FUNCTIONS STARLIKE OF ORDER \( \alpha \)

BY

MELVYN KLEIN

1. Introduction. Let \( f(z) \) be regular in the unit disk, \( \mathbb{U} \), with an expansion of the form:

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n.
\]

We will say that \( f(z) \) is starlike of order \( \alpha \) in \( \mathbb{U} \) if:

\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad \text{for all } z \in \mathbb{U};
\]

we will denote by \( S(\alpha) \) the class of all such functions for fixed \( \alpha \). We consider all real \( \alpha : -\infty < \alpha \leq 1 \). The class \( S(0) \) will be recognized as the class of functions starlike with respect to the origin.

For the natural number \( p \) we define the related classes.

\[
S_p(\alpha) = \{ f_p(z) = f(z^p) : f(z) \in S(\alpha) \}.
\]

Extremal problems for the coefficients in the power series expansions of functions in \( S_p(0) \) and powers of such functions were recently investigated by J. T. Poole [4]. In this paper we will show that these coefficient problems are equivalent to extremal problems for the coefficients in the power series expansion of functions in \( S(\alpha) \), \( \alpha < 0 \). This alternative approach leads to a substantial generalization.

We will also prove several theorems of the type commonly called “distortion theorems” for functions in the classes \( S(\alpha) \). One such result is the:

**Theorem.** Let \( f(z) \in S(\alpha), -\infty < \alpha \leq 1 \). Then for each natural number \( n \) there is a point \( z = e^{i\theta} \) on the unit circle such that:

\[
\sum_{k=1}^{n} |f(e^{i\theta} \cdot \eta_k)| \geq n,
\]

where \( \{\eta_k\} k = 1, 2, \ldots, n \) denote the \( n \)th roots of unity.

Let \( f(z) \) be regular in the exterior of the unit circle \( \mathbb{V} \), except for a simple pole at infinity, with an expansion of the form:

\[
f(z) = z + \sum_{n=0}^{\infty} a_n z^{-n}.
\]

We will say that \( f(z) \) is starlike of order \( \alpha \) in \( \mathbb{V} \) if:

\[
\text{Re} \left\{ \frac{zf''(z)}{f'(z)} \right\} > \alpha \quad \text{for all } z \in \mathbb{V};
\]

Received by the editors December 16, 1966.

99
we will denote by $\Sigma(\alpha)$ the class of all such functions for fixed $\alpha$. The class $\Sigma(0)$ is the class of functions whose compact complement is starlike with respect to the origin.

For the natural number $p$ we define the related classes:

$$\Sigma_p(\alpha) = \{ f_p(z) = \hat{f}(z^p)^{1/p} : \hat{f}(z) \in \Sigma(\alpha) \}.$$ 

We will obtain partial results for the coefficients in the power series expansions of functions in $\Sigma_p(\alpha)$ and powers of such functions by methods analogous to those employed with $S_p(\alpha)$.

2. Two lemmas for the classes $S(\alpha)$. The first lemma is, essentially, a result of M. S. Robertson [6, p. 386]. Though in his paper he speaks only of $S(\alpha)$, $0 \leq \alpha \leq 1$, the same result holds for all $-\infty < \alpha \leq 1$; his proof goes through without modification.

First a word about notation. If $x$ is a positive real number and $n$ a natural number, we will write:

$$\binom{-x}{n} = \frac{(-x)(-x-1)\cdots(-x-n+1)}{n!}.$$ 

**Lemma 2.1.** If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in $S(\alpha)$, $-\infty < \alpha \leq 1$, then:

$$|a_{n+1}| \leq \left| \binom{-2(1-\alpha)}{n} \right|.$$ 

The second lemma is an extension of a theorem of Merkes, Robertson and Scott [1].

**Lemma 2.2.** Suppose

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

$$h(z) = z + \sum_{k=2}^{\infty} b_k z^k$$

and that

(2) \quad f(z) = z [h(z)/z]^{1/(1-\beta)}.

Then $f(z) \in S(\alpha)$ iff $h(z) \in S(\alpha + \beta - \alpha \beta)$.

**Proof.** From (2) we calculate:

$$\text{Re} \{z f'(z)/f(z)\} = 1 - (1/1-\beta) + (1/1-\beta) \cdot \text{Re} \{z h'(z)/h(z)\}$$

so that

$$(1-\beta) \text{Re} \{z f'(z)/f(z)\} + \beta = \text{Re} \{z h'(z)/h(z)\}.$$ 

Thus $f(z) \in S(\alpha)$, i.e., $\text{Re} \{z f'(z)/f(z)\} > \alpha$ iff $h(z) \in S(\alpha + \beta - \alpha \beta)$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
3. A coefficient theorem for functions in $S_p(\alpha)$.

**Theorem 3.1.** Let $f_p(z) \in S_p(\alpha)$ and let $t$ be any positive real number. Then the coefficients of

$$[f_p(z)]^t = z^t + a_{p+1}z^{p+1} + a_{2p+1}z^{2p+1} + \ldots$$

are subject to the sharp bounds

$$|a_{n+p+t}| \leq \left| \left( -\frac{2(t/p)(1-\alpha)}{n} \right) \right|, \quad n = 1, 2, \ldots.$$

**Proof.** By definition, $f_p(z) = f(z^p)^{1/p}$, where $f(z) \in S(\alpha)$.

We use the representation (2) for the function $f(z)$, with $\beta = 1 - t/p$:

$$f(z) = z^{1-\beta} h(z)^{1/p}$$

or

$$f(z)^{1/p} = z^{(1-\beta)/p} h(z).$$

Let $\xi = z$. This becomes

$$[f(\xi^p)^{1/p}]^t = \xi^{(1-\beta)/p} h(\xi).$$

Equation (3) shows that the coefficients of $[f_p(z)]^t$ are identical with those of a function:

$$h(z) \in S(\alpha + 1 - t/p - \alpha [1 - t/p]) = S(1 + t/p[\alpha - 1])$$

by Lemma 2.2.

Applying Lemma 2.1 to $h(z)$ we find that it is subject to the sharp coefficient bounds:

$$|b_{n+1}| \leq \left| \left( -\frac{2(t/p)(1-\alpha)}{n} \right) \right|.$$

Thus these are sharp bounds for the coefficients $|a_{n+p+t}|$ of the function $[f_p(z)]^t$ as asserted.

It should be remarked that the hypothesis: $f_p \in S_p(\alpha)$ is equivalent to the condition: $f_p \in S(\alpha)$. For a straightforward calculation shows:

$$\text{Re} \left\{ z f_p'(z)/f_p(z) \right\} = \text{Re} \left\{ z [f(z^{2p})^{1/p}]'/f(z^{2p})^{1/p} \right\}$$

$$= \text{Re} \left\{ z^p f'(z^p)/f(z^p) \right\} = \text{Re} \left\{ \xi f'(\xi)/f(\xi) \right\},$$

where $|z| < 1$ iff $|\xi| < 1$. Thus we have the

**Corollary 3.1.** If $f_p(z)$ is starlike with respect to the origin, the coefficients of $[f_p(z)]^t$ are subject to the sharp bounds:

$$|a_{n+p+t}| \leq \left| \left( -\frac{2t/p}{n} \right) \right|, \quad n = 1, 2, \ldots$$

**Proof.** Set $\alpha = 0$ in Theorem 3.1. This is Poole's theorem, if we restrict $t > 0$ to integral values.
Corollary 3.2. If \( f_p(z) \) is convex, the coefficients of \( [f_p(z)]' \) are subject to the sharp bounds:

\[
|a_{np+1}| \leq \left| \frac{-t/p}{n} \right|, \quad n = 1, 2, \ldots
\]

**Proof.** The well-known condition that a function \( g(z) \) be convex is:

\[
\text{Re} \left\{ \frac{z}{g'(z)} + 1 \right\} > 0 \quad \text{for all } z \in \mathbb{U}.
\]

Evaluating this functional for \( f_p(z) \) we find:

\[
\text{Re} \left\{ z \frac{f_p(z)}{f'(z)} + 1 \right\} = \text{Re} \left\{ z \frac{f'(z) + 1}{f(z)p} \right\} = \text{Re} \left\{ z \frac{f''(z) + 1}{f(z)} \right\}
\]

where \( |z| < 1 \) iff \( |\xi| < 1 \).

Our hypothesis that \( f_p(z) \) is convex together with (5) and (6) means that in our case:

\[
p \cdot \text{Re} \left\{ \xi f''(\zeta) + 1 \right\} > (p - 1) \cdot \text{Re} \left\{ \xi f'(\zeta) \right\}.
\]

Furthermore, \( f_p(z) \) convex implies, a fortiori, that \( f_p(z) \) is starlike; therefore by (4), \( f(z) \) is starlike so that:

\[
p \cdot \text{Re} \left\{ \xi f''(\zeta) + 1 \right\} > 0.
\]

We have thus shown that \( f_p(z) \) convex implies that \( f(z) \) is convex.

E. Strohhacker has shown [8] that if \( f(z) \) is a convex function then:

\[
\text{Re} \left\{ z f'(z) / f(z) \right\} > 1/2 \quad \text{for all } z \in \mathbb{U}.
\]

Consequently, \( f(z) \in S(1/2) \) so that by definition, \( f_p(z) \in S_p(1/2) \). Setting \( \alpha = 1/2 \) in Theorem 3.1 we obtain the stated result.

In the paper of Strohhacker referred to just above he proves that a convex function is also a star function of order not less than 1/2, but that the converse is false. This he illustrates with the example: \( f_1(z) = z + z^2/3 \); he shows that \( f_1(z) \in S(1/2) \) but nevertheless \( f_1(z) \) is not convex.

It is natural to ask whether there is an order of starlikeness \( \alpha > 1/2 \) which will guarantee convexity. We will show there is none; that is to say:

**Theorem 3.2.** For every \( \alpha < 1 \) there exists a function \( g(z) \in S(\alpha) \) which is not convex.

**Proof.** As we shall show presently, if \( f(z) \in S(\alpha) \), \( \alpha < 1 \), there is some point \( \zeta \in \mathbb{U} \) for which

\[
\text{Re} \left\{ \xi f''(\zeta) + 1 \right\} - \text{Re} \left\{ \xi f'(\zeta) \right\} = \delta < 0 \quad \text{and} \quad \text{Re} \left\{ \xi f'(\zeta) \right\} < 2.
\]

Choose \( p \) so large that: \( 2 + p \cdot \delta < 0 \).
Now let \( f(z) \in S(\alpha) \) and consider the function \( f_p(z) \). By (6):

\[
\text{Re} \left\{ z \left[ f_p(z) \right]'/\left[ f_p(z) \right] + 1 \right\} = p \cdot \text{Re} \left\{ f'(\zeta) / f(\zeta) + 1 \right\} - p \cdot \text{Re} \left\{ \xi f'(\xi) / f(\xi) + \text{Re} \left\{ f'(\xi) / f(\xi) \right\} \right\}
\]

\[
< p \cdot \delta + 2 < 0 \quad \text{for} \quad z^p = \zeta.
\]

Thus \( f_p(z) \) is not convex. On the other hand we have by (4) that \( f_p(z) \in S(\alpha) \). This is the sought after function, \( g(z) \).

It remains to establish the inequality (7).

The difference on the left side can be expressed as:

\[
\text{Re} \left\{ z \frac{d}{dz} \log \left( \frac{f'(z)}{f(z)} \right) \right\} = \text{Re} \left\{ z \frac{d}{dz} \log \left( 1 + b_1 z + b_2 z^2 + \cdots \right) \right\}
\]

The only function for which \( b_1 = b_2 = \cdots = 0 \) is \( f(z) = z \); this function is excluded by the hypothesis: \( \alpha < 1 \). The expression in braces is therefore a MacLaurin series equal to zero at \( z = 0 \) but not identically zero. By continuity it maps a neighborhood of the origin onto a neighborhood of the origin. Since \( \text{Re} \left\{ \xi f'(\xi) / f(\xi) \right\} = 1 \) at \( \xi = 0 \), this neighborhood can be chosen small enough that \( \text{Re} \left\{ \xi f'(\xi) / f(\xi) \right\} < 2 \) at every point of it. This proves (7).

4. A coefficient theorem for functions in \( \Sigma_p(\alpha) \). In this section we obtain partial results for \( \Sigma_p(\alpha) \) of the same type as those which were obtained for \( S_p(\alpha) \) in Theorem 3.1. Our methods are modified owing to the lack of an equivalent to Lemma 2.1 for the classes \( \Sigma(\alpha) \). We have only a theorem of Pommerenke [3] which gives sharp coefficient bounds for \( f \in \Sigma(\alpha) \), \( \alpha \geq 0 \). His proof does not lend itself to an extension to cases where \( \alpha < 0 \). Pommerenke’s theorem is:

**Lemma 4.1.** Let \( \tilde{f}(z) = z + \sum_{n=0}^{\infty} \tilde{a}_n z^{-n} \) be in \( \Sigma(\alpha) \), \( 0 \leq \alpha \leq 1 \).

Then \( |\tilde{a}_n| \leq 2(1-\alpha)/(n+1) \), \( n = 0, 1, 2, \ldots \), with equality for the functions:

\[
\tilde{f}(z) = z(1+z^{-n-1})^{\alpha(1-\alpha)/(n+1)}.
\]

**Lemma 4.2.** Suppose

\[
\tilde{f}(z) = z[\tilde{h}(z)/z]^{1/(1-\beta)}.
\]

Then \( \tilde{f}(z) \in \Sigma(\alpha) \) if and only if \( \tilde{h}(z) \in \Sigma(\alpha+\beta-\alpha\beta) \).

**Proof.** See the proof of Lemma 2.2.

**Theorem 4.1.** The coefficients of \( [f_p(z)]^t \), \( f_p \in \Sigma_p(\alpha) \) where \( t > 0 \) and \( (1-\alpha)t \leq p \) are subject to the sharp bounds:

\[
|\tilde{a}_{(n+1)p-t}| \leq t/p \cdot 2(1-\alpha)/(n+1), \quad n = 0, 1, 2, \ldots
\]
Proof. In a manner analogous to that of the proof of Theorem 3.1 we arrive at:

\[ f_p(z) = z^{t-p} \cdot h(z^p) = z^t + b_0 z^{t-p} + b_1 z^{t-2p} + \ldots \]

where \( h(z) \in \Sigma(1 + (t/p)(\alpha - 1)) \). When \( 1 + (t/p)(\alpha - 1) \geq 0 \), that is, when \( (1-\alpha)t \leq p \), the coefficients of \( h(z) \) are, by Lemma 4.1, subject to the sharp bounds:

\[ |b_n| \leq \frac{2(t/p)(1-\alpha)}{n+1}. \]

Thus these are sharp bounds for the coefficients: \( |a_{(n+1)p-1}| \) of the function \( [f'_p(z)] \).

5. Covering theorems for the classes \( S(\alpha) \). A problem in conformal mapping is determining the largest disk about the origin \( w=0 \), covered by every mapping in a particular class. The classical result of Koebe-Bieberbach [2, p. 214] states that every mapping \( w=f(z) \in S \) covers the disk: \( |w| < 1/4 \). The next theorem settles this problem for the classes under consideration here. Since radial limits exist for functions of these classes, we can speak about the value of a function at a point on the unit circle without any ambiguity.

**Theorem 5.1.** Suppose \( f(z) \in S(\alpha) \), \( -\infty < \alpha \leq 1 \). Then the image of the circle \( |z| = 1 \) under \( w=f(z) \) lies exterior to the disk:

\[ |w| < (1/4)^{1-\alpha}. \]

**Proof.** We write

\[ f(z) = z[h(z)/z]^{1-\alpha}. \]

By Lemma 2.2 \( h(z) \in S(0) \). It is known that \( S(0) \subset S \) [7] where \( S \) denotes the class of functions regular and univalent in \( \mathbb{U} \), normalized as in (1).

If \( z = e^{i\theta} \) is a point on the circle \( |z| = 1 \) we have \( |f(e^{i\theta})| = |h(e^{i\theta})|^{1-\alpha} \geq (1/4)^{1-\alpha} \) by the Koebe-Bieberbach theorem.

Suppose the hypothesis \( f(z) \) is convex is inserted in Theorem 5.1. Then according to the result of E. Strohhacker [8] \( f(z) \in S(1/2) \). Consequently, the image of \( \mathbb{U} \) under \( w=f(z) \) covers the disk \( |w| < (1/4)^{1-1/2} = 1/2 \); a well-known result.

We have established a sharp lower bound for \( |f(e^{i\theta})| \) where

\[ f \in S(\alpha) : |f(e^{i\theta})| \geq (1/4)^{1-\alpha}. \]

Let \( \eta_1, \eta_2, \ldots, \eta_n \) be the set of \( n \)th roots of unity. It follows trivially that:

\[ \sum_{k=1}^{n} |f(e^{i\theta}, \eta_k)| \geq n(1/4)^{1-\alpha}. \]

The next theorem will improve on this lower bound. We will need two lemmas.

**Lemma 5.1.** If \( f_1(z), f_2(z), \ldots, f_n(z) \) are each in \( S(\alpha) \), then:

\[ \left[ \prod_{k=1}^{n} f_k(z) \right]^{1/n} \in S(\alpha). \]
Proof. Let
\[ g(z) = \left[ \prod_{k=1}^{n} f_k(z) \right]^{1/n}. \]

Straightforward calculation shows:
\[ \Re \left\{ z \frac{g'(z)}{g(z)} \right\} = \frac{1}{n} \sum_{k=1}^{n} \Re \left\{ z \frac{f_k'(z)}{f_k(z)} \right\} > \frac{1}{n} (n\alpha) = \alpha. \]

Lemma 5.2 (Rengel [5]). Let \( w = f(z) \) belong to class \( S \) and consider any system of \( n \) rays emerging from \( w = 0 \) at equal angles. Then the maximal distance from \( w = 0 \) of the nearest boundary points in the \( w \)-plane on these \( n \) rays is not less than \((1/4)^{1/n}\).

Theorem 5.2. Let \( f(z) \in S(\alpha) \) and \( z = e^{i\theta} \) a point on the circle \( |z| = 1 \). Let \( \eta_1, \eta_2, \ldots, \eta_n \) be the set of \( n \)-th roots of unity. Then
\[ \sum_{k=1}^{n} |f(e^{i\theta} \cdot \eta_k)| \geq n(1/4)^{(1-\alpha)/n}. \]

Proof. Consider the function
\[ g(z) = \left[ \prod_{k=1}^{n} \frac{f(z \cdot \eta_k)}{\eta_k} \right]^{1/n}. \]

Each of the factors is in \( S(\alpha) \) and therefore by Lemma 5.1, \( g(z) \in S(\alpha) \). Consequently we can write, as we did in (8):
\[ g(z) = z \left[ h(z) / |z| \right]^{1-\alpha} \quad \text{where} \quad h(z) \in S(0) \subseteq S. \]

The mapping \( w = g(z) \) has the property that for every point \( r \cdot e^{i\theta} \) in the closed disk \(|z| \leq 1|:\)
\[ g(re^{i\theta} \cdot \eta_k) = \eta_k \cdot g(re^{i\theta}), \quad k = 1, \ldots, n. \]

In other words, the set of \( n \)-fold symmetry \( \{re^{i\theta} \cdot \eta_k\}, k = 1, \ldots, n \) maps onto a set of \( n \)-fold symmetry. Using this fact, (9) and Lemma 5.2 we may conclude:
\[ |g(e^{i\theta})| = |h(e^{i\theta})|^{1-\alpha} \geq (1/4)^{(1-\alpha)/n} \]
or
\[ \left| \prod_{k=1}^{n} f(e^{i\theta} \cdot \eta_k) \right|^{1/n} \geq (1/4)^{(1-\alpha)/n}. \]

The inequality relating the arithmetic and geometric means then implies the inequality of the theorem.

Theorem 5.3. If \( \Re \{z f'(z) / f(z)\} \) is bounded from below; that is to say, if \( f(z) \in S(\alpha) \) for some \( \alpha \), there is a point \( z = e^{i\theta} \) for which:
\[ \sum_{k=1}^{n} |f(e^{i\theta} \cdot \eta_k)| \geq n, \]
\[ n = 1, 2, \ldots, z = \phi(n). \]
Proof. Given arbitrary $\epsilon > 0$ choose $s$ so large that $(1/4)^{(1-a)/ns} > 1 - \epsilon$. By the preceding theorem:
\[
\sum_{k=1}^{ns} |f(e^{i\theta} \cdot \beta_k)| \geq ns(1/4)^{(1-a)/ns}
\]
where $\beta_1, \beta_2, \ldots, \beta_{ns}$ are the $ns$ roots of unity. The set of points $\{e^{i\theta} \cdot \beta_k\}$, $k = 1, 2, \ldots, ns$ on the unit circle can be considered as $s$ disjoint sets of $n$ points each, every one of which is a set of $n$th roots of unity (rotated on the unit circle). The above inequality can therefore be written as:
\[
\sum_{j=1}^{s} \sum_{k=1}^{n} |f(e^{i\theta} \cdot \phi_j \cdot \eta_k)| \geq ns(1/4)^{(1-a)/ns}
\]
where $\{\eta_k\}$, $k = 1, \ldots, n$, are the $n$th roots of unity and $\{\phi_j\}$, $j = 1, \ldots, s$, are the first $s$ of the $ns$ roots of unity. This means that for some $\phi_j$:
\[
\sum_{k=1}^{n} |f(e^{i\theta} \cdot \phi_j \cdot \eta_k)| \geq n(1/4)^{(1-a)/ns} > n(1-\epsilon).
\]
Since $\epsilon$ is arbitrary the result follows.

References

New York University,
University Heights