CONDITIONS IMPLYING THAT A 2 SPHERE IS ALMOST TAME

BY

L. D. LOVELAND

1. Introduction. Recently, Hempel [15] proved that a 2-sphere $S$ in $S^3$ is tame if $S$ is free and satisfies an additional Condition (A). It is not known whether $S$ is tame if it is free; however, each complementary domain of a free 2-sphere must be an open 3-cell [18]. We show that $S$ has at most two wild points if $S$ satisfies (A) and each component of $S^3 - S$ is an open 3-cell. Thus it appears that Hempel needed the full force of freeness only to rid $S$ of two wild points.

Burgess [11] proved that a 2-sphere $S$ in $S^3$ is tame modulo two points if each component of $S^3 - S$ is an open 3-cell and $S$ can be locally peripherally collared. In the next section we define a surface to be locally annular if it satisfies certain conditions more general than those in Burgess’ definition of locally peripherally collared. We show in §5 that a surface is locally annular if it satisfies Hempel’s Condition (A). Furthermore Burgess’ result remains valid with “locally annular” replacing “locally peripherally collared” (see §4).

No examples were given in [15] of surfaces which fail to satisfy Condition (A). It follows from our results in §5 that the examples described in [1] and [13] each fail to satisfy (A). These examples also fail to be locally annular.

We show that a closed connected 2-manifold in $S^3$ is tame if it is free and locally annular (see §6). All the main results of [15] follow as corollaries, and in addition we remove the condition in the hypothesis of Theorem 4 of [15] that the 2-manifold $M$ be a 2-sphere. In §3 we develop a characterization of tame surfaces in $S^3$ which turns out to be useful in both §§4 and 6.

2. Definitions and notation. If $e$ is a positive number we use “$e$-disk”, “$e$-set”, “$e$-annulus”, etc., to mean that the given set has diameter less than $e$. However, when we say $f$ is an $e$-map we mean that $d(x, f(x))$ is less than $e$ for all $x$ in the domain of the continuous function $f$. An open disk is any set homeomorphic to the interior of a circle in the plane. An open annulus is a set homeomorphic to $\{(x, y) | \frac{1}{2} < x^2 + y^2 < 1\}$. We use $N(x, e)$ to denote the set of all points within a distance $e$ of the point $x$, and we use $N(M, e)$ to denote the set of all points within a distance $e$ of some point of the point set $M$.

In the following definitions we consider $M$ to be a closed connected 2-manifold in $S^3$, $U$ to be a component of $S^3 - M$, and $p$ to be a point of $M$.

CONDITION (A). We say that $M$ satisfies Condition (A) provided that whenever
D is a polyhedral disk in $S^3$ with $\text{Bd} \; D$ in $S^3-M$ and $V$ is any neighborhood of $D$ there is a disk $D'$ (not necessarily tame) such that 
(i) $\text{Bd} \; D = \text{Bd} \; D'$,
(ii) $D' \subset V$, and
(iii) if $C$ is the component of $D' - M$ which contains $\text{Bd} \; D'$, then $D' - C$ has only a finite number of components.

Also $M$ satisfies Condition (A) relative to $U$ if the above definition holds for those disks $D$ with $\text{Bd} \; D$ in $U$.

A simple closed curve $\gamma$ is said to be essential on an annulus (or on an open annulus) $A$ provided $\gamma$ lies on $A$ and bounds no disk on $A$. We say that a simple closed curve $\gamma$ spans $M$ if and only if $\gamma \cap M$ consists of two points and $\gamma$ intersects each component of $S^3-M$.

Locally Annular. We say that $M$ is locally annular in $U$ at $p$ if and only if for each $\varepsilon > 0$ and for each simple closed curve $\gamma$ which spans $M$ and contains $p$, there is an open annulus $A$ in $U \cap N(p, \varepsilon)$ such that
(i) $\gamma \cap \overline{A} = \emptyset$,
(ii) one component of $\text{Bd} \; A$ is a simple closed curve $K$ in $U$ so that $\gamma$ links $K$, and
(iii) $\text{Bd} \; A - K \subset M$.

In case the above definition holds at each point of $M$ we say $M$ is locally annular in $U$. If the above definition holds for each point $p$ of $M$ and for each component $U$ of $S^3-M$ we say that $M$ is locally annular.

The above definition is similar to what is meant by $M$ being locally peripherally collared from $U$ at $p$ [11]. The major difference is that in [11] the annulus $A$ has two simple closed curves as its boundary components. Thus a locally peripherally collared surface is locally annular. In §4 of this paper we modify the proofs in [11] to get similar results for locally annular surfaces.

We say that $M$ is free relative to $U$ if and only if for each $\varepsilon > 0$ there is an $\varepsilon$-map of $M$ into $U$. Then $M$ is free in $S^3$ if and only if $M$ is free relative to each complementary domain of $M$.

The manifold $M$ is tame in $S^3$ if there is a homeomorphism of $S^3$ onto itself which takes $M$ onto a polyhedron. Also $M$ is tame from $U$ if $M \cup U$ is a 3-manifold-with-boundary. We define $M$ to be locally tame from $U$ at $p$ if $p$ lies in a subset $V$ of $M \cup U$ such that $V$ is open relative to $M \cup U$ and the closure of $V$ is a topological cube. It follows from [2], [3], and [17] that $M$ is tame if and only if $M$ is tame from each component of $S^3-M$; and that $M$ is tame from $U$ if and only if $M$ is locally tame from $U$ at each point of $M$. If $K$ is a subset of $M$ we say that $M$ is locally tame from $U$ modulo $K$ if and only if $M$ is locally tame from $U$ at each point of $M-K$.

For other definitions see [4], [10], or [15].

3. A characterization of tame surfaces. Let $M$ be a closed connected 2-manifold in $S^3$, let $U$ be a component of $S^3-M$, and let $p$ be a point of $M$. We say that $M$ can be locally $a$-capped from $U$ at $p$ if and only if for each $\varepsilon > 0$ there is an $\varepsilon$-disk
R on M so that \( p \in \text{Int } R \) and for each \( \alpha > 0 \) there is an open \( \epsilon \)-disk \( D \) in \( U \cap N(M, \alpha) \) such that \( \text{Bd } D \) lies in \( M - R \) and \( R \) lies on the boundary of an \( \epsilon \)-component of \( U - D \). If the above definition holds for each \( p \) in \( M \) we say that \( M \) can be \textit{locally} \( \alpha \text{-capped from } U \); and if the above definition is independent of both \( p \) and \( U \) we say that \( M \) can be \textit{locally} \( \alpha \text{-capped} \).

**Theorem 1.** If \( M \) can be locally \( \alpha \text{-capped from } U \) at \( p \in M \), then \( U \) is locally simply connected at \( p \).

**Proof.** Let \( N \) be an open set containing \( p \), and choose a positive number \( \epsilon \) such that \( 2\epsilon < d(p, S^3 - N) \). Let \( R \) be an \( \epsilon \)-disk on \( M \) such that \( p \in \text{Int } R \) and for each \( \alpha > 0 \) there exists an open \( \epsilon \)-disk \( D \) in \( U \cap N(M, \alpha) \) where \( \text{Bd } D \) lies in \( M - R \) and \( R \) lies on the boundary of an \( \epsilon \)-component of \( U - D \). Such a disk \( D \) will lie in \( N \).

Let \( V \) be an open set such that \( p \in V \), \( \text{Bd } V \) is a tame 2-sphere, and \( V \cap M \) lies in \( \text{Int } R \), and let \( f \) be a map of \( R \) into \( V \) such that \( f(\text{Bd } R) \subseteq U \). Choose a positive number \( \alpha \) such that \( \alpha < d(M, f(\text{Bd } R)) \), and let \( D \) be an open \( \epsilon \)-disk chosen as above relative to this \( \alpha \). Let \( C \) be the \( \epsilon \)-component of \( U - D \) having \( R \) on its boundary.

We shall show that \( f|_{\text{Bd } R} \) can be extended to map \( R \) into \( U \cap N \). This will show that \( U \) is locally simply connected at \( p \).

Now \( D \) separates \( f(\text{Bd } R) \) from \( f(R) \cap M \) on \( f(R) - D \) (for otherwise an arc \( A \) in \( f(R) - D \) from a point in \( f(\text{Bd } R) \) to a point in \( R \) would lie in \( C \), contrary to the choice of \( \alpha \)). Let \( Q \) be the component of \( R - f^{-1}(D) \) containing \( \text{Bd } R \). Then \( f(Q) \subseteq N \cap U \) and \( f|_Q \) can be extended to a map of \( R \) into \( f(Q) \cup D \) which lies in \( N \cap U \).

**Theorem 2.** If \( V \) is an open subset of \( M \) so that \( M \) can be locally \( \alpha \text{-capped from } U \) at each point of \( V \), then \( M \) is locally tame from \( U \) at each point of \( V \).

**Proof.** From Theorem 1, \( U \) is locally simply connected at each point of \( V \). Thus Theorem 2 follows from the proof of Theorem 4 of [10].

**Theorem 3.** If \( M \) can be locally \( \alpha \text{-capped, then } M \) is tame.

**Proof.** It follows from Theorem 1 that each component of \( S^3 - M \) is locally simply connected at each point of \( M \). Thus each such component is 1-ULC, so Theorem 3 follows from [4].

**Remark.** We define \( M \) to be \textit{locally capped} if and only if for each \( \epsilon > 0 \), for each \( p \in M \), and for each component \( U \) of \( S^3 - M \) there exists an \( \epsilon \)-disk \( R \) on \( M \) and an \( \epsilon \)-open disk \( D \) in \( U \) such that

1. \( p \in \text{Int } R \),
2. \( \text{Bd } D \subseteq M - R \), and
3. \( R \) lies on the boundary of an \( \epsilon \)-component of \( U - D \).

We were unable to answer the following

**Question.** Is a closed connected 2-manifold \( M \) tame if it can be locally capped?

Results related to this question can be found in [4], [10], and [16].
4. Sufficient conditions for 2-spheres to be tame modulo two points. The next two lemmas are used in several places in the remainder of the paper in connection with the definition of locally annular.

**Lemma 1.** Suppose $S$ is a 2-sphere in $S^3$, $U$ is a component of $S^3 - S$, $E$ is a disk on $S$, $J$ is a simple closed curve such that $J \cap E$ is a single point $p$, $A$ is an open annulus in $U$ such that one component of $\partial A$ is a simple closed curve $K$ in $U$, $J$ links $K$, $J \cap A = \emptyset$, and $\partial A - K \subseteq E$. Then $\partial A - K$ separates $J \cap S - \{p\}$ from $p$ on $S$.

**Proof.** Let us denote $S - \text{Int} E$ by $B$. It will be sufficient to prove that $\partial A - K$ separates $p$ from $B$ on $S$. Suppose this is not the case. Then there exists an arc $H$ from $p$ to a point in $B$ such that $H \cap \partial A = \emptyset$. Let $r$ be a retraction of $S \cup (U \text{ minus a point})$ onto $S - \{p\}$. Now there exists a simple closed curve $K'$ essential in $A$ such that the identity map on $K'$ is homotopic to $r|K'$ in $S^3 - (J \cup H \cup B)$. Since $r(K') \subseteq E - H$, $r|K'$ is null homotopic in $E - H$ and $K$ can be shrunk to a point in $S^3 - J$. This is a contradiction since $J$ links $K$.

**Lemma 2.** Suppose we have the same conditions as in Lemma 1 and $J'$ is a simple closed curve containing a subarc that pierces $E$ at $p$ such that $J' \cap E = \{p\}$ and $J' \cap A = \emptyset$; then $K$ cannot be shrunk to a point in the complement of $J'$.

**Proof.** As in the proof of Lemma 1 there exists a loop $f$ homotopic in $S^3 - (J \cup J')$ to the identity map on $K$. If $K$ can be shrunk to a point in $S^3 - J'$, then $f$ can also be contracted there. But $f$ is homotopic in $S^3 - J'$ to some odd multiple of $\partial E$, so we have the contradiction that some odd multiple of $\partial E$ can be contracted in $S^3 - J'$.

**Theorem 4.** If $C$ is a cellular crumpled cube in $S^3$ and $V$ is an open subset of $\partial C$ such that $\partial C$ is locally annular in $S^3 - C$ at each point of $V$, then there is a point $p$ in $V$ such that $\partial C$ is locally tame from $S^3 - C$ at each point of $V - \{p\}$.

**Proof.** We let $S$ denote $\partial C$. If $S$ can be locally $\alpha$-capped from $S^3 - C$ at each point of $V$, then Theorem 4 follows from Theorem 2. Otherwise there is a point $p_1$ of $V$ so that $S$ cannot be locally $\alpha$-capped from $S^3 - C$ at $p_1$. We will show that under these circumstances $S$ can be locally $\alpha$-capped from $S^3 - C$ at each point of $V - \{p_1\}$. Then Theorem 4 will also follow from Theorem 2 in this case.

Let $p_2$ be a point of $V$ different from $p_1$, let $\epsilon > 0$ be given, and let $\delta$ be a positive number such that

(1) $7\delta < \epsilon$.

Let $J$ be a simple closed curve which spans $S$ and intersects $S$ at $\{p_1, p_2\}$. Using Lemma 1 and the hypothesis that $S$ is locally annular in $S^3 - C$ at $p_1$ and $p_2$, we obtain two disjoint open annuli $A_1$ and $A_2$, two simple closed curves $J_1$ and $J_2$, and two open sets $D_1$ and $D_2$ such that for each $i$,

(2) $J_i$ is a boundary component of $A_i$ in $S^3 - C$,

(3) $J$ links $J_i$,
(4) $D_i$ is the component of $S - \text{Bd } A_i$ containing $p_i$.

(5) $A_i \subset S^3 - C$, and

(6) $S \cap A_i$ lies in a disk $E_i$ on $S$ so that $D_i \subset E_i$ and diam $(E_i \cup A_i) < \delta$.

Let $R_1$ and $R_2$ be disjoint disks in $S$ so that $p_i$ is in $\text{Int } R_i$ and $R_i \subset D_i$, for each $i$. Then each $R_i$ is an $\varepsilon$-disk which does not intersect $\text{Bd } A_i$. We will eventually show that $R_2$ can be used as the "$R$" in the definition of "locally $\alpha$-capped from $S^3 - C$ at $p_2$". Thus we assume $\alpha$ to be a given positive number.

For each $i (i = 1, 2)$ we let $X_i$ be an arc in $J - \text{Int } C$ from $p_i$ to a point $t_i$ such that $X_2 \cap X_2 = \emptyset$ and diam $(X_i) < \delta$. With no loss in generality we assume that

(7) $\alpha < \delta$,  

(8) $\alpha < d(S, \{t_1, t_2\})$, and  

(9) $\alpha < d(S, J_i)$.

Since $C$ is cellular there is a 2-sphere $S_1$ in $S^3 - C$ such that

(10) $S_1 \subset N(S, \alpha)$.

From (8), (9), and (10) we see that $J_1 \cup J_2 \cup \{t_1, t_2\}$ and $S$ are in different components of $S^3 - S_1$.

Let $\sigma$ be a positive number chosen so small that

(11) $\sigma < \alpha$ and  

(12) $N(S, \sigma) \cap S_1 = \emptyset$.

Let $A'_1$ and $A'_2$ be open annuli in $A_1$ and $A_2$, respectively, such that

(13) $A'_i \subset N(S, \sigma)$ and  

(14) the boundary of $A_i - A'_i$ consists of two simple closed curves $J_i$ and $J'_i$.

We see from (12) and (13) that $S \cup A'_1 \cup A'_2$ and $J_1 \cup J_2$ lie in different components of $S^3 - S_1$.

Let $S_2$ be a 2-sphere such that $S$ and $S_1 \cup (A_1 - A'_1) \cup (A_2 - A'_2)$ are in different components of $S^3 - S_2$ and

(15) $S_2 \subset N(S, \sigma)$.

Using [5] we may assume that each $A'_i$ is locally polyhedral and in general position with respect to $S_2$ so that each component of $S_2 \cap (A_1 \cup A_2)$ is a simple closed curve and there are only finitely many such components. Because $S_2$ separates $S$ from $A_i - A'_i$ there must be an essential simple closed curve in each $S_2 \cap A'_i$. We choose a disk $D'$ in $S_2$ so that $\text{Bd } D'$ is an essential simple closed curve on either $A'_1$ or $A'_2$ and there is no simple closed curve in $\text{Int } D'$ which is essential on either $A'_1$ or $A'_2$.

The remainder of the proof is devoted to showing that if $\text{Bd } D'$ lies on $A'_1$ then $\text{Bd } C$ can be locally $\alpha$-capped in $S^3 - C$ at $p_i$. Once this is established we are able to conclude that $\text{Bd } D'$ lies on $A'_2$ since $p_1$ was chosen as a point where $S$ cannot be locally $\alpha$-capped from $S^3 - C$. Thus it will follow that $S$ can be locally $\alpha$-capped from $S^3 - C$ at $p_2$, which is what we set out to prove. We will assume, just to be definite, that $\text{Bd } D' \subset A'_1$.

Using the usual disk replacement process [4, p. 297] we obtain a disk $D$ so that $\text{Bd } D = \text{Bd } D'$,
(16) $\text{Int } D \subseteq S^3 - (C \cup A_1 \cup A_2 \cup S_1)$, and

(17) $D \subseteq D' \cup [N(A'_1 \cup A_2, \delta) \cap N(S, \sigma)]$.

In the next paragraph we show there is an arc $Y$ in $X_1$ from $p_1$ to a point $a$ in $\text{Int } D$ so that

(18) $\text{diam } Y < \delta$ and

(19) $Y \cap D = \{a\}$.

Let $Z$ be an arc in $S^3 - \text{Int } S_1$ from $t_1$ to a point $w$ of $J_2$ so that the interior of $Z$ does not intersect $X_1$, and let $W$ be an arc from $w$ to a point $v$ in $S$ so that $W$ is close enough to $A_2$ to insure that $W \cap (D_1 \cup D) = Z \cap \text{Int } W = \emptyset$. This is possible by (16). Let $J''$ be a simple closed curve contained in the union of $X_1 \cup Z \cup W$ with an open arc in $\text{Int } C$ from $v$ to $p_1$. It follows from Lemmas 1 and 2 that neither $J''_1$ nor $\text{Bd } D$ can be shrunk to a point in $S^3 - J''$. This means that $J''$ must intersect $D$, but $J'' - X_1$ was chosen in the complement of $D$. Thus it is apparent how to construct $Y$ in $X_1$.

Suppose that the diameter of $D$ is larger than $6 \delta$. We will obtain a contradiction using an argument similar to that given in [11]. There must be a point $b$ in $\text{Int } D$ so that

(20) $d(a, b) \geq 3 \delta$.

It follows from (15), (17), and the fact that $D' \subseteq S_2$ that there is a point $c$ of $S$ such that the line segment $cb$ has diameter less than $\sigma$. Then from (7), (11), and (12),

(21) $d(c, b) < \delta$ and

(22) $cb \cap S_1 = \emptyset$.

From (18) and (6) it follows that $D_1 \cup A_1$ lies in a $2 \delta$ neighborhood of $a$, so that from (20) and (21) we have

(23) $cb \cap (D_1 \cup A_1) = \emptyset$.

Let $B$ be an arc from $a$ to $c$ such that $B$ contains $cb$ and $B - cb$ lies in $D$, and let $L$ be a simple closed curve containing $Y \cup B$ such that $L - Y \cup B \subseteq \text{Int } C$. Then $L$ pierces $S$ at $p_1$, $L \cap (A_1 \cup S_1) = \emptyset$, and $L \cap E_1 = \{p_1\}$. Since $L$ and $J_1$ lie in different components of $S^3 - S_1$ we have a contradiction to Lemma 2.

Therefore,

(24) $\text{diam } D < 6 \delta$.

Now one component, say $F$, of $A'_1 - \text{Bd } D$ is an open annulus with one boundary component on $S$. We let $E = D \cup F$ so that $E$ is an open disk in $S^3 - S$ such that $\text{Bd } E \subseteq S - R_1$. It follows from (6), (24), and (1) that

(25) $\text{diam } E < \epsilon$.

All that remains in showing that $S$ can be locally $\alpha$-capped from $S^3 - C$ at $p_1$ is to show that

(26) $E \subseteq N(S, \sigma)$ and

(27) $R_1$ lies on the boundary of an $\epsilon$-component of $(S^3 - C) - E$.

Condition (26) follows from (11), (13), (15), and (17).

Let $H$ be the component of $(S^3 - C) - E$ such that $R_1 \subseteq \text{Bd } H$. Condition (27) will follow from (6), (24), and (1) once we show that $\text{Bd } H \subseteq E_1 \cup A_1 \cup D$. If this
is not the case there must be a point \( q \) of \( \text{Bd} \ H \) so that \( q \) is not in \( E_1 \cup A_1 \cup D \). Then \( q \) is in \( S \) and there is a simple closed curve \( J' \) which spans \( S \), intersects \( S \) only at \( p_i \) and \( q \), and which does not intersect \( F \cup D \). This contradicts Lemma 2 since \( J' \) does not intersect \( D \).

**Corollary 1.** If \( C \) is a cellular crumpled cube and \( W \) is the set of points where \( \text{Bd} \ C \) is wild from \( S^2 - C \), then either

1. \( W = \emptyset \),
2. \( W \) is degenerate, or
3. \( W \) contains a nondegenerate continuum.

**Corollary 2.** With the same hypothesis as in Corollary 1, \( W \) does not contain two isolated points.

**Corollary 3 (Burgess).** If \( S \) is a 2-sphere in \( S^3 \) such that each component of \( S^3 - S \) is an open 3-cell and the set \( W \) of wild points of \( S \) is 0-dimensional, then \( W \) contains at most two points.

5. **Condition (A) implies that \( M \) is locally annular.** If \( D \) is a disk and \( D_1, D_2, \ldots, D_n \) is a finite collection of disjoint disks in \( \text{Int} \ D \), then \( D - \bigcup \text{Int} \ D_i \) is called a disk-with-holes.

**Lemma 3.** If \( D \) is a disk-with-holes in \( S^3 \). \( J \) is a simple closed curve which links one component of \( \text{Bd} \ D \), and \( J \) does not intersect \( D \), then \( J \) links at least two components of \( \text{Bd} \ D \).

**Proof.** The proof follows by repeated use of Theorems 9 and 10 of [5].

**Theorem 5.** If \( C \) is a crumpled cube in \( S^3 \) such that \( \text{Bd} \ C \) satisfies Condition (A) relative to \( S^3 - C \), then \( \text{Bd} \ C \) is locally annular in \( S^3 - C \).

**Proof.** We will let \( S \) denote the 2-sphere that is the boundary of \( C \). Let \( J \) be a simple closed curve that spans \( S \) such that \( J \cap S \) consists of the two points \( p \) and \( q \), and let \( \varepsilon \) be a positive number. We must show that there is an open annulus \( A \) in \( (S^3 - C) \cap N(p, \varepsilon) \) such that \( J \cap A = \emptyset \), one component \( K \) of \( \text{Bd} \ A \) is a simple closed curve in \( U \) such that \( J \) links \( K \), and \( \text{Bd} \ A - K \) lies in \( S \).

Let \( N \) be an open \( \varepsilon \)-set such that \( p \) lies in \( N \) and \( q \in S - \overline{N} \). Let \( G \) be an annulus in \( N \cap S \) so that each component of \( \text{Bd} \ G \) is a tame simple closed curve which separates \( p \) from \( q \) [8]. Let the closure of the component of \( S - G \) which contains \( p \) be called \( R \). Then \( R \) is a disk on \( S \) so that \( p \in \text{Int} \ R \).

Let \( \sigma \) be a positive number such that

1. \( \sigma < d(J, G) \) and
2. \( \sigma < d(G, \text{Bd} \ N) \).

Using Bing's Side Approximation Theorem for the open set \( \text{Int} \ G \) [6] we obtain an annulus \( B \), a homeomorphism \( h \) from \( B \) into \( G \), and a collection of disjoint \( \sigma \)-disks \( Q_1, Q_2, Q_3, \ldots \) (locally finite modulo \( \text{Bd} \ R \)) in \( \text{Int} \ B \) such that

3. \( h \) moves no point more than a distance \( \sigma \),
License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
1968] IMPLYING THAT A 2 SPHERE IS ALMOST TAME 177

(4) \( B - \text{Int} Q_1 - \text{Bd} R \subset S^3 - C \),
(5) \( B \cap \text{Int} R = \emptyset \),
(6) \( \text{Bd} R \subset \text{Bd} B \), and
(7) \( B - \text{Bd} R \) is locally polyhedral.

Let \( F = \text{Bd} B - \text{Bd} R \). It follows from (1), (6), and (3) that the identity maps on 
\( F \) and \( \text{Bd} R \) are homotopic in the complement of \( J \); hence, from [5],
(8) \( J \) links \( F \).

Also from (4), (2), (3), and (6),
(9) \( F \subset S^3 - C \) and
(10) \( B \subset N \).

Now we apply the Side Approximation Theorem again, this time relative to the 
open set \( \text{Int} R \), to obtain a disk \( E \) "almost in \( \text{Int} C \)" (in the sense of (4)) such that
\( \text{Bd} E = \text{Bd} R \), \( E \subset N \), and \( E \cap B = \text{Bd} R \). Then \( E \cup B \) is a disk \( D' \) which is locally 
polyhedral modulo the tame simple closed curve \( \text{Bd} R \). Since \( D' \) is tame [7], [12] we 
assume with no loss in generality that \( D' \) is polyhedral.

Let \( V \) be an open set such that \( D' \subset V \) and the closure \( X \) of the component of
\( V - S \) containing \( \text{Bd} D' \) does not intersect \( J \). From condition (A) there is a disk 
\( D \) in \( V \) such that
(11) \( \text{Bd} D = \text{Bd} D' \),
(12) \( D \subset V \), and
(13) the component \( T \) of \( D - S \) which contains \( \text{Bd} D \) has the property that
\( D - T \) has only a finite number of components.

As in the proof of Theorem 2 of [15] we choose a finite collection of disjoint 
subdisks \( D_1, D_2, \ldots, D_n \) of \( D \) so that, for each \( i \), the boundary \( J_i \) of \( D_i \) lies in \( T \) 
and the interior of the component \( A_i \) of \( D_i - S \) which contains \( J_i \) is an open annulus 
in \( S^3 - C \). It is Condition (13) that makes the interior of \( A_i \) an open annulus.

Now \( D - \bigcup \text{Int} D_i \) is a disk-with-holes and \( J \) links \( \text{Bd} D \) from (8), so it follows 
from Lemma 3 that \( J \) links some \( J_i \). From (12) and the fact that \( V \subset N \), we see that
\( \text{Int} A_i \) is an open \( \varepsilon \)-annulus.

Since \( A_i \subset T \subset X \), it follows that \( A_i \cap J = \emptyset \). Now the open annulus \( \text{Int} A_i \) fits 
the definition of \( S \) being locally annular in \( S^3 - C \) at \( p \). Because \( p \) was chosen as an 
arbitrary point of \( S \), \( S \) is locally annular in \( S^3 - C \).

**Theorem 6.** If \( M \) is a closed connected 2-manifold in \( S^3 \), \( U \) is a component of 
\( S^3 - M \), and \( M \) satisfies Condition (A) relative to \( U \), then \( M \) is locally annular in \( U \).

**Proof.** Let \( p \in M \). Using Theorem 5 of [4] we obtain a disk \( D \) in \( M \) and a 2-
sphere \( S \) such that \( p \in \text{Int} D \subset S \). One component \( V \) of \( S^3 - S \) has the property 
that \( D \) lies on the boundary of \( V \cap U \). Now relative to any simple closed curve \( J \) 
which spans \( M \) and contains \( p \) we are able to apply the argument of the proof of 
Theorem 5 to show that \( S \) is locally annular in \( V \) at \( p \). This implies that \( M \) is locally 
annular in \( U \) at \( p \).

**Theorem 7.** If a cellular crumpled cube \( C \) in \( S^3 \) has a boundary which satisfies

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Condition (A) relative to $S^3 - C$, then there is a point $p$ in $\text{Bd} \ C$ so that $\text{Bd} \ C$ is locally tame from $S^3 - C$ modulo $\{p\}$.

**Proof.** It follows from Theorem 5 that $\text{Bd} \ C$ is locally annular in $S^3 - C$. Thus Theorem 7 follows from Theorem 4.

**Corollary 4.** If $S$ is a 2-sphere in $S^3$ such that each component of $S^3 - S$ is an open 3-cell and $S$ satisfies Condition (A), then $S$ has at most two wild points.

**Corollary 5.** If $C$ is a cellular 3-cell in $S^3$ such that $\text{Bd} \ C$ satisfies Condition (A), then $\text{Bd} \ C$ is locally tame modulo a point.

**Remark.** No examples were given in [15] of surfaces which fail to satisfy Condition (A). It follows from Corollary 5 that the examples in [1] and [13] fail to satisfy (A); also neither of these examples is locally annular. The following question remains open:

**Question.** If a closed connected 2-manifold $M$ in $S^3$ is free relative to a component $U$ of $S^3 - M$, then does $M$ satisfy Condition (A) relative to $U$?

An answer to this question would also provide an answer to the question raised on p. 280 of [14].

6. A 2-manifold is tame if it is locally free and locally annular. Throughout this section we let $M$ denote a closed connected 2-manifold in $S^3$, we let $U$ denote a component of $S^3 - M$, and we let $p$ be a point of $M$. We say that $M$ is locally free in $U$ at $p$ if and only if there is a disk $G$ in $M$ so that $p \in \text{Int} \ G$ and for each $\varepsilon > 0$ there is an $\varepsilon$-map of $G$ into $U$. Also $M$ is locally free in $U$ if the above definition is independent of the point $p$ in $M$, and $M$ is locally free if $M$ is locally free in each component of $S^3 - M$.

Hempel's proof of Theorem 2 of [15] needs only slight modification to establish the following result.

**Theorem 8 (Hempel).** If $M$ is locally free in $U$ and $M$ satisfies Condition (A) relative to $U$, then $M$ is tame from $U$.

We show in Theorem 12 that Theorem 8 remains valid if "$M$ is locally annular in $U$" is substituted for "$M$ satisfies Condition (A) relative to $U$" in the hypothesis. Then the results of §§4 and 5 of [15] follow as corollaries. Also we obtain Hempel's Theorem 4 without the restriction that the closed connected 2-manifold $M$ be a 2-sphere (see Corollary 7).

We begin by outlining a proof for Theorem 9 although it is a corollary to Theorem 12 which follows. The reason for including this special case separately is that there is an interesting short proof based on Theorem 2 of [15]. Corollary 6 is Theorem 4 of [15].

**Theorem 9.** If a 2-sphere $S$ in $S^3$ is free relative to a component $U$ of $S^3 - S$ and $S$ is locally annular in $U$, then $S$ is tame from $U$. 

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Proof. Using the hypothesis that $S$ is free relative to $U$ together with the Sphere Theorem [18] we see that $S^3 - U$ is cellular. Then it follows from Theorem 4 that $S$ is locally tame from $U$ except at possibly one point. This means that $S$ satisfies Condition (A) relative to $U$, so it follows from Theorem 2 of [15] that $S$ is tame from $U$.

COROLLARY 6 (Hempel). If a 2-sphere $S$ in $S^3$ is locally tame from a component $U$ of $S^3 - S$ modulo a 0-dimensional set and $S$ is free relative to $U$, then $S$ is tame from $U$.

THEOREM 10. If $M$ is locally free in $U$ at $p$ and $M$ is locally annular in $U$, then $U$ is locally simply connected at $p$.

Proof. Let $J$ be a simple closed curve that spans $M$ such that $J \cap M$ consists of the two points $p$ and $q$. Since $M$ is locally free in $U$ at $p$, there is a disk $G$ on $M$ such that $p \in \text{Int } G$, $q \in M - G$, and for each $\eta > 0$ there is an $\eta$-map of $G$ into $U$. We may assume without loss in generality that $G$ lies on a 2-sphere $S$ such that $G$ lies on the boundary of the intersection of $U$ with one component, which we call $\text{Int } S$, of $S^3 - S$ [4, Theorem 5]. We shall show that $S$ can be locally $\alpha$-capped in $\text{Int } S$ at $p$ so that it will follow from Theorem 1 that $\text{Int } S$ is locally simply connected at $p$. This will imply that $U$ is locally simply connected at $p$ since $G \subset \text{Bd } (U \cap \text{Int } S)$.

Let $\varepsilon > 0$ be given, and let $E$ be a disk such that $p \in \text{Int } E \subset \text{Int } G$ and $\text{diam } E < \varepsilon/3$. As in the definition of "locally annular", let $A$ be an open annulus in $U \cap \text{Int } S \cap N(p, \varepsilon)$ such that $J \cap \overline{A} = \emptyset$, one component of $\text{Bd } A$ is a simple closed curve $K$ that links $J$, and $\text{Bd } A - K$ lies in $\text{Int } E$. We also assume that $A$ is locally polyhedral [5]. Following the definition of "locally $\alpha$-capped" we identify a disk $R$ on $S$ such that $p \in \text{Int } R, R \subset \text{Int } E$, and $R \cap \overline{A} = \emptyset$; and we suppose that $\alpha$ is a given positive number. We shall indicate how to obtain an open $\varepsilon$-disk $D$ in $N(E, \alpha) \cap \text{Int } S$ such that $R$ lies on the boundary of an $\varepsilon$-component of $\text{Int } S - D$.

In the annulus $A$ we choose a subannulus $A_1$ with boundary components $\text{Bd } A - K$ and a simple closed curve $J_1$ such that $\text{diam } A_1 < \alpha$ and $A_1 \subset N(E, \alpha)$. Now a proof much like Hempel gave for Theorem 2 of [15] shows the existence of a disk $E_1$ in $\text{Int } E$ and a map $F_1$ such that $F_1$ takes $\text{Bd } E_1$ essentially into $A_1$, $F_1(E_1) = D_1$ has diameter less than $\varepsilon/3$, and $D_1 \subset U$. Little additional argument is needed to show that $D_1$ can be chosen in $N(E, \alpha)$.

Using Dehn's Lemma [18] relative to a singular disk in $A_1 \cup D_1$ having a nonsingular neighborhood of its boundary and containing $D_1$, we obtain the desired disk $D$. An argument like that given in the last paragraph of the proof of Theorem 4 shows that the component of $\text{Int } S - D$ whose boundary contains $R$ has small diameter.

THEOREM 11. If $V$ is an open subset of $M$ such that $M$ is locally free in $U$ at each point of $V$ and $M$ is locally annular in $U$ at each point of $V$, then $M$ is locally tame from $U$ at each point of $V$. 

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Proof. From Theorem 10 it follows that $U$ is locally simply connected at each point of $V$. Then Theorem 11 follows from the proof of Theorem 4 of [10].

**Theorem 12.** If $M$ is locally free in $U$ and locally annular in $U$, then $M$ is tame from $U$.

**Proof.** If we apply Theorem 11, where $V = M$, we see that $M$ is locally tame from $U$. Then $M$ is tame from $U$ [2], [17].

**Corollary 7.** If $M$ is locally free in $U$ and $M$ is locally tame from $U$ modulo a 0-dimensional set, then $M$ is tame from $U$.

**Corollary 8.** If $M$ is locally free and locally annular, then $M$ is tame.

**Corollary 9.** If $M$ is locally free and locally tame modulo a 0-dimensional set, then $M$ is tame.

**Remark.** Burgess has defined "$M$ is locally peripherally collared" to mean that for each $e > 0$, for each point $p \in M$, and for each component $U$ of $S^3 - M$, there is a disk $D$ and an annulus $A$ such that $p \in \text{Int} D \subseteq M$, $\text{diam } D < e$, $\text{Bd } D \subseteq \text{Bd } A$, and $A - \text{Bd } D \subseteq U$. Since a 2-manifold satisfying this condition is locally annular we have the following result.

**Corollary 10.** If $M$ is locally free and locally peripherally collared, then $M$ is tame.

**References**

3. ———, *An alternative proof that 3-manifolds can be triangulated*, Ann. of Math. (2) 69 (1959), 37–65.


**Utah State University,**

**Logan, Utah**