π.

It is an immediate consequence of assertion (iii) in Proposition 5 that the graph G associated with a simple system determines the set Δ of all nonzero roots (since G obviously determines the Cartan matrix A).

DEFINITION 7. A map ψ of an L^*-algebra L onto an L^*-algebra L' is called an *L^*-isomorphism* if ψ is an isomorphism between L and L' as Lie algebras and also an isomorphism between L and L' as Hilbert spaces.

LEMMA 4. *If ψ is an L^*-isomorphism of L onto L', then $(\psi x)^* = \psi x^*$ for all x in L.*

Proof. For $x, y, z \in L$ we have

$$\begin{aligned}\langle[\psi x, \psi y], \psi z\rangle &= \langle[x, y], z\rangle = \langle y, [x^*, z]\rangle \\ &= \langle \psi y, [\psi x^*, \psi z]\rangle \\ &= \langle[(\psi x^*)^*, \psi y], \psi z\rangle.\end{aligned}$$

Hence $\psi x = (\psi x^*)^*$, or $\psi x^* = (\psi x)^*$.

THEOREM 3 (ISOMORPHISM THEOREM). *Let L, L' be regular L^*-algebras with Cartan subalgebras H, H' and corresponding graphs G, G'. If there is a map $\rho \to \rho'$ making G and G' isomorphic, then the map $\psi\colon h_\rho \to h_{\rho'}$ can be extended to an L^*-isomorphism of L onto L'.*

This isomorphism theorem has been obtained by Schue [7, p. 76] under the additional restrictions that L and L' are separable and simple. However, his proof goes over without change to the present general case since we have with us now the necessary extensions (like Theorem 1 and Proposition 5) of the auxiliary results on which it depends.

COROLLARY. *Let L be a regular L^*-algebra with a Cartan subalgebra H. Then there is an L^*-isomorphism ψ of period 2 (i.e., $\psi^2=1$) such that $\psi(h)=-h$ for all h in H.*

9. Weyl basis.

DEFINITION 8. Let L be a regular L^*-algebra with a simple system of roots, Π, and Δ^+ the positive roots under the lexicographic ordering induced by Π. For each $\alpha \in \Delta^+$ choose $v_\alpha \in V_\alpha$ with $\|v_\alpha\|=1$. Then

$$v_{-\alpha} = v_\alpha^* \in V_{-\alpha}, \quad \|v_{-\alpha}\| = 1, \quad \text{and} \quad [v_\alpha, v_{-\alpha}] = h_\alpha. \tag{7}$$

For $\alpha, \beta \in \Delta$ and $\alpha+\beta \neq 0$, we define $N_{\alpha,\beta}$ to be zero if $\alpha+\beta$ is not a root and by the equation $[v_\alpha, v_\beta]=N_{\alpha,\beta}v_{\alpha+\beta}$ if $\alpha+\beta$ is a root.

LEMMA 5. *Suppose $\alpha, \beta \in \Delta$ and $\beta+i\alpha$ $(p \leqq i \leqq q)$ is the α-series containing β. Then*

$$N_{\alpha,\beta}N_{-\alpha,-\beta} = -\{q(1-p)/2\}\langle\alpha, \alpha\rangle. \tag{8}$$

For proof see [7, p. 74].

THEOREM 4. *Let V_α $(\alpha \in \Delta)$ be the root subspaces of L. Then for each $\alpha \in \Delta^+$ we can choose $e_\alpha \in V_\alpha$ satisfying the following conditions:*

(i) $\|e_\alpha\|=1$,

(ii) $e_\alpha^*=e_{-\alpha}$,

(iii) $[e_\alpha, e_{-\alpha}]=h_\alpha$,

(iv) $N_{\alpha,\beta}=-N_{-\alpha,-\beta}$; *the $N_{\alpha,\beta}$ are real, being determined by the equation*

$$N_{\alpha,\beta}^2 = \{q(1-p)/2\}\langle\alpha, \alpha\rangle.$$

Proof. The proof of this theorem will make use of the automorphism ψ in the Corollary to Theorem 3 and runs on lines similar to that for the corresponding result in finite-dimensional Lie algebras (cf. [4, p. 152]).

Choose the v_α as in Definition 8. Since $\psi v_\alpha \in V_{-\alpha}$, we can write $\psi v_\alpha=c_{-\alpha}v_{-\alpha}$. Now

$$\bar{c}_{-\alpha}v_\alpha = (c_{-\alpha}v_{-\alpha})^* = (\psi v_\alpha)^* = \psi v_\alpha^* = \psi v_{-\alpha} = c_\alpha v_\alpha,$$

so that $c_{-\alpha}=\bar{c}_\alpha$. Also $|c_\alpha|=|c_{-\alpha}|=1$ since $\|v_\alpha\|=1$ and ψ is an isometry.

Choose constants d_α, $d_{-\alpha}$ such that $d_\alpha d_{-\alpha}=1$ and $d_\alpha/d_{-\alpha}=-c_\alpha$; then $|d_\alpha|=1$ and $d_{-\alpha}=\bar{d}_\alpha$. If we write $e_\alpha=d_\alpha v_\alpha$, $e_{-\alpha}=d_{-\alpha}v_{-\alpha}$ then $\|e_\alpha\|=1$, $e_\alpha^*=e_{-\alpha}$ and $[e_\alpha, e_{-\alpha}]=h_\alpha$ (cf. (7)). Again

$$\psi e_{-\alpha} = \psi(d_{-\alpha}v_{-\alpha}) = d_{-\alpha}c_\alpha v_\alpha = -e_\alpha.$$

Hence

$$\begin{aligned} N_{\alpha,\beta}e_{\alpha+\beta} &= [e_\alpha, e_\beta] = \psi[e_{-\alpha}, e_{-\beta}] \\ &= \psi(N_{-\alpha,-\beta}e_{-\alpha-\beta}) = -N_{-\alpha,-\beta}e_{\alpha+\beta}, \end{aligned}$$

or $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$. Finally the expression for $N^2_{\alpha,\beta}$ in (iv) follows from (8) (Lemma 5).

REMARK 6. Let Π be a simple system of roots for L (relative to H). Consider the total basis T_W for L—in the sense of Remark 5—comprising h_ρ ($\rho \in \Pi$) and the e_α, $e_{-\alpha} = e^*_\alpha$ ($\alpha \in \Delta^+$) chosen as above. The structure constants associated with this basis are all real. (Note that $[h_\rho, e_\alpha] = \langle h_\rho, h_\alpha \rangle e_\alpha$, and $\langle h_\rho, h_\alpha \rangle$ is real since $h^*_\rho = h_\rho$, $h^*_\alpha = h_\alpha$). We may call T_W a *Weyl basis* for L (relative to H).

10. Compact forms.

DEFINITION 9. If in the definition of an L^*-algebra [7, p. 70] the underlying complex Hilbert space is replaced by a real Hilbert space, the resulting system we get is called a *real* L^*-algebra.

DEFINITION 10. Let L be a semisimple L^*-algebra. By considering L as a complex Lie algebra we have the associated real Lie algebra L^R as in [4, p. 153]. Suppose now L_0 is a subalgebra of L^R with the following properties:

(i) L_0 is closed for the *-operation of L,

(ii) The inner product $\langle \cdot \rangle$ of L is real over L_0,

(iii) L_0 is a closed subset of L,

(iv) L as a vector space is a complexification of L_0: $L = L_0 + \sqrt{-1} L_0$.

Then L_0 is called a *real form* of the L^*-algebra L.

REMARK 7. Since a complete orthonormal set for L_0 is easily seen to be a complete orthonormal set for L also, we have

$$\dim L_0 = \dim L. \tag{9}$$

Further L is semisimple in the sense that its center is zero.

By (iv) every element z in L can be uniquely written as

$$z = x + \sqrt{-1}\, y (x, y \in L_0). \tag{10}$$

The map $\sigma: x + \sqrt{-1}\, y \to x - \sqrt{-1}\, y$ of L onto itself is called the conjugation of L with respect to L_0. σ has the properties

$$\sigma(\sigma z) = z, \qquad \sigma(z + z') = \sigma z + \sigma z',$$
$$\sigma(\alpha z) = \bar{\alpha} z, \qquad \sigma[z, z'] = [\sigma z, \sigma z'],$$
$$\langle \sigma z, \sigma z' \rangle = \langle z', z \rangle,$$

for $z, z' \in L$ and $\alpha \in C$. Conversely, if σ is any map of L onto itself with the above properties, the set L_0 of fixed points of σ is a real form of L having σ as the associated conjugation. (Cf. [4, p. 154].)

DEFINITION 11. A real form L_0 is called a *compact* form if

$$\langle x, x^* \rangle \leqq 0 \tag{11}$$

for every x in L_0.

THEOREM 5. *L has a unique compact form L_k consisting of all skew-adjoint elements of L, with the associated conjugation σ in L being given by $\sigma x = -x^*$ for all x in L.*

Proof. Define $L_k = \{x \in L : x^* = -x\}$. It is easy to verify that L_k is a real form. Further, for $x \in L_k$

$$\langle x, x^* \rangle = -\langle x, x \rangle \leqq 0$$

so that L_k is compact. Again, for $z \in L$ write

$$x = \tfrac{1}{2}(z - z^*), \qquad y = (z + z^*)/2\sqrt{-1};$$

then $x, y \in L_k$ and $z = x + \sqrt{-1}\, y$. Hence

$$\sigma z = x - \sqrt{-1}\, y = -x^* + \sqrt{-1}\, y^* = -(x + \sqrt{-1}\, y)^* = -z^*.$$

It remains to prove the uniqueness of the compact form L_k. Suppose now L_0 is any compact form of L. If $x \in L_0$, then x^* and hence also $z = x + x^* \in L_0$. Since $z^* = z$, by (11),

$$\langle z, z \rangle = \langle z, z^* \rangle \leqq 0,$$

whence $z = 0$, or $x^* = -x$. Further, for any $z \in L$ by (10) $z = x + \sqrt{-1}\, y$ with $x, y \in L_0$. Hence

$$z^* = x^* - \sqrt{-1}\, y^* = -x + \sqrt{-1}\, y = -z.$$

This proves $L_0 = L_k$.

Remark 7. Let L be a regular L^*-algebra with a Cartan subalgebra H. Let

$$T_W = \{h_\rho;\, e_\alpha, e_\alpha^* : \rho \in \Pi, \alpha \in \Delta^+\}$$

be a Weyl basis relative to H (Remark 6). Write

$$B_0 = \{\sqrt{-1}\, h_\rho, e_\alpha - e_\alpha^*, \sqrt{-1}(e_\alpha + e_\alpha^*) : \rho \in \Pi, \alpha \in \Delta^+\}, \qquad L_1 = \text{closure of } \mathrm{Sp}_R\, B_0.$$

Then L_1 is a real L^*-algebra having B_0 for a total basis (as defined in Remark 5) and the multiplication table for B_0 is completely determined by the graph G associated with the simple system Π. Further it is not difficult to see that L_1 coincides with the compact real form L_k of L. Thus, as in the finite-dimensional case the Weyl basis can be used to determine the compact form.

Bibliography

1. V. K. Balachandran, *The Weyl group of an L^*-algebra*, Math. Ann. **154** (1964), 157–165.

2. V. K. Balachandran and P. R. Parthasarathy, *Cartan subalgebras of an L^*-algebra*, Math. Ann. **166** (1966), 300–301.

3. H. Freudenthal, *Lie groups*, Mimeographed lecture notes, Yale Univ., New Haven, Conn., 1961.

4. S. Helgason, *Differential geometry and symmetric spaces*, Academic Press, New York, 1962.

5. N. Jacobson, *Lie algebras*, Interscience, New York, 1962.

6. I. Satake, *On a theorem of E. Cartan*, J. Math. Soc. Japan **2** (1951), 284–305.

7. J. R. Schue, *Hilbert space methods in the theory of Lie algebras*, Trans. Amer. Math. Soc. **95** (1960), 69–80.

8. ———, *Cartan decompositions for L^*-algebras*, Trans. Amer. Math. Soc. **98** (1961), 334–349.

University of Madras, The Ramanujan Institute,
Madras-5, India