THE CHARACTERS OF THE FINITE SYMPLECTIC GROUP \( Sp(4, q) \)

BY

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1. Introduction. In this paper we calculate all the (complex) irreducible characters of the group \( Sp(4, q) \) where \( q \) is odd. The conjugacy classes of this group have been determined by Dickson [1a], Springer [4], and Wall [7]. We show that the irreducible characters of the group fall into families in a natural way, just as the conjugacy classes of the group do. Also involved in our work are certain polynomials in \( q \) which have properties similar to those of the polynomials \( Q_a \) defined by Green [2] in his work on the characters of the groups \( GL(n, q) \).

I thank Dr. R. Ree for many valuable and stimulating discussions.

Notation. \( G \) is the group of all nonsingular \( 4 \times 4 \) matrices \( X \) over \( F = GF(q) \) (\( q \) a power of the odd prime \( p \)) satisfying \( XAX' = A \), where

\[
A = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix}.
\]

The order of \( G \) is \( q^4(q^2 - 1)(q^4 - 1) \), and the center \( Z \) is of order 2. (See e.g. Dickson [1].)

Let \( \kappa \) be a generator of the multiplicative group of \( GF(q^4) \), and let \( \zeta = \kappa^{q^2 - 1}, \ \theta = \kappa^{q^2 + 1}, \ \eta = \theta^{q - 1}, \ \gamma = \theta^{q + 1} \). Choose a fixed isomorphism from the multiplicative group of \( GF(q^4) \) into the multiplicative group of complex numbers, and let \( \xi, \ \theta, \ \eta, \ \gamma \) be the images of \( \zeta, \ \theta, \ \eta, \ \gamma \) respectively under this isomorphism.

By a character of a finite group we mean a rational integral combination of the complex irreducible characters of the group. If \( \chi, \phi \) are class functions on the group, the scalar product \( \langle \chi, \phi \rangle \) is defined as usual.

If \( \phi \) is a character of a subgroup \( H \) of \( G \), \( \phi^G \) denotes the character of \( G \) induced from \( \phi \).

Conjugacy Classes of \( G \). Each element of \( G \) is an element of \( GL(4, q) \), and so there correspond to it its characteristic polynomial \( f_1 f_2 \ldots \) where \( f_1, f_2, \ldots \) are distinct irreducible polynomials over \( F \), and certain partitions \( \nu_1, \nu_2, \ldots \) of the positive integers \( n_1, n_2, \ldots \) (see [2, p. 406]). Using the results of Wall [7] we see the conjugacy classes of \( G \) are given by the table below.

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<table>
<thead>
<tr>
<th>Class representative</th>
<th>Number of classes</th>
<th>Order of centralizer</th>
<th>Notation</th>
</tr>
</thead>
</table>
| \[
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1
\end{pmatrix},
\begin{pmatrix}
-1 & 1 \\
1 & -1 \\
-1 & -1
\end{pmatrix}
\] | 1, 1 | \(q^4(q^2-1)(q^4-1)\) | \(A_1, A_1'\) |
| \[
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1
\end{pmatrix},
\begin{pmatrix}
-1 & -1 \\
1 & -1 \\
1 & -1
\end{pmatrix}
\] | 1, 1 | \(2q^4(q^2-1)\) | \(A_{21}, A_{21}'\) |
| \[
\begin{pmatrix}
1 & \gamma \\
1 & 1 \\
1 & 1
\end{pmatrix},
\begin{pmatrix}
-1 & -\gamma \\
1 & -1 \\
1 & -1
\end{pmatrix}
\] | 1, 1 | \(2q^4(q-1)\) | \(A_{31}, A_{31}'\) |
| \[
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1
\end{pmatrix},
\begin{pmatrix}
-1 & -1 \\
-1 & -1 \\
1 & 1
\end{pmatrix}
\] | 1, 1 | \(2q^4(q+1)\) | \(A_{32}, A_{32}'\) |
| \[
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1
\end{pmatrix},
\begin{pmatrix}
-1 & -1 \\
-1 & -1 \\
1 & 1
\end{pmatrix}
\] | 1, 1 | \(2q^2\) | \(A_{41}, A_{41}'\) |
| \[
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1
\end{pmatrix},
\begin{pmatrix}
-1 & -1 \\
-1 & -1 \\
1 & 1
\end{pmatrix}
\] | 1, 1 | \(2q^2\) | \(A_{42}, A_{42}'\) |
| \[
\begin{pmatrix}
\zeta^i \\
\zeta^{-1} \\
\zeta^i
\end{pmatrix},
\begin{pmatrix}
\zeta^{-i} \\
\zeta^i \\
\zeta^{-i}
\end{pmatrix}
\] | \(\frac{1}{2}(q^2-1)\) | \(q^2+1\) | \(B_i(l)\) |
<table>
<thead>
<tr>
<th>Class representative</th>
<th>Number of classes</th>
<th>Order of centralizer</th>
<th>Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma^{i}$</td>
<td>$\frac{1}{2}(q - 1)^2$</td>
<td>$q^2 - 1$</td>
<td>$B_2(i)$</td>
</tr>
<tr>
<td>$\eta^{i}$</td>
<td>$\frac{1}{2}(q - 1)(q + 1)$</td>
<td>$(q - 1)^2$</td>
<td>$B_2(i, j)$</td>
</tr>
<tr>
<td>$\eta^{i}$</td>
<td>$\frac{1}{2}(q - 1)$</td>
<td>$q^2 - 1$</td>
<td>$B_2(i, j)$</td>
</tr>
<tr>
<td>$\eta^{i}$</td>
<td>$\frac{1}{2}(q - 1)$</td>
<td>$q^2 + 1$</td>
<td>$B_2(i)$</td>
</tr>
<tr>
<td>$\eta^{i}$</td>
<td>$\frac{1}{2}(q - 1)$</td>
<td>$q^2 - 1$</td>
<td>$B_2(i)$</td>
</tr>
<tr>
<td>$\eta^{i}$</td>
<td>$\frac{1}{2}(q - 1)$</td>
<td>$q^2 + 1$</td>
<td>$B_2(i)$</td>
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<td>$\eta^{i}$</td>
<td>$\frac{1}{2}(q - 1)$</td>
<td>$q^2 - 1$</td>
<td>$B_2(i)$</td>
</tr>
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<td>$\frac{1}{2}(q - 1)$</td>
<td>$q^2 + 1$</td>
<td>$B_2(i)$</td>
</tr>
<tr>
<td>$\eta^{i}$</td>
<td>$\frac{1}{2}(q - 1)$</td>
<td>$q^2 - 1$</td>
<td>$B_2(i)$</td>
</tr>
<tr>
<td>$\eta^{i}$</td>
<td>$\frac{1}{2}(q - 1)$</td>
<td>$q^2 + 1$</td>
<td>$B_2(i)$</td>
</tr>
</tbody>
</table>

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| $\begin{pmatrix} \eta^{-1} \\ 1 \\ 1 \\ 1 \end{pmatrix}$ | $\begin{pmatrix} \eta^{-1} \\ -1 \\ -1 \\ -1 \end{pmatrix}$ | $\frac{1}{2}(q-1)$, $\frac{1}{4}(q-1)$ | $C_{21}(i)$, $C_{21}'(i)$ |
| $\begin{pmatrix} \eta^{-1} \\ 1 \\ \gamma \\ 1 \end{pmatrix}$ | $\begin{pmatrix} \eta^{-1} \\ -1 \\ -\gamma \\ -1 \end{pmatrix}$ | $\frac{1}{2}(q-1)$, $\frac{1}{4}(q-1)$ | $i \in T_2$ |
| $\begin{pmatrix} \gamma^{-1} \\ 1 \\ 1 \end{pmatrix}$ | $\begin{pmatrix} \gamma^{-1} \\ -1 \\ -1 \end{pmatrix}$ | $\frac{1}{2}(q-3)$, $\frac{1}{4}(q-3)$ | $i \in T_1$ |
| $\begin{pmatrix} \gamma^{-1} \\ 1 \\ \gamma \\ 1 \end{pmatrix}$ | $\begin{pmatrix} \gamma^{-1} \\ -1 \\ -\gamma \\ -1 \end{pmatrix}$ | $\frac{1}{2}(q-3)$, $\frac{1}{4}(q-3)$ | $i \in T_1$ |
| $\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$ | | | $q^2(q^2-1)^2$ |
| $\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$ | $\begin{pmatrix} 1 \\ -1 \\ -\gamma \\ -1 \end{pmatrix}$ | | $D_{21}, D_{22}$ |
| $\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$ | $\begin{pmatrix} 1 \\ \gamma \\ 1 \\ -1 \end{pmatrix}$ | | $D_{23}, D_{24}$ |
| $\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$ | $\begin{pmatrix} 1 \\ -1 \\ -\gamma \\ -1 \end{pmatrix}$ | | $D_{31}, D_{32}$ |
| $\begin{pmatrix} 1 \\ \gamma \\ 1 \\ -1 \\ -1 \end{pmatrix}$ | $\begin{pmatrix} 1 \\ \gamma \\ 1 \\ -1 \\ -1 \end{pmatrix}$ | | $D_{23}, D_{24}$ |
(We remark that in the first column we give as class representatives not necessarily elements of $G$, but their canonical forms in an extension field of $F$.)

**Remarks.** The sets $R_1$, $R_2$, $T_1$, $T_2$ of positive integers mentioned in the table are defined as follows.

$R_1$ = \{1, 2, \ldots, \frac{1}{2}q^2 - 1\},
R_2$ is a set of $\frac{1}{2}(q-1)^2$ distinct positive integers $i$ such that $\phi_i$, $\theta^{-i}$, $\theta^{qi}$, $\theta^{-qi}$ are all distinct,

$T_1$ = \{1, 2, \ldots, \frac{1}{2}(q-3)\}, and $T_2$ = \{1, 2, \ldots, \frac{1}{2}(q-1)\}.

The elements of the classes $B_1(i), \ldots, B_9(i), B_8(i), B_9(i)$ are $p$-regular. The elements of $B_7(i)$ and $B_9(i)$ have their $\phi$-regular factors in $B_8(i)$ and $B_9(i)$ respectively.

**Lemma 1.1.** Let $A$ be the additive group of $F$, i.e. an elementary abelian group of order $q$. Then there exist irreducible characters $\alpha \to \epsilon(\alpha)$ and $\alpha \to \epsilon'(\alpha)$ of $A$ such that

$$\sum_{\alpha \in S} \epsilon(\alpha) = \sum_{\alpha \in S'} \epsilon'(\alpha) = -\frac{s}{2} (s + (sq)^{1/2}),$$
$$\sum_{\alpha \in S} \epsilon(\alpha) = \sum_{\alpha \in S'} \epsilon'(\alpha) = -\frac{s}{2} (s - (sq)^{1/2}),$$

where $s = (-1)^{(q-1)/2}$, $S$ is the set of nonzero elements of $F$ which are squares, and $S'$ is the set of elements of $F$ which are not squares in $F$.

**Proof.** We know (see e.g. [5, p. 103]) that there exist characters $\phi, \phi'$ of $A$ such that

$$\phi(0) = \phi'(0) = \frac{1}{2}(q+s),$$
$$\phi(\alpha) = \frac{1}{2}(s + (sq)^{1/2}) \quad \text{if } \alpha \in S,$$
$$= \frac{1}{2}(s - (sq)^{1/2}) \quad \text{if } \alpha \in S',$$

and

$$\phi'(\alpha) = \frac{1}{2}(s - (sq)^{1/2}) \quad \text{if } \alpha \in S,$$
$$= \frac{1}{2}(s + (sq)^{1/2}) \quad \text{if } \alpha \in S'.$$

Then $(\phi', \phi') = (\phi, \phi) = \frac{1}{2}(q+s)$, and

$$(\phi, \phi') = 1 \quad \text{if } q \equiv 1 \pmod{4},$$
$$= 0 \quad \text{if } q \equiv -1 \pmod{4}.$$

$\phi$ and $\phi'$ contain the identity character if $q \equiv 1 \pmod{4}$ and have no irreducible constituent in common if $q \equiv -1 \pmod{4}$. They are each the sum of $\frac{1}{2}(q+s)$ distinct irreducible characters.

Let $\epsilon$ ($\epsilon'$) be a nonidentity irreducible character of $A$ occurring in $\phi'$ ($\phi$) if $q \equiv 1 \pmod{4}$, and in $\phi$ ($\phi'$) if $q \equiv -1 \pmod{4}$. Solving the equations

$$(\epsilon, \phi') = 1, \quad (\epsilon, \phi) = 0 \quad \text{if } q \equiv 1 \pmod{4},$$
$$(\epsilon, \phi) = 1, \quad (\epsilon, \phi') = 0 \quad \text{if } q \equiv -1 \pmod{4},$$

we find that $\epsilon$, and similarly $\epsilon'$, have the required properties.
Let $\varepsilon = -s(s+(sq)^{1/2})/2$, $\varepsilon' = -s(s-(sq)^{1/2})/2$, where $s = (-1)^{q-1}/2$

We have the identities

$$\varepsilon^2 + \varepsilon'^2 = \frac{1}{4}(1+sq),$$
$$2\varepsilon\varepsilon' = \frac{1}{4}(1-sq).$$

2. The Sylow $p$-subgroup of $G$. Consider a Sylow $p$-subgroup $U$ of $G$ of order $q^4$ consisting of all matrices of the form

$$\begin{pmatrix}
1 & \lambda & 0 & \lambda\alpha + \beta \\
0 & 1 & 0 & \alpha \\
-\alpha & \beta & 1 & \mu \\
0 & 0 & 0 & 1
\end{pmatrix} \quad (\lambda, \alpha, \mu, \beta \in F).$$

This element of $U$ will be denoted by $(\lambda, \alpha, \mu, \beta)$. The elements of the form $(\lambda, 0, \mu, \beta)$ form a subgroup $W$ of $U$ of order $q^3$. The center is of order $q$. The commutator subgroup is of order $q^2$ and consists of elements of the form $(0, 0, \mu, \beta)$.

The conjugacy classes of $U$ are given in the table below. Again we use the results of Wall [7] to determine the class of $G$ in which each class of $U$ lies.

<table>
<thead>
<tr>
<th>Class representative</th>
<th>No. of classes</th>
<th>Order of Centralizer in $U$</th>
<th>Class in G</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0, 0, 0, 0)$</td>
<td>1</td>
<td>$q^4$</td>
<td>$A_1$</td>
</tr>
<tr>
<td>$(0, 0, \mu, 0), \mu \neq 0$</td>
<td>$q-1$</td>
<td>$q^4$</td>
<td>$A_{21}$ if $\mu \in S$, $A_{22}$ if $\mu \in S'$</td>
</tr>
<tr>
<td>$(0, 0, \beta, \beta \neq 0)$</td>
<td>$q-1$</td>
<td>$q^3$</td>
<td>$A_{21}$</td>
</tr>
<tr>
<td>$(\lambda, 0, 0, 0), \lambda \neq 0$</td>
<td>$q-1$</td>
<td>$q^3$</td>
<td>$A_{21}$ if $\lambda \in S$, $A_{22}$ if $\lambda \in S'$</td>
</tr>
<tr>
<td>$(\lambda, 0, \mu, 0), \lambda \neq 0, \mu \neq 0$</td>
<td>$(q-1)^2$</td>
<td>$q^3$</td>
<td>$A_{31}$ if $-\lambda \mu \in S$, $A_{32}$ if $-\lambda \mu \in S'$</td>
</tr>
<tr>
<td>$(0, \alpha, 0, 0), \alpha \neq 0$</td>
<td>$q-1$</td>
<td>$q^2$</td>
<td>$A_{31}$</td>
</tr>
<tr>
<td>$(\lambda, \alpha, 0, 0), \lambda \neq 0, \alpha \neq 0$</td>
<td>$(q-1)^2$</td>
<td>$q^2$</td>
<td>$A_{41}$ if $\lambda \in S$, $A_{42}$ if $\lambda \in S'$</td>
</tr>
</tbody>
</table>

We now consider certain characters of $UZ$ and induce them to $G$. Now the “one-parameter subgroups” $\{(\lambda, 0, 0, 0)\}_{\lambda \in F}$, $\{(0, \alpha, 0, 0)\}_{\alpha \in F}$, etc. are all isomorphic to the additive group of $F$. By making use of these isomorphisms we see that there exist characters $\varepsilon : (\lambda, 0, 0, 0) \rightarrow e(\lambda), \quad \varepsilon' : (\lambda, 0, 0, 0) \rightarrow e'(\lambda)$ of the subgroup $\{(\lambda, 0, 0, 0)\}$ having the properties stated in Lemma 1.1. Similarly we define characters $\varepsilon, \varepsilon'$ of each of the three other subgroups.

(2.1) Consider the character of $U$ defined by

$$\varepsilon : (\lambda, 0, \mu, \beta) \rightarrow 1, \quad (0, \alpha, 0, 0) \rightarrow e(\alpha).$$
This character can be extended to a linear character of $UZ$ in two ways. We introduce these two characters of $UZ$ to $G$, and denote the characters obtained by $\psi_1$ and $\psi'_1$. Thus the representation associated with $\psi_1 (\psi'_1)$ maps $-I$ on $I (-I)$. (The same convention will be followed in the rest of this section.)

(2.2) Take the characters
\[
(0, \alpha, \mu, \beta) \rightarrow 1, \quad (\lambda, 0, 0, 0) \rightarrow \varepsilon(\lambda); \quad (0, \alpha, \mu, \beta) \rightarrow 1, \quad (\lambda, 0, 0, 0) \rightarrow \varepsilon'(\lambda)
\]
and apply the same procedure as in (2.1). The two characters of $G$ obtained from the first character will be denoted by $\psi_{21}$, $\psi'_{21}$, and those obtained from the second character will be denoted by $\psi_{22}$, $\psi'_{22}$. Let $\psi_2 = \psi_{21} + \psi_{22}$, $\psi'_2 = \psi'_{21} + \psi'_{22}$.

(2.3) Consider the characters
\[
(0, 0, \mu, \beta) \rightarrow 1, \quad (0, \alpha, 0, 0) \rightarrow \varepsilon(\alpha), \quad (\lambda, 0, 0, 0) \rightarrow \varepsilon(\lambda)
\]
and
\[
(0, 0, \mu, \beta) \rightarrow 1, \quad (0, \alpha, 0, 0) \rightarrow \varepsilon(\alpha), \quad (\lambda, 0, 0, 0) \rightarrow \varepsilon'(\lambda).
\]
Using the same procedure and convention as in (2.2), the four characters of $G$ obtained will be denoted by $\psi_{31}$, $\psi'_{31}$, $\psi_{32}$, $\psi'_{32}$.

Let $\psi_3 = \psi_{31} + \psi_{32}$, $\psi'_3 = \psi'_{31} + \psi'_{32}$.

(2.4) Consider the character
\[
(\lambda, 0, \mu, 0) \rightarrow 1, \quad (0, 0, 0, \beta) \rightarrow \varepsilon(\beta),
\]
of the subgroup $W$. Extend this in two ways to $WZ$, and induce to $G$. We denote the two characters obtained by $\psi_4$, $\psi'_4$.

(2.5) Consider the character of $W$ which is the sum of the two characters
\[
(0, 0, \beta) \rightarrow 1, \quad (\lambda, 0, 0, 0) \rightarrow \varepsilon(\lambda), \quad (0, 0, \mu, 0) \rightarrow \varepsilon(\mu)
\]
and
\[
(0, 0, \beta) \rightarrow 1, \quad (\lambda, 0, 0, 0) \rightarrow \varepsilon(\lambda), \quad (0, 0, \mu, 0) \rightarrow \varepsilon'(\mu).
\]
Again adopting the procedure of (2.4), we denote the two characters of $G$ obtained by $\psi_5$ and $\psi'_5$.

We now give the values of the characters of $G$ that we have constructed at the classes of $G$. We have omitted the characters $\psi_1$, $\ldots$, and also the values of $\psi_1$, $\ldots$ at $A'_1$, $\ldots$, $A'_{42}$. At all other classes which are not mentioned the values of the characters are zero.

Each entry in the table is to be multiplied by an integer which depends on the column in which the entry is found. Thus, the value of $\psi_i$ at $A_1$ is $\frac{1}{2}(q^2 - 1)(q^4 - 1)$, etc.

In computing these characters, we have made use of the identities (1.3), as well as the following facts.

(2.6) The elements of $W$ which lie in $A_{21}$ ($A_{22}$) are the elements of the form $(0, 0, \mu, 0)$ where $\mu \in S$ ($S'$), and the elements of the form $(\lambda, 0, \nu^2 \lambda, -\nu \lambda)$ where $\lambda \in S$ ($S'$) and $\nu$ is any element of $F$. 

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<table>
<thead>
<tr>
<th>Class</th>
<th>$A_1$</th>
<th>$A_{21}$</th>
<th>$A_{22}$</th>
<th>$A_{31}$</th>
<th>$A_{32}$</th>
<th>$A_{41}$</th>
<th>$A_{42}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multiplying factor</td>
<td>$(q^3 - 1)$</td>
<td>$q^3 - 1$</td>
<td>$q^3 - 1$</td>
<td>$q - 1$</td>
<td>$q + 1$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\psi_1$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}(q^3 - 1)$</td>
<td>$\frac{1}{2}(q^3 - 1)$</td>
<td>$\frac{1}{2}(q^3 - 2q - 1)$</td>
<td>$\frac{1}{2}(q^3 - 1)^2$</td>
<td>$\frac{1}{2}(1 - q)$</td>
<td>$\frac{1}{2}(1 - q)$</td>
</tr>
<tr>
<td>$\psi_{21}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}(q^3 - 1)$</td>
<td>$\frac{1}{2}(q^3 - 1)$</td>
<td>$\frac{1}{2}(q^3 - 2q - 1)$</td>
<td>$\frac{1}{2}(q^3 - 1)^2$</td>
<td>$\frac{1}{2}(1 - q)$</td>
<td>$\frac{1}{2}(1 - q)$</td>
</tr>
<tr>
<td>$\psi_{22}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}(q^3 - 1)$</td>
<td>$\frac{1}{2}(q^3 - 1)$</td>
<td>$\frac{1}{2}(q^3 - 2q - 1)$</td>
<td>$\frac{1}{2}(q^3 - 1)^2$</td>
<td>$\frac{1}{2}(1 - q)$</td>
<td>$\frac{1}{2}(1 - q)$</td>
</tr>
<tr>
<td>$\psi_2 (\psi_{21} + \psi_{22})$</td>
<td>1</td>
<td>$-1$</td>
<td>$-1$</td>
<td>$2q^3 - q^2 - 1$</td>
<td>$1 - q$</td>
<td>$1 - q$</td>
<td>$1 - q$</td>
</tr>
<tr>
<td>$\psi_{31}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}(q^3 - 1)$</td>
<td>$\frac{1}{2}(q^3 - 1)$</td>
<td>$\frac{1}{2}(q^3 - 2q - 1)$</td>
<td>$\frac{1}{2}(q^3 - 1)^2$</td>
<td>$\frac{1}{2}(1 - q)$</td>
<td>$\frac{1}{2}(1 - q)$</td>
</tr>
<tr>
<td>$\psi_{32}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}(q^3 - 1)$</td>
<td>$\frac{1}{2}(q^3 - 1)$</td>
<td>$\frac{1}{2}(q^3 - 2q - 1)$</td>
<td>$\frac{1}{2}(q^3 - 1)^2$</td>
<td>$\frac{1}{2}(1 - q)$</td>
<td>$\frac{1}{2}(1 - q)$</td>
</tr>
<tr>
<td>$\psi_3 (\psi_{31} + \psi_{32})$</td>
<td>1</td>
<td>$-1$</td>
<td>$-1$</td>
<td>$1 - q$</td>
<td>$1 - q$</td>
<td>$1 - q$</td>
<td>$1 - q$</td>
</tr>
<tr>
<td>$\psi_4$</td>
<td>$\frac{1}{2}q$</td>
<td>$\frac{1}{2}q(q - 1)$</td>
<td>$\frac{1}{2}q(q - 1)$</td>
<td>$-q$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\psi_5$</td>
<td>$q$</td>
<td>$-q$</td>
<td>$-q$</td>
<td>$-q$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

(2.7) The elements of $W$ which lie in $A_{31}$ are the elements of the form $(0, 0, -2v\beta, \beta) \ (v, \beta \in F)$, and the elements of the form $(\lambda, 0, v^2\lambda + \mu, -v\lambda)$ where $v$ is any element of $F$ and $-\lambda\mu \in S$.

(2.8) The elements of $W$ which lie in $A_{32}$ are the elements of the form $(\lambda, 0, v^2\lambda + \mu, -v\lambda)$ where $v$ is any element of $F$ and $-\lambda\mu \in S'$.

3. Some subgroups of $G$. We first prove a lemma which will be used to show the existence of certain types of subgroups of $G$.

**Lemma 3.1.** Let $\bar{G}$ be the algebraic group (over $F$) of all nonsingular $4 \times 4$ matrices $X$ over a universal domain $\Omega$ containing $F$ as a subfield, satisfying $XAX' = A$ where $A$ is as in §1. If $a \in \bar{G}$, let $a^{(q)}$ be the element of $\bar{G}$ obtained by raising every entry in the matrix $a$ to its $q$th power. Let $H$ be a subgroup of $\bar{G}$. Then there exists an element $y \in \bar{G}$ such that $y^{-1}Hy \subseteq G$, if and only if there exists an element $z \in \bar{G}$ such that $z^{-1}az = a^{(q)}$ for all $a \in H$.

**Proof.** Clearly $G$ is the subgroup of $\bar{G}$ consisting of all $a \in \bar{G}$ such that $a^{(q)} = a$. The lemma then follows from a theorem of Lang [3] which asserts that for any $z \in \bar{G}$ there exists a $y \in \bar{G}$ such that $z = yxy^{-q}$.

We now consider the following subgroups of $G$.

(3.2) Consider the elements

$$a = \begin{pmatrix} \zeta & \zeta^{-1} \\ \zeta^q & \zeta^{-q} \end{pmatrix}, \quad b = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$
in $\tilde{G}$. Then $b$ transforms every element $c$ of $\{a, b\}$ to $c^{(q)}$, and hence, by Lemma 3.1, $G$ contains a subgroup $M_1$ conjugate to $\{a, b\}$ in $\tilde{G}$. Let $M_1 = \{a_1, x_1\}$, where $a_1^{q^2 + 1} = x_1^q = 1$, $x_1^{-1} a_1 x_1 = a_1^q$. Let $H_1 = \{a_1\}$.

We give below the conjugacy classes of $M_1$ and the classes of $G$ containing them. Sometimes the class of $G$ to which an element belongs depends on $q$. In these cases we have just indicated the characteristic polynomial of the element. The elements in question are always $p$-regular.

<table>
<thead>
<tr>
<th>$1, a_1^{(q^2 + 1)}$</th>
<th>$1, 1$</th>
<th>$4(q^2 + 1)$</th>
<th>$A_1, A_1^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1^i$</td>
<td>$1(q^2 - 1)$, $i \in R_1$</td>
<td>$q^2 + 1$</td>
<td>$B_1(i)$</td>
</tr>
<tr>
<td>$x_1, x_1^2, a_1^2 x_1, a_1^2 x_1^2$</td>
<td>$1, 1, 1, 1$</td>
<td>$8$</td>
<td>$(x^4 + 1)$</td>
</tr>
<tr>
<td>$x_1^2, a_1 x_1$</td>
<td>$1, 1$</td>
<td>$8$</td>
<td>$(x^2 + 1)^2$</td>
</tr>
</tbody>
</table>

In the tables of conjugacy classes of subgroups in this section the columns contain, from left to right, (i) class representative, (ii) number of classes, (iii) order of centralizer in the subgroup, and (iv) class in $G$. In the following, the use of Lemma 3.1 to prove the existence of subgroups will not be explicitly mentioned unless the details are not straightforward.

(3.3) Let $M_2$ be a subgroup of $G$ which is conjugate in $\tilde{G}$ to the subgroup of $\tilde{G}$ generated by

$$\left(\begin{array}{ccc}
\theta & -1 & 1 \\
\theta^{-q} & -1 & -1 \\
\theta^{-q} & 1 & 1
\end{array}\right), \quad \left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right), \quad \text{and} \quad \left(\begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{array}\right).$$

Then $M_2 = \{a_2, x_2, y_2\}$, with

$$a_2^{q^2 - 1} = x_2^q = y_2^q = [x_2, y_2](2) = 1, \quad x_2^{-1} a_2 x_2 = a_2^q, \quad y_2^{-1} a_2 y_2 = a_2^{-q}.$$ 

Let $H_2 = \{a_2\}$. The conjugacy classes of $M_2$ are given in the next table.

(3.4) Let $H_3 = \{a_3, b_3\}$, $M_3 = \{a_3, b_3, x_3, y_3, z_3\}$, where

$$a_3 = \left(\begin{array}{ccc}
\gamma & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right), \quad b_3 = \left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & \gamma & 0 \\
0 & 0 & \gamma^{-1}
\end{array}\right), \quad x_3 = \left(\begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{array}\right),$$

$$y_3 = \left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right), \quad z_3 = \left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right).$$

(3) If $a, b \in G$, $[a, b] = a^{-1} b^{-1} a b$. 

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Then we have the relations

\[ a_3^{-1} = b_3^{-1} = [a_3, b_3] = x_3^4 = y_3^4 = z_3^4 = [y_3, z_3] = [a_3, z_3] = [b_3, y_3] = 1, \]

\[ x_3^{-1} y_3 x_3 = z_3, \quad x_3^{-1} a_3 x_3 = b_3, \quad y_3^{-1} a_3 y_3 = a_3^{-1}, \quad z_3^{-1} b_3 z_3 = b_3^{-1}. \]

The conjugacy classes of \( M_3 \) are given below.

<table>
<thead>
<tr>
<th>1, ( (a_3 b_3)^{q-13/2} )</th>
<th>1, 1</th>
<th>8(q-1)^2</th>
<th>( A_1, A'_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_3^{-1/2} )</td>
<td>1</td>
<td>4(q-1)^2</td>
<td>( D_1 )</td>
</tr>
<tr>
<td>( a_3, a_3 b_3^{-13/2} )</td>
<td>( 1/4(q-3), 1/4(q-3), i \in T_1 )</td>
<td>2(q-1)^2</td>
<td>( C_3(i), C_3(i') )</td>
</tr>
<tr>
<td>( (a_3 b_3)^4 )</td>
<td>( 1/4(q-3), i \in T_1 )</td>
<td>2(q-1)^2</td>
<td>( B_6(i) )</td>
</tr>
<tr>
<td>( a_3 b_3 )</td>
<td>( (q-3)(q-5)/8, i, j \in T_1, i \neq j )</td>
<td>(q-1)^2</td>
<td>( B_6(i, j) )</td>
</tr>
<tr>
<td>( y_3, a_3 y_3 )</td>
<td>1, 1</td>
<td>8(q-1)</td>
<td>( (x^2 + 1)(x-1)^2 )</td>
</tr>
<tr>
<td>( (a_3 b_3)^{q-13/2} y_3, (a_3 b_3)^{q-13/2} a_3 y_3 )</td>
<td>1, 1</td>
<td>8(q-1)</td>
<td>( (x^2 + 1)(x+1)^2 )</td>
</tr>
<tr>
<td>( a_3 z_3, a_3 b_3 z_3 )</td>
<td>( 1/4(q-3), 1/4(q-3), i \in T_1 )</td>
<td>4(q-1)</td>
<td>( (x-\gamma)(x-\gamma^{-1}) ) ( (x^2 + 1) )</td>
</tr>
<tr>
<td>( y_3 z_3, a_3 y_3 z_3 )</td>
<td>1, 1</td>
<td>16</td>
<td>( (x^2 + 1)^2 )</td>
</tr>
<tr>
<td>( x_3 )</td>
<td>1</td>
<td>4(q-1)</td>
<td>( (x^2 + 1)^2 )</td>
</tr>
<tr>
<td>( a_3^{-1/2} x_3 )</td>
<td>1</td>
<td>4(q-1)</td>
<td>( D_1 )</td>
</tr>
<tr>
<td>( a_3 x_3 )</td>
<td>( 1/4(q-3), i \in T_1 )</td>
<td>2(q-1)</td>
<td>( (x^2 + \gamma)(x^2 + \gamma^{-1}) )</td>
</tr>
<tr>
<td>( x_3 y_3, a_3 x_3 y_3 )</td>
<td>1, 1</td>
<td>8</td>
<td>( (x^4 + 1) )</td>
</tr>
</tbody>
</table>

(3.5) Let \( M_4 \) be a subgroup of \( G \) which is conjugate in \( \mathcal{G} \) to the subgroup of \( \mathcal{G} \) generated by
Then $M_4 = \{a_4, b_4, x_4, y_4, z_4\}$, where

\[
a_4^{-1} b_4^{-1} = [a_4, b_4] = x_4^2 = y_4^2 = z_4^2 = [y_4, z_4] = [a_4, z_4] = [y_4, b_4] = 1,
\]

\[
x_4^{-1} y_4 x_4 = z_4, \quad x_4^{-1} a_4 x_4 = b_4, \quad y_4^{-1} a_4 y_4 = a_4^{-1}, \quad z_4^{-1} b_4 z_4 = b_4^{-1}.
\]

Let $H_4 = \{a_4, b_4\}$. The conjugacy classes of $M_4$ are given below.

<table>
<thead>
<tr>
<th>Class Description</th>
<th>Type</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1, (a_4b_4)^{q-1/2}$</td>
<td>$A_1$, $A_1'$</td>
<td>$8(q+1)^2$</td>
</tr>
<tr>
<td>$a_4^q$</td>
<td>$D_1$</td>
<td>$4(q+1)^2$</td>
</tr>
<tr>
<td>$a_4 b_4^q$</td>
<td>$C_i$, $C_i'$</td>
<td>$2(q+1)^2$</td>
</tr>
<tr>
<td>$a_4 b_4^q$</td>
<td>$B_4(i)$</td>
<td>$(q+1)^2$</td>
</tr>
<tr>
<td>$(a_4b_4)^{q+1} y_4, (a_4b_4)^{(q+1)/2} a_4 y_4$</td>
<td></td>
<td>$8(q+1)$</td>
</tr>
<tr>
<td>$y_4, a_4 y_4$</td>
<td></td>
<td>$8(q+1)$</td>
</tr>
<tr>
<td>$a_4 z_4, a_4 b_4 z_4$</td>
<td></td>
<td>$4(q+1)$</td>
</tr>
<tr>
<td>$y_4 z_4, y_4 a_4 z_4$</td>
<td></td>
<td>$16$</td>
</tr>
<tr>
<td>$x_4, a_4 x_4 z_4$</td>
<td></td>
<td>$4(q+1)$</td>
</tr>
<tr>
<td>$a_4^{q+1/2} x_4$</td>
<td></td>
<td>$4(q+1)$</td>
</tr>
<tr>
<td>$a_4 x_4$</td>
<td></td>
<td>$(x^2+1)^3$</td>
</tr>
<tr>
<td>$x_4 y_4, a_4 x_4 y_4$</td>
<td></td>
<td>$8$</td>
</tr>
</tbody>
</table>

(3.6) Let $M_6$ be a subgroup of $G$ which is conjugate in $G$ to the subgroup generated by

\[
\begin{pmatrix}
  \eta & \eta^{-1} \\
  1 & 1 \\
\end{pmatrix},
\quad
\begin{pmatrix}
  1 & 1 \\
  \gamma & \gamma^{-1} \\
\end{pmatrix},
\quad
\begin{pmatrix}
  1 & 1 \\
  -1 & 1 \\
\end{pmatrix}, \text{ and }
\begin{pmatrix}
  1 & 1 \\
  -1 & 1 \\
\end{pmatrix}.
\]
Then $M_5 = \{a_5, b_5, y_5, z_5\}$, where $a_5^{-1} = b_5^{-1} = y_5^{-1} = z_5^{-1} = [a_5, b_5] = [y_5, z_5] = [a_5, z_5] = [b_5, y_5] = 1$, $y_5^{-1}a_5y_5 = a_5^{-1}$, $z_5^{-1}b_5z_5 = b_5^{-1}$. Let $H_5 = \{a_5, b_5\}$.

The conjugacy classes of $M_5$ are given below.

<table>
<thead>
<tr>
<th>$1, a_5^{g+1/2}b_5^{g-1/2}$</th>
<th>$1, z$</th>
<th>$4(q^2-1)$</th>
<th>$A_1, A_1'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_5^{g+1/2}, b_5^{g-1/2}$</td>
<td>$1, 1$</td>
<td>$4(q^2-1)$</td>
<td>$D_1$</td>
</tr>
<tr>
<td>$a_5, b_5b_5^{g-1/2}$</td>
<td>$\frac{1}{2}(q-1), \frac{1}{2}(q-1), i \in T_2$</td>
<td>$2(q^2-1)$</td>
<td>$C_1(i), C_1'(i)$</td>
</tr>
<tr>
<td>$b_5b_5b_5^{g+1/2}$</td>
<td>$\frac{1}{2}(q-3), \frac{1}{2}(q-3), i \in T_1$</td>
<td>$2(q^2-1)$</td>
<td>$C_2(i), C_2'(i)$</td>
</tr>
<tr>
<td>$a_5b_5$</td>
<td>$\frac{1}{2}(q-1)(q-3), i \in T_2,$ $j \in T_1$</td>
<td>$q^2-1$</td>
<td>$B_5(i, j)$</td>
</tr>
<tr>
<td>$y_5, a_5y_5$</td>
<td>$1, 1$</td>
<td>$8(q-1)$</td>
<td>$(x^2+1)(x-1)^2$</td>
</tr>
<tr>
<td>$y_5b_5^{g-1/2}, a_5y_5b_5^{g-1/2}$</td>
<td>$1, 1$</td>
<td>$8(q-1)$</td>
<td>$(x^2+1)(x+1)^2$</td>
</tr>
<tr>
<td>$z_5, b_5z_5$</td>
<td>$1, 1$</td>
<td>$8(q+1)$</td>
<td>$(x^2+1)(x-1)^2$</td>
</tr>
<tr>
<td>$z_5a_5^{g+1/2}, b_5z_5a_5^{g-1/2}$</td>
<td>$1, 1$</td>
<td>$8(q+1)$</td>
<td>$(x^2+1)(x+1)^2$</td>
</tr>
<tr>
<td>$a_5z_5, a_5b_5z_5$</td>
<td>$\frac{1}{2}(q-1), \frac{1}{2}(q-1), i \in T_2$</td>
<td>$4(q-1)$</td>
<td>$(x-\eta)(x-\eta^{-1})$ $((x^2+1))$</td>
</tr>
<tr>
<td>$b_5y_5, b_5a_5y_5$</td>
<td>$\frac{1}{2}(q-3), \frac{1}{2}(q-3), i \in T_1$</td>
<td>$4(q-1)$</td>
<td>$(x-\eta')(x-\eta'^{-1})$ $((x^2+1))$</td>
</tr>
<tr>
<td>$y_5z_5, y_5b_5z_5, a_5y_5z_5, a_5b_5y_5z_5$</td>
<td>$1, 1, 1, 1$</td>
<td>$16$</td>
<td>$(x^2+1)^2$</td>
</tr>
</tbody>
</table>

(3.7) Let $K$ be the subgroup of $G$ consisting of all elements of the form

$$ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} $$

where $A$, $B$ are $2 \times 2$ matrices. Then $K$ is of order $q^2(q^2-1)^2$, and $K \cong S_p(2, q) \times S_p(2, q) \cong SL(2, q) \times SL(2, q)$.

(3.8) Let $K_1$ be the subgroup of $G$ consisting of all elements of the form

$$ \begin{pmatrix} \gamma^t & \beta \gamma^t \\ \gamma^{-t} & \beta \gamma^{-t} \\ \gamma^{-t} & \gamma^t \end{pmatrix} \quad (\beta \in F). $$

Then $|K_1| = q(q-1)$, and $K_1$ is the direct product of a cyclic group of order $q-1$ and an elementary abelian group of order $q$. Let

$$ u = \begin{pmatrix} \gamma \\ \gamma^{-1} \\ \gamma^{-1} \end{pmatrix}, \quad b_\beta = \begin{pmatrix} 1 & \cdots & \beta \\ \cdots & 1 & \cdots \\ \beta & 1 \end{pmatrix}, \quad v = \begin{pmatrix} \cdots & 1 \\ \cdots & 1 \\ 1 & \cdots \end{pmatrix}. $$
Let $K'_1 = \{K_1, v\}$. Then $v$ centralizes each $b_\beta$ and transforms $u$ into $u^{-1}$. We now give the classes of $K'_1$.

<table>
<thead>
<tr>
<th>Class</th>
<th>Description</th>
<th>Characteristic</th>
<th>Class</th>
<th>Characteristic</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1, u^{(a-1)}$</td>
<td>$q-1, q-1, 0 \neq \beta \in F$</td>
<td>$2q(q-1)$</td>
<td>$A_3, A'_3$</td>
<td></td>
</tr>
<tr>
<td>$b_\beta, u^{(a-1)}b_\beta$</td>
<td>$\frac{1}{2}(q-3), i \in T_1, \beta \neq 0$</td>
<td>$q(q-1)$</td>
<td>$B_8(i)$</td>
<td></td>
</tr>
<tr>
<td>$u^i$</td>
<td>$\frac{1}{2}(q-3)(q-1), i \in T_1, \beta \neq 0$</td>
<td>$q(q-1)$</td>
<td>$B_8(i)$</td>
<td></td>
</tr>
<tr>
<td>$v, uv$</td>
<td>$1, 1$</td>
<td>$4q, 4q$</td>
<td>$D_1$</td>
<td></td>
</tr>
<tr>
<td>$b_\beta v, b_\beta uv$</td>
<td>$q-1, q-1, \beta \neq 0$</td>
<td>$4q, 4q$</td>
<td>See remark below</td>
<td></td>
</tr>
</tbody>
</table>

**Remark.** $b_\beta uv, b_\beta v$ are conjugate in $G$ to

$$
\begin{pmatrix}
1 & 2\beta y^{-1} \\
-1 & 2\beta y^{-1}
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
1 & 2\beta \\
-1 & 2\beta
\end{pmatrix}
$$

respectively.

Hence, of the $q-1$ elements $b_\beta u$, $\frac{1}{2}(q-1)$ elements belong to $D_{31}$ and $\frac{1}{2}(q-1)$ to $D_{34}$ if $q \equiv 1 \pmod{4}$. Similarly, $\frac{1}{2}(q-1)$ of the elements $b_\beta uv$ belong to $D_{31}$ and $\frac{1}{2}(q-1)$ to $D_{34}$ if $q \equiv 1 \pmod{4}$. If $q \equiv -1 \pmod{4}$, $D_{31}$ and $D_{34}$ are to be replaced by $D_{32}$ and $D_{33}$ in the above statements.

(3.9) Let $K_2$ be the subgroup of $G$ of all elements of the form

$$
\begin{pmatrix}
\eta^i & \delta \beta \eta^i \\
\eta^{-i} & \eta^{-i} \delta \beta \eta^{-i} \\
\eta^i & 
\end{pmatrix}
$$

($\beta \in F$, $\delta$ an element of $\Omega$ such that $\delta^2 = \gamma$).

Now $\delta^2 = -\delta$. Let $\tilde{K}_2 = \{\tilde{K}_2, v\}$ where

$$
v = \begin{pmatrix}
\ddots & \ddots & 1 \\
\ddots & \ddots & 1 \\
1 & \ddots & \\
\ddots & 1 & \\
1 & \ddots & \\
\ddots & 1 & \\
\ddots & 1 & \\
\ddots & \ddots & \ddots \\
\end{pmatrix}
$$

Then the element

$$
\begin{pmatrix}
\ddots & \ddots & 1 \\
\ddots & \ddots & 1 \\
1 & \ddots & \\
\ddots & 1 & \\
1 & \ddots & \\
\ddots & 1 & \\
\ddots & \ddots & \ddots \\
\end{pmatrix}
$$
transforms every element $c$ of $K'_2$ into $c^{(q)}$. Hence there exist $z \in G$ such that $z^{-1}K'_2z = K'_2 \equiv G$. Let $z^{-1}K'_2z = K_2$.

$$
\begin{pmatrix}
\eta & \eta^{-1} \\
\eta^{-1} & \eta
\end{pmatrix}
\begin{pmatrix}
z
\end{pmatrix}
= w,
$$

$$
\begin{pmatrix}
1 & \delta \beta \\
\delta \beta & 1
\end{pmatrix}
\begin{pmatrix}
z
\end{pmatrix}
= d_8, \quad z^{-1}y = y.
$$

The conjugacy classes of $K'_2$ are given below.

<table>
<thead>
<tr>
<th>$1, w^{(q+1)/2}$</th>
<th>$1, 1$</th>
<th>$2q(q+1)$</th>
<th>$A_3, A'_3$</th>
<th>$A_{32}, A'_{32}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_s, w^{q+1/2}d_s$</td>
<td>$q - 1, q - 1, \beta \neq 0$</td>
<td>$2q(q+1)$</td>
<td>$A_{32}, A'_{32}$</td>
<td>$B_4(i)$</td>
</tr>
<tr>
<td>$w^i$</td>
<td>$\frac{1}{2}(q - 1), i \in T_2$</td>
<td>$q(q+1)$</td>
<td>$B_4(i)$</td>
<td>$D_1$</td>
</tr>
<tr>
<td>$w^j d_g$</td>
<td>$\frac{1}{2}(q - 1)^2, i \in T_2, \beta \neq 0$</td>
<td>$q(q+1)$</td>
<td>$B_4(i)$</td>
<td>$D_1$</td>
</tr>
<tr>
<td>$w, yw$</td>
<td>$1, 1$</td>
<td>$4q$</td>
<td>$D_1$</td>
<td>$4q$</td>
</tr>
<tr>
<td>$d_sw, d_s yw$</td>
<td>$q - 1, q - 1, \beta \neq 0$</td>
<td>$4q$</td>
<td>$D_1$</td>
<td>See remark below</td>
</tr>
</tbody>
</table>

Remark. Here $\frac{1}{2}(q - 1)$ of the elements $d_sw$ belong to $D_{32}$ and $\frac{1}{2}(q - 1)$ to $D_{33}$ if $q \equiv 1 \mod 4$. Similarly $\frac{1}{2}(q - 1)$ of the elements $d_s yw$ belong to $D_{32}$ and $\frac{1}{2}(q - 1)$ to $D_{33}$ if $q \equiv 1 \mod 4$. If $q \equiv -1 \mod 4$, replace $D_{32}$ and $D_{33}$ by $D_{31}$ and $D_{34}$ in the above statements.

(3.10) Let $L_1$ be the subgroup of $G$ consisting of all elements of the form

$$
\begin{pmatrix}
\gamma^i & \cdots & \cdots \\
\cdots & \gamma^{-i} & \cdots \\
\cdots & \pm 1 & \beta \\
\cdots & \cdots & \pm 1
\end{pmatrix}
$$

$(\beta \in F)$. Then $|L_1| = 2q(q - 1)$. Let

$$
u_1 = \begin{pmatrix}
1 & \cdots & \cdots \\
-1 & \cdots & \cdots \\
\cdots & 1 & \cdots \\
\cdots & \cdots & 1
\end{pmatrix}, \quad \text{and} \quad L'_1 = \{L_1, u_1\}.
We put
\[
g_1 = \begin{pmatrix}
\gamma & \cdot & \cdot \\
\cdot & \gamma^{-1} & \cdot \\
\cdot & \cdot & 1 \\
\cdot & \cdot & \cdot
\end{pmatrix}, \quad h_\beta = \begin{pmatrix}
1 & \cdot & \cdot \\
\cdot & 1 & \cdot \\
\cdot & \cdot & 1 & \beta \\
\cdot & \cdot & \cdot & 1
\end{pmatrix}, \quad c_1 = \begin{pmatrix}
1 & \cdot \\
\cdot & -1 \\
\cdot & \cdot \\
\cdot & \cdot
\end{pmatrix}.
\]

The conjugacy classes of \( L'_1 \) are easily written down and they will be omitted here.

(3.11) Let \( L_2 \) be the subgroup of \( G \) generated by all elements of the form
\[
\begin{pmatrix}
\eta \\
\eta^{-1} \\
\pm 1 \\
\beta
\end{pmatrix} \quad (\beta \in F)
\]
and
\[
\begin{pmatrix}
1 \\
-1 \\
1 \\
1
\end{pmatrix}.
\]

Then \( z^{-1} L_2 z = L'_2 \subseteq G \), for some \( z \in G \). Let
\[
g_2 = z^{-1} \begin{pmatrix}
\eta \\
\eta^{-1} \\
1 \\
1
\end{pmatrix} z, \quad k_\beta = z^{-1} \begin{pmatrix}
1 \\
1 \\
1 \\
1
\end{pmatrix} z,
\]
\[
c_2 = z^{-1} \begin{pmatrix}
1 \\
1 \\
-1 \\
-1
\end{pmatrix} z, \quad \text{and} \quad u_2 = z^{-1} \begin{pmatrix}
1 \\
-1 \\
1 \\
1
\end{pmatrix} z.
\]

Let \( L_2 = \{ g_2, c_2, k_\beta (\beta \in F) \} \). Again, the conjugacy classes of \( L'_2 \) will be omitted. We note that, in fact, \( z \) can be chosen such that \( L'_2 \) is a subgroup of the subgroup \( K \) of (3.7).

**Certain characters of subgroups of \( G \).**

(3.12) Consider the subgroups \( K_1, K'_1 \) of (3.8), and the character \( u \rightarrow \gamma^j \), \( b_\beta \rightarrow \epsilon(\beta) \) of \( K_1 \), where \( j \) is any integer.

[Here we make use of the isomorphism \( \beta \rightarrow b_\beta \) between the additive group of \( F \) and the subgroup of \( K_1 \) consisting of all the \( b_\beta \).]

Induce this character to \( G \) and let \( \rho(j) \) be the character of \( G \) obtained in this way. Now consider the characters
\[
\alpha_{11}: u \rightarrow 1, \quad b_\beta \rightarrow \epsilon(\beta), \quad v \rightarrow -1,
\]
\[
\alpha_{12}: u \rightarrow 1, \quad b_\beta \rightarrow \epsilon(\beta), \quad v \rightarrow 1,
\]
of \( K'_1 \), and induce the characters \( \alpha_{11}, \alpha_{12} \) to \( G \).

(3.13) Similarly we construct a character \( \sigma(j) \) of \( G \) which is the induced character
of the following character of $K^\circ$: $w \mapsto \eta^j$, $d_\beta \mapsto \epsilon(\beta)$ ($j$ any integer). We also consider the following characters of $K^\circ_2$, and induce them to $G$.

\begin{align*}
\alpha_{11}: w &\mapsto 1, \quad d_\beta \mapsto \epsilon(\beta), \quad y \mapsto -1, \\
\alpha_{22}: w &\mapsto 1, \quad d_\beta \mapsto \epsilon(\beta), \quad y \mapsto 1.
\end{align*}

The values of $\rho(j)$, $\sigma(j)$, $\alpha_{11}^0$, $\alpha_{12}^0$, $\alpha_{21}^0$, $\alpha_{22}^0$ at the classes of $G$ are given below. At classes not mentioned the values are zero. Also, where there is no entry in the table the value is zero. (We will stick to this convention throughout this paper.)

<table>
<thead>
<tr>
<th>$\rho(j)$</th>
<th>$A_1$</th>
<th>$A_4$</th>
<th>$A_{11}$</th>
<th>$A_{12}$</th>
<th>$A_{21}$</th>
<th>$A_{22}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q^3(q+1)(q^4-1)$</td>
<td>$-2q^2$</td>
<td>$-q^2$</td>
<td>$-2q^2$</td>
<td>$-q^2$</td>
<td>$-q^2$</td>
<td>$-q^2$</td>
</tr>
<tr>
<td>$q^4(q-1)(q^4-1)$</td>
<td>$(-1)^j q^3(q-1)(q^4-1)$</td>
<td>$-q^2$</td>
<td>$-q^2$</td>
<td>$-q^2$</td>
<td>$-q^2$</td>
<td>$-q^2$</td>
</tr>
<tr>
<td>$\frac{1}{4}q^3(q+1)(q^4-1)$</td>
<td>$\frac{1}{4}q^3(q+1)(q^4-1)$</td>
<td>$-q^2$</td>
<td>$-q^2$</td>
<td>$-q^2$</td>
<td>$-q^2$</td>
<td>$-q^2$</td>
</tr>
<tr>
<td>$\frac{1}{4}q^3(q+1)(q^4-1)$</td>
<td>$\frac{1}{4}q^3(q+1)(q^4-1)$</td>
<td>$-q^2$</td>
<td>$-q^2$</td>
<td>$-q^2$</td>
<td>$-q^2$</td>
<td>$-q^2$</td>
</tr>
<tr>
<td>$\frac{1}{4}q^3(q-1)(q^4-1)$</td>
<td>$\frac{1}{4}q^3(q-1)(q^4-1)$</td>
<td>$-q^2$</td>
<td>$-q^2$</td>
<td>$-q^2$</td>
<td>$-q^2$</td>
<td>$-q^2$</td>
</tr>
<tr>
<td>$\frac{1}{4}q^3(q-1)(q^4-1)$</td>
<td>$\frac{1}{4}q^3(q-1)(q^4-1)$</td>
<td>$-q^2$</td>
<td>$-q^2$</td>
<td>$-q^2$</td>
<td>$-q^2$</td>
<td>$-q^2$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\sigma(j)$</th>
<th>$B_1(i)$</th>
<th>$B_1(i)$</th>
<th>$B_3(i)$</th>
<th>$B_3(i)$</th>
<th>$D_1$</th>
<th>$D_{31}$</th>
<th>$D_{32}$</th>
<th>$D_{33}$</th>
<th>$D_{34}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q^2-1$</td>
<td>$\frac{q^2-1}{(\eta^i+\eta^{-1})}$</td>
<td>$-\frac{q^2-1}{(\eta^i+\eta^{-1})}$</td>
<td>$q$</td>
<td>$q$</td>
<td>$-q$</td>
<td>$-q$</td>
<td>$q$</td>
<td>$q$</td>
<td></td>
</tr>
<tr>
<td>$q^2-1$</td>
<td>$-1$</td>
<td>$-1$</td>
<td>$q$</td>
<td>$q$</td>
<td>$-q$</td>
<td>$-q$</td>
<td>$q$</td>
<td>$q$</td>
<td></td>
</tr>
<tr>
<td>$q^2-1$</td>
<td>$-1$</td>
<td>$-1$</td>
<td>$q$</td>
<td>$q$</td>
<td>$-q$</td>
<td>$-q$</td>
<td>$q$</td>
<td>$q$</td>
<td></td>
</tr>
<tr>
<td>$q^2-1$</td>
<td>$-1$</td>
<td>$-1$</td>
<td>$q$</td>
<td>$q$</td>
<td>$-q$</td>
<td>$-q$</td>
<td>$q$</td>
<td>$q$</td>
<td></td>
</tr>
<tr>
<td>$q^2-1$</td>
<td>$-1$</td>
<td>$-1$</td>
<td>$q$</td>
<td>$q$</td>
<td>$-q$</td>
<td>$-q$</td>
<td>$q$</td>
<td>$q$</td>
<td></td>
</tr>
</tbody>
</table>

**Remark.** In reading the values at $D_{31}$ to $D_{34}$, the upper entry is to be taken if $q \equiv 1 \pmod{4}$ and the lower entry if $q \equiv -1 \pmod{4}$.

(3.14) We now introduce certain characters of $K$. Since $K \cong SL(2, q) \times SL(2, q)$, any two characters $\rho$ and $\sigma$ of $SL(2, q)$ give rise to a character $\rho \times \sigma$ of $K$ which has as its value at $A = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$.
the value of $\rho$ at $A$ multiplied by the value of $\sigma$ at $B$. Now there exist families of characters $\lambda(i)$, $\lambda'(j)$, $(i, j$ are any integers) of degree $1 + q$, $1 - q$ respectively of $SL(2, q)$. (See e.g. [2]). Hence we have characters $\lambda(i) \times \lambda'(j)$, $\lambda(i) \times \lambda(j)$, $\lambda'(i) \times \lambda'(j)$ of $K$.

We also give below four characters $\mu_i$ $(i = 1, 2, 3, 4)$ of $SL(2, q)$ which will be used to construct characters of $K$ (see [5, p. 103]). Let $t = \frac{1}{4}(q - 1)$.

<table>
<thead>
<tr>
<th>$(1\quad 1)$</th>
<th>$(-1\quad -1)$</th>
<th>$\left(\begin{array}{ll} 1 &amp; \gamma' \ \gamma' &amp; 1 \end{array}\right)$, $i \in T_1$</th>
<th>$\left(\begin{array}{ll} 1 &amp; \gamma' \ \gamma' &amp; 1 \end{array}\right)$, $i \in T_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda(j)$</td>
<td>$\lambda'(j)$</td>
<td>$(1 + q)$</td>
<td>$(-1)'(1 + q)$</td>
</tr>
<tr>
<td>$\lambda(j)$</td>
<td>$\lambda'(j)$</td>
<td>$(-1)(1 - q)$</td>
<td>$(-1)'(1 - q)$</td>
</tr>
<tr>
<td>$\mu_1$</td>
<td>$\mu_2$</td>
<td>$\frac{1}{2}(1 + q)$</td>
<td>$(-1)'\frac{1}{2}(1 + q)$</td>
</tr>
<tr>
<td>$\mu_3$</td>
<td>$\mu_4$</td>
<td>$\frac{1}{2}(1 - q)$</td>
<td>$(-1)'\frac{1}{2}(1 - q)$</td>
</tr>
</tbody>
</table>

(3.15) We now consider the subgroups $L'_1$, $L'_2$ of (3.10) and (3.11). Let $\delta_i$ $(i = 1, 2, 3, 4)$ be the characters of $L'_1$ given by

$$
\begin{align*}
\delta_1 & : g_1 \to -1, \quad c_1 \to (-1)', \quad u_1 \to 1, \quad h_\beta \to \epsilon(\beta), \\
\delta_2 & : g_1 \to -1, \quad c_1 \to (-1)', \quad u_1 \to 1, \quad h_\beta \to \epsilon'(\beta), \\
\delta_3 & : g_1 \to -1, \quad c_1 \to (-1)'^t, \quad u_1 \to 1, \quad h_\beta \to \epsilon(\beta), \\
\delta_4 & : g_1 \to -1, \quad c_1 \to (-1)'^{t+1}, \quad u_1 \to 1, \quad h_\beta \to \epsilon'(\beta).
\end{align*}
$$

Let $\delta'_i$ $(i = 1, 2, 3, 4)$ be the characters of $L'_2$ obtained by replacing $g_1$ by $g_2$, $c_1$ by $c_2$, $u_1$ by $u_2$, and $h_\beta$ by $k_\beta$ in the definitions of $\delta_i$ $(i = 1, 2, 3, 4)$.

We give below the induced character $\delta'_1$. The others can be obtained in a similar way.

<table>
<thead>
<tr>
<th>$A_1$, $A'_1$</th>
<th>$A_{21}$, $A'_{21}$</th>
<th>$A_{22}$, $A'_{22}$</th>
<th>$C_3(i)$</th>
<th>$C_3(i)$</th>
<th>$C_{41}(i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{4}q^2(q + 1)(q^4 - 1)$</td>
<td>$\frac{1}{4}q^2(q + 1)\varepsilon$</td>
<td>$\frac{1}{4}q^2(q + 1)\varepsilon'$</td>
<td>$(-1)'\frac{1}{2}(q^2 - 1)$</td>
<td>$(-1)'^t\frac{1}{2}(q^2 - 1)$</td>
<td>$(-1)'\varepsilon'$</td>
</tr>
</tbody>
</table>
We will also need characters $\delta_5, \delta_6$ of degree 2 of $L_1', L_2$ respectively defined as follows.

$\delta_5$ is the sum of the characters

$$g_1 \rightarrow 1, \quad u_1 \rightarrow 1, \quad c_1 \rightarrow 1, \quad h_\beta \rightarrow e^{\beta};$$

$$g_1 \rightarrow 1, \quad u_1 \rightarrow 1, \quad c_1 \rightarrow 1, \quad h_\beta \rightarrow e^{\beta}\sigma(\beta)$$

$\delta_6$ is the character of $L_2'$ obtained by replacing $g_1$ by $g_2$, $u_1$ by $u_2$, $c_1$ by $c_2$ and $h_\beta$ by $k_\beta$ in the above definition.

4. Families of characters of $G$ corresponding to the families of classes $B_1(i)$ to $B_6(i)$.

(4.1) Consider the subgroups $H_1, H_2$ defined in (3.2) and (3.3), and the linear characters $\beta_1(j), \beta_2(j)$ of $H_1, H_2$ respectively given by

$$\beta_1(j) : a_1 \rightarrow \xi^j, \quad \beta_2(j) : a_2 \rightarrow \theta^j$$

(j any integer).

We define characters $\chi_1(j), \chi_2(j)$ (for any integer $j$) of $G$ by

$$\chi_1(j) = \beta_1^2(j) - \psi_1 - \psi_2 - \frac{1}{2}(3q-5)\psi_3 - \psi_4 - (q-2)\psi_5, \quad \text{if } j \text{ is even},$$

$$= \beta_1^2(j) - \psi_1 - \psi_2 - \frac{1}{2}(3q-5)\psi_3 - \psi_4 - (q-2)\psi_5, \quad \text{if } j \text{ is odd}.$$

$$\chi_2(j) = \beta_2^2(j) - \rho(j) - \sigma(j) + \psi_1 + \psi_2 + \frac{1}{2}(3q-5)\psi_3 + \psi_4 + (q-2)\psi_5, \quad \text{if } j \text{ is even},$$

$$= \beta_2^2(j) - \rho(j) - \sigma(j) + \psi_1 + \psi_2 + \frac{1}{2}(3q-5)\psi_3 + \psi_4 + (q-2)\psi_5, \quad \text{if } j \text{ is odd}.$$
These three families of characters are of degrees \((1 + q)^2(q^2 + 1), (1 - q)^2(1 + q^2)\) and \(1 - q^4\) respectively. The values of these characters at the classes of \(G\) are given in §8. We note that \(\chi_i(j) = \chi_i(k)\) if \(j \equiv k \pmod{q^2 + 1}\), etc.

In order to consider the irreducibility of these characters, we prove the following two lemmas.

**Lemma 4.3.** Consider the \(5 \times 5\) matrix

\[
(R_{ab}) = \begin{pmatrix}
(1-q^2)^2 & 1-q^4 & (1+q)^2(1+q^2) & (1-q)^2(1+q^2) & 1-q^4 \\
1-q^2 & 1-q^2 & (1+q)^2 & (1-q)^2 & 1+q^2 \\
1+q & 1-q & 1+q & 1-3q & 1+q \\
1-q & 1+q & 1+3q & 1-q & 1-q \\
1 & 1 & 1 & 1 & 1
\end{pmatrix}
\]

Between the columns of this matrix we have the relations

\[
\sum_a \frac{1}{s_a} R_{ab} R_{ay} = \delta_{by} \frac{z_b}{t_b},
\]

where \(s_1 = q^4(q^2 - 1)(q^4 - 1), \ s_2 = q^4(q^2 - 1), \ s_3 = 2q^6(q - 1), \ s_4 = 2q^6(q + 1), \ s_5 = q^2, \ z_1 = z_2 = z_5 = 4, \ z_3 = z_4 = 8, \ t_1 = q^2 + 1, \ t_2 = q^2 - 1, \ t_3 = (q - 1)^2, \ t_4 = (q + 1)^2, \ t_5 = q^2 - 1.\)

**Lemma 4.4.** Consider the \(2 \times 2\) matrix

\[
(K_{ab}) = \begin{pmatrix}
1+q & 1-q \\
1 & 1
\end{pmatrix}
\]

Between the columns of this matrix we have the relations

\[
\sum_a \frac{1}{a_a} K_{ab} K_{ay} = \delta_{by} \frac{2}{b_y},
\]

where \(a_1 = q(q^2 - 1), \ a_2 = q, \ b_1 = q - 1, \ b_2 = q + 1.\) (Compare [2, p. 431].)

Since the proofs of these lemmas are a routine verification of their statements, they will be omitted. The meaning of these lemmas will become clear when we prove the next lemma.

Consider the linear characters

\[
a_1 \rightarrow \xi_1; \quad a_2 \rightarrow \xi_1; \quad a_3 \rightarrow \xi^k; \quad b_3 \rightarrow \xi^l; \\
a_4 \rightarrow \eta^k; \quad b_4 \rightarrow \eta^l; \quad a_5 \rightarrow \eta^k; \quad b_5 \rightarrow \eta^l;
\]

of \(H_1, \ H_2, \ H_3, \ H_4, \ H_5\) respectively (\(j, k, l\) are any integers). Let \(\phi_1(j), \ \phi_2(j), \ \phi_3(k, l), \ \phi_4(k, l), \ \phi_5(k, l)\) be the characters of \(M_1, \ M_2, \ M_3, \ M_4, \ M_5\) respectively induced from these characters.

**Lemma 4.5.** The scalar product of two characters belonging to distinct families
\( \{ \chi_1(j) \}, \ldots \) is zero. For characters belonging to the same family, we have the following relations.

\[
(\chi_i(j), \chi(k)) = (\phi_i(j), \phi_i(k)) \quad (i = 1, 2) \\
(\chi_i(k, l), \chi(m, n)) = (\phi_i(k, l), \phi_i(m, n)) \quad (i = 3, 4, 5).
\]

(Compare [2, p. 431].)

**Proof.** We first remark that the entries in the five columns of the matrix \( (R_{xj}) \) of (4.3) are just the values of the five families of characters \( \{ \chi_1(j) \}, \{ \chi_2(j) \}, \{ \chi_3(k, l) \}, \{ \chi_4(k, l) \}, \{ \chi_5(k, l) \} \) at the classes \( A_1, A_2 \) (or \( A_{22} \)), \( A_{31}, A_{32}, \) and \( A_{41} \) (or \( A_{42} \)) respectively. Further,

\[
s_1 = \frac{|G|}{|A_1|} \quad s_2 = \frac{|G|}{|A_{21}| + |A_{22}|} \quad s_3 = \frac{|G|}{|A_{31}|} \quad s_4 = \frac{|G|}{|A_{32}|} \quad s_5 = \frac{|G|}{|A_{41}| + |A_{42}|}
\]

\(|t_i| = |H_i|, \ z_i|M_i : H_i| \). Next, we see that since each \( H_i \) is normal in \( M_i \), \( \phi_1(j), \ldots, \phi_5(k, l) \) vanish outside \( H_1, \ldots, H_5 \) respectively. Also \( \chi_1(j), \ldots, \chi_5(k, l) \) vanish on all elements of \( G \), whose \( p \)-regular factors do not lie in \( H_1, \ldots, H_5 \) respectively.

To prove the first assertion of the lemma consider a scalar product of two characters from different families \( \{ \chi_i(j) \} (i = 1, 2), \{ \chi_6(k, l) \} (i = 3, 4, 5) \). We consider the contribution to the scalar product from all elements of \( G \) whose \( p \)-regular factors are conjugate to a fixed element of \( G \). In other words, we consider separately the contributions from \( \{ A_1, \ldots, A_{42} \}, \{ A'_1, \ldots, A'_{42} \}, \{ B_1(i) \}, \ldots, \{ B_6(i, j) \}, \{ B_7(i) \}, \{ B_6(i) \}, \{ B_5(i) \}, \{ C_1(i) \}, \{ C_{21}(i) \}, \{ C_{22}(i) \}, \{ C'_1(i) \}, \{ C'_2(i) \}, \{ C'_3(i) \}, \{ C_4(i) \}, \{ C_4(i) \}, \{ C_5(i) \}, \{ C_5(i) \}, \{ D_1, \ldots, D_9 \} \) (where \( i, j, \ldots \) run over suitable index sets). Using Lemmas 4.3 and 4.4 we see that each such contribution must be zero.

Similarly we compare the contribution from each of the sets to the left-hand side of (4.6), with the contribution from the intersection with \( M_i \) of the \( p \)-regular class contained in the set to the right hand side of (4.6). We see then, again using Lemmas 4.3 and 4.4, that (4.6) holds. The details are omitted.

Using Lemma 4.5 we can now construct irreducible characters of \( G \). It is easy to see that \( \phi_1(j) (\phi_2(j)) \) is an irreducible character of \( M_1 (M_2) \) if and only if \( j \in R_1 \) (\( j \in R_2 \)). Hence, corresponding to these values of \( j \), we get \( \frac{1}{2}(q^2 - 1) \) irreducible characters in the family \( \{ \chi_1(j) \} \) and \( \frac{1}{4}(q - 1)^2 \) irreducible characters in the family \( \{ -\chi_5(j) \} \). Similarly \( \chi_3(k, l), \chi_4(k, l), \chi_5(k, l) \) are irreducible if and only if \( k, l \in T_1, k, l \in T_2, k \in T_1, l \in T_1 \) respectively. We have thus constructed families of irreducible characters which correspond to the families of classes \( \{ B_1(i) \}, \ldots, \{ B_6(i, j) \} \), and in each family there are as many irreducible characters as there are conjugacy classes in the corresponding family.

We now construct four families of characters which correspond to the families of classes \( \{ B_k(i) \} \) (\( k = 6, 7, 8, 9 \)).

**Lemma 4.7.** For each integer \( k \), the function \( f(k) \) which takes values on \( G \) as given in the table below is a character of \( G \).
BHAMA SRINIVASAN

<table>
<thead>
<tr>
<th>$A_1, A'_1$</th>
<th>$A_{21}, A_{22}, A'<em>{21}, A'</em>{22}$</th>
<th>$A_{23}, A_{32}$</th>
<th>$A_{41}, A_{42}, A_{42}, A_{42}$</th>
<th>$B_0(i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2(1+q^2)$</td>
<td>2</td>
<td>$2(1+q)$</td>
<td>$2(1-q)$</td>
<td>$\tilde{\gamma}<em>{ik} + \tilde{\gamma}</em>{-ik} + \tilde{\eta}<em>{ik} + \tilde{\eta}</em>{-ik}$</td>
</tr>
</tbody>
</table>

Proof. Let $\tau(k)$ be the character of degree 6 of $G$ which has as its value at an element $a$ of $G$, the image under the isomorphism (from the multiplicative group of $F$ into the multiplicative group of complex numbers) described in §1, of the second elementary symmetric function in the $k$th powers of the characteristic roots of $a$. (To see that such a character exists, see [2, p. 415]. This is $\alpha_k^G$ in Green’s notation.)

Then we see that $f(k) = \tau(k) - \chi_0(k, k) - \chi_1(k(q+1) - 2\theta_0)$ where $\theta_0$ is the identity character of $G$. Formally,

$$f(k) = \frac{1}{2}\chi_2(k(q+1)) + \frac{1}{2}\chi_2(k(q-1)) + \frac{1}{2}\chi_3(k, k) + \frac{1}{2}\chi_4(k, k).$$

Suppose $k$ is not a multiple of $\frac{1}{2}(q+1)$ or $\frac{1}{2}(q-1)$. Then, using Lemma 4.5 we see that $(f(k), f(k)) = 2$. Hence $f(k) = \zeta_1 + \zeta_2$ where either $\zeta_1$ or $-\zeta_1$ is irreducible $(i=1, 2)$. Now let, for a fixed $k$,

$$g_1 = \chi_4(k, k), \quad g_2 = \chi_2(k(q-1)),$$

$$g_3 = \chi_3(k, k), \quad g_4 = \chi_2(k(q+1)).$$

Then $(g_1, g_2) = 2\delta_{ij}$ and $(g_1, f(k)) = 1$, each $i$. Hence the $g_i$ must be of the form $\zeta_1 + \alpha, \zeta_1 - \alpha, \zeta_2 + \beta, \zeta_2 - \beta$ where $\alpha$ and $\beta$ are characters distinct from the $\zeta_i$ and from each other, and either $\alpha(\beta)$ or $-\alpha(-\beta)$ is irreducible. We can assume that $g_1 = \zeta_1 + \alpha$.

Case 1. Suppose $g_4 = \zeta_1 - \alpha$; then $\frac{1}{2}(g_1 + g_4)$ is a character. Consider the restriction of $g_1 + g_4$ to the cyclic subgroup $H_2$.

<table>
<thead>
<tr>
<th>$g_1 + g_4$</th>
<th>$h_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2(1-q)(1+q^2)$</td>
<td>$4 + 2(q-1)(1+q^2)$</td>
</tr>
</tbody>
</table>

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Since the function in the second row of the table is a character, the function \( h_1 \) in the third row is a character of \( H_2 \). If \( \frac{1}{2}(g_1 + g_4) \) is a character of \( G \), then \( \frac{1}{2}h_1 \) is a character of \( H_2 \). We shall prove that this is impossible. Now \( h_1 = 4(q-1)h_2 + h_3 + h_4 \), where \( h_2, h_3, h_4 \) are the characters of \( H_2 \) given below.

\[
\begin{array}{|c|c|}
\hline
h_2 & \frac{1}{2}(q^2-1) \\
\hline
h_3 & 2(q+1) \\
\hline
h_4 & 2(q-1) \\
\hline
\end{array}
\]

\[
(q+1)(\xi^{2ik} + \bar{\xi}^{-2ik}) \quad (q-1)(\xi^{2ik} + \bar{\xi}^{-2ik})
\]

Suppose \( \bar{\xi}^{2ik} \neq 1, -1 \) and \( \bar{\eta}^{2ik} \neq 1, -1 \). Then there is an irreducible character of \( H_2 \) occurring in \( h_3 \) with multiplicity 1 which does not occur in \( h_4 \) (e.g. the character defined by \( a_2 \to \theta^{2ik} + \bar{\theta}^{-1} \)). This shows that \( \frac{1}{2}h_1 \) cannot be a character of \( H_2 \).

If \( \bar{\xi}^{2ik} = 1 \) then \( k \) is a multiple of \( \frac{1}{2}(q-1) \) and if \( \bar{\eta}^{2ik} = 1 \) then \( k \) is a multiple of \( \frac{1}{2}(q+1) \). But, since \( \xi^{2ik} = \theta^{k(q+1)} \), \( \eta^{2ik} = \bar{\theta}^{k(q-1)} \), the case \( \bar{\xi}^{2ik} = \bar{\eta}^{2ik} = -1 \) is impossible.

**Case 2.** Suppose \( \frac{1}{2}(g_1 + g_3) \) is a character. We restrict \( g_1 + g_3 \) to the subgroup \( H_2 \).

\[
\begin{array}{|c|c|}
\hline
g_1 + g_3 & 2(1+q^2)^2 \\
\hline
h_5 & 2(1+q^2)^2 -4 \\
\hline
(1+q)(\xi^{2ik} + \bar{\xi}^{2ik}) & (1+q)(\xi^{2ik} + \bar{\xi}^{-2ik}) \\
\hline
(1-q)(\xi^{2ik} + \bar{\xi}^{-2ik}) & (1-q)(\xi^{2ik} + \bar{\xi}^{-2ik}) \\
\hline
\end{array}
\]

The function \( h_5 \) is a character of \( H_2 \). If \( \frac{1}{2}(g_1 + g_3) \) is a character of \( G \), then \( \frac{1}{2}h_5 \) is a character of \( H_2 \). An argument similar to that of Case 1 shows that this is impossible.

Hence we have shown

**Lemma 4.9.** Let \( k \) be an integer which is not a multiple of \( \frac{1}{2}(q+1) \) or \( \frac{1}{2}(q-1) \). Then the class functions \( \frac{1}{2}X_4(k,k)+\frac{1}{2}X_2(k(q-1)), \frac{1}{2}X_4(k,k)-\frac{1}{2}X_2(k(q-1)), \frac{1}{2}X_0(k,k)+\frac{1}{2}X_2(k(q+1)), \frac{1}{2}X_0(k,k)-\frac{1}{2}X_2(k(q+1)) \) are characters of \( G \). We denote these families by \( \{X_0(k,k), \{x_0(k,k)\} \) and \( \{x_6(k,k)\} \). They are of degrees \( (1-q)(1+q^2), q(q-1)(1+q^2), (1+q)(1+q^2) \) and \( (1+q)(1+q^2) \) respectively. We get \( \frac{j}{q-3} \) irreducible characters in each of the families \( \{-X_0(k), \{x_0(k), \{x_0(k), \{x_0(k)\} \}

The remaining values of \( k \) will be considered in (7.4). We shall then prove that \( \frac{1}{2}X_4(\frac{1}{2}(q-1), \frac{1}{2}(q-1)) \pm \frac{1}{2}X_2(\frac{1}{2}(q-1)^2) \) are also irreducible characters.

5. Characters corresponding to the families of classes \( \{C_1(i), \ldots, \{C_{42}(i)\}, \{C_1(i), \ldots, \{C_{42}(i)\} \).

(5.1) Consider the subgroup \( K \), and, for any integer \( k \), the characters \( \lambda(k) \times \lambda_0, \lambda'(k) \times \lambda_0 \) of \( K \), where \( \lambda_0 \) is the identity character of \( SL(2, q) \). Define characters \( \xi_1(k), \xi_2(k) \) of \( G \) by

\[
\xi_1(k) = \left[ \lambda'(k) \times \lambda_0 \right] - \sigma(k) + \psi_1 + \psi_2 - \frac{1}{2}(1-3q)\psi_2 + \psi_4 + (q-3)\psi_5, \quad \text{if } k \text{ is even,}
\]

\[
\xi_2(k) = \left[ \lambda(k) \times \lambda_0 \right] - \rho(k) + \psi_1 + \psi_2 - \frac{1}{2}(1-3q)\psi_2 + \psi_4 + (q-1)\psi_5, \quad \text{if } k \text{ is even.}
\]

If \( k \) is odd, replace \( \psi_i \) in the above definitions by \( \psi_i' \). The same convention will be followed in the rest of the paper.
Then we see that

\[
\xi_1(k) = \frac{1}{2} x_4(k, q + 1) + \frac{1}{2} x_5(k, q - 1),
\]

\[
\xi_2(k) = \frac{1}{2} x_9(k, q - 1) + \frac{1}{2} x_8(q + 1, k).
\]

Define \( \xi'_1(k) \) and \( \xi'_2(k) \) by

\[
\xi'_1(k) = -x_4(k, q + 1) + \xi_1(k),
\]

\[
\xi'_2(k) = x_9(k, q - 1) - \xi_2(k).
\]

Again by using Lemma 4.5 we see that for \( k \in T_2 \), \( \{-\xi'_1(k)\} \) and \( \{-\xi'_2(k)\} \) are irreducible characters of degrees \((q - 1)(q^2 + 1), q(q - 1)(q^2 + 1)\) respectively. For \( k \in T_1 \), \( \{\xi_8(k)\} \), \( \{\xi_9(k)\} \) are irreducible of degrees \((q + 1)(q^2 + 1), q(q + 1)(q^2 + 1)\) respectively.

(5.3) Consider the characters \( \mu_i \times \lambda^i(k) \), \( \mu_i \times \lambda(k) \) \( (i=1, 2, 3, 4) \) of \( K \), for any integer \( k \). Let

\[
\xi_{21}(k) = [\mu_1 \times \lambda(k)]^0 + \psi_{32}, \quad \text{if } k + t \text{ is even},
\]

\[
\xi_{22}(k) = [\mu_2 \times \lambda(k)]^0 + \psi_{31}, \quad \text{if } k + t \text{ is even},
\]

\[
\xi'_{21}(k) = [\mu_3 \times \lambda(k)]^0 - \sigma(k + t + 1) + \psi_1 + \psi_2 - \frac{1}{2}(1 - 3q)\psi_3 + \psi_4 + (q - 3)\psi_5 - \psi_{31},
\]

\[
\text{if } k + t + 1 \text{ is even},
\]

\[
\xi'_{22}(k) = [\mu_4 \times \lambda(k)]^0 - \sigma(k + t + 1) + \psi_1 + \psi_2 - \frac{1}{2}(1 - 3q)\psi_3 + \psi_4 + (q - 3)\psi_5 - \psi_{32},
\]

\[
\text{if } k + t + 1 \text{ is even},
\]

\[
\xi_{41}(k) = [\mu_1 \times \lambda(k)]^0 - \rho(k + t) + \psi_1 + \psi_2 - \frac{1}{2}(1 - 3q)\psi_3 + \psi_4 + (q - 1)\psi_5 - \psi_{32},
\]

\[
\text{if } k + t \text{ is even},
\]

\[
\xi_{42}(k) = [\mu_2 \times \lambda(k)]^0 - \rho(k + t) + \psi_1 + \psi_2 - \frac{1}{2}(1 - 3q)\psi_3 + \psi_4 + (q - 1)\psi_5 - \psi_{31},
\]

\[
\text{if } k + t \text{ is even},
\]

\[
\xi'_{41}(k) = [\mu_3 \times \lambda(k)]^0 + \psi_{31}, \quad \text{if } k + t + 1 \text{ is even},
\]

\[
\xi'_{42}(k) = [\mu_4 \times \lambda(k)]^0 + \psi_{32}, \quad \text{if } k + t + 1 \text{ is even}.
\]

Then we have equations

\[
(\xi_{21}(k) + \xi_{22}(k)) = x_5(k, \frac{1}{2}(q - 1)),
\]

\[
(\xi'_{21}(k) + \xi'_{22}(k)) = x_4(k, \frac{1}{2}(q + 1)),
\]

\[
(\xi_{41}(k) + \xi_{42}(k)) = x_5(\frac{1}{2}(q - 1), k),
\]

\[
(\xi'_{41}(k) + \xi'_{42}(k)) = x_4(\frac{1}{2}(q + 1), k).
\]

We can verify directly that

\[
(\xi_{21}(k), \xi_{22}(k)) = 0, \quad (\xi'_{21}(k), \xi'_{22}(k)) = 0,
\]

\[
(\xi_{41}(k), \xi_{42}(k)) = 0, \quad (\xi'_{41}(k), \xi'_{42}(k)) = 0.
\]

Hence, by a further application of Lemma 4.5, we see that for \( k \in T_1 \), we get four families \( \{-\xi'_{41}(k)\}, \{-\xi'_{42}(k)\}, \{\xi_{41}(k)\}, \{\xi_{42}(k)\} \) of irreducible characters of degrees \( \frac{1}{2}(q^4 - 1), \frac{1}{2}(q^4 - 1), \frac{1}{2}(q + 1)(q^2 + 1), \frac{1}{2}(q + 1)(q^2 + 1) \) respectively. For \( k \in T_2 \) we get
four families \{-\xi_{41}(k)\}, \{-\xi_{22}(k)\}, \{\xi_{21}(k)\}, \{\xi_{42}(k)\} of irreducible characters of degree \(\frac{1}{2}(q^4-1)\), \(\frac{1}{2}(q^4-1)\), \(\frac{1}{2}(q-1)(q^2+1)\), \(\frac{1}{2}(q-1)(q^2+1)\) respectively.

We remark that there does not appear to be a clearly defined correspondence between irreducible characters and conjugacy classes in this and subsequent sections. However, we can say that the set of characters \{\xi_{1}(k)\}, \{\xi_{1}'(k)\}, \ldots., corresponds to the set of classes \{C_{1}(i)\}, \{C_{1}'(i)\}, \ldots.

6. Characters corresponding to the classes \(D_{1}, \ldots, D_{34}\). Consider the following characters of \(G\):

\[
\begin{align*}
  f_1 &= \xi_{41}(q-1), \\
  f_2 &= \xi_{21}(q+1), \\
  f_3 &= \xi_{42}(q-1), \\
  f_4 &= \xi_{22}(q+1), \\
  g &= \xi_{1}(\frac{1}{2}(q+1)), \\
  h &= \xi_{1}'(\frac{1}{2}(q+1)).
\end{align*}
\]

Using (5.2), (5.4) and (5.5) we can show that

\[
\begin{align*}
  (f_i, f_j) &= 2\delta_{ij}, \\
  (g, f_i) &= 1 \quad \text{(all } i\text{)}, \\
  (h, f_i) &= (h, f_3) = 1, \\
  (h, f_2) &= (h, f_4) = -1.
\end{align*}
\]

It then follows that the \(f_i\) must be of the form \(\xi_1 + \xi_2, \xi_1 - \xi_2, \xi_3 + \xi_4, \xi_3 - \xi_4\), where either \(\xi_i\) or \(-\xi_i\) is irreducible \((i=1, 2, 3, 4)\). We consider the values of \(\frac{1}{2}(f_i + f_j)\) \((i=2, 3, 4)\) at the class \(A_{4i}\) of \(G\). We find that only \(\frac{1}{2}(f_1 + f_4)\) is integral at \(A_{4i}\). Hence the \(\xi_i\) must be the characters \(\frac{1}{2}(f_i \pm f_4)\) and \(\frac{1}{2}(f_2 \pm f_3)\). Thus, we have characters

\[
\begin{align*}
  \frac{1}{2}\xi_{41}(q-1) + \xi_{22}(q+1), \\
  \frac{1}{2}\xi_{42}(q-1) + \xi_{21}(q+1), \\
  \frac{1}{2}\xi_{41}(q-1) - \xi_{22}(q+1), \\
  \frac{1}{2}\xi_{42}(q-1) - \xi_{21}(q+1),
\end{align*}
\]

which will be denoted by \(\phi_1, \phi_2, \phi_3, \phi_4\), and are of degrees \(\frac{1}{2}(1-q)(1+q^2)\), \(\frac{1}{2}(1-q)(1+q^2)\), \(\frac{1}{2}q(1-q)(1+q^2)\), \(\frac{1}{2}q(1-q)(1+q^2)\) respectively. The \((-\phi_i)\) are irreducible, for each \(i\).

Similarly we can show that the functions

\[
\begin{align*}
  \frac{1}{2}\xi_{41}(q-1) + \xi_{22}(q+1), \\
  \frac{1}{2}\xi_{42}(q-1) + \xi_{21}(q+1), \\
  \frac{1}{2}\xi_{41}(q-1) - \xi_{22}(q+1), \\
  \frac{1}{2}\xi_{42}(q-1) - \xi_{21}(q+1),
\end{align*}
\]

are irreducible characters of degrees \(\frac{1}{2}(1+q)(1+q^2)\), \(\frac{1}{2}(1+q)(1+q^2)\), \(\frac{1}{2}q(1+q)(1+q^2)\), \(\frac{1}{2}q(1+q)(1+q^2)\) respectively. They will be denoted by \(\phi_5, \phi_6, \phi_7, \phi_8\) respectively. We now construct a further irreducible character of degree \(q(q^2+1)\). Consider the characters \(\mu_1 \times \mu_2, \mu_3 \times \mu_4\) of \(K\). Let \(\tau_1\) be the character \(\alpha_{11}\) (see (3.12)) of \(K'\) if \(q \equiv -1 \pmod{4}\), and \(\alpha_{12}\) if \(q \equiv 1 \pmod{4}\). Let \(\tau_2\) be the character \(\alpha_{21}\) of \(K'\) if \(q \equiv -1 \pmod{4}\) and \(\alpha_{22}\) if \(q \equiv 1 \pmod{4}\). Then let

\[
\Phi_9 = [\mu_1 \times \mu_2]^0 - [\mu_3 \times \mu_4]^0 + \tau_2^0 - \tau_1^0 + \phi_5.
\]

Then

\[
\Phi_9 = \frac{1}{2}X_0(\frac{1}{2}(q-1), \frac{1}{2}(q-1)) - \frac{1}{2}X_2(\frac{1}{2}(q+1), \frac{1}{2}(q+1)),
\]

showing that \(\Phi_9\) is an irreducible character of \(G\).
7. Characters corresponding to the classes $A_1, \ldots, A_{42}, A_1', \ldots, A_{42}'.$ There are fourteen irreducible characters of this type, of which two are the identity character and the Steinberg character [6].

(7.1) Consider the subgroups $M_2, M_3, M_4,$ and characters $\xi_2, \xi_3, \xi_4$ of $M_2, M_3, M_4$ respectively defined by

$$\begin{align*}
\xi_2 &: a_2 \rightarrow -1, \quad x_2 \rightarrow 1, \quad y_2 \rightarrow 1, \\
\xi_3 &: a_3 \rightarrow -1, \quad b_3 \rightarrow -1, \quad x_3 \rightarrow 1, \quad y_3 \rightarrow 1, \quad z_3 \rightarrow 1, \\
\xi_4 &: a_4 \rightarrow -1, \quad b_4 \rightarrow -1, \quad x_4 \rightarrow 1, \quad y_4 \rightarrow 1, \quad z_4 \rightarrow 1.
\end{align*}$$

We compute $-\xi_2^0 + \xi_3^0 + \xi_4^0.$ By the choice of the subgroups and the characters, the contribution to this character from the elements of $M_2, M_3, M_4$ outside $H_2, H_3, H_4$ is zero. (This situation will be encountered several times in this section.)

Consider the characters $\delta_1, \delta_2, \delta_3, \delta_4$ of (3.15). Let

$$\begin{align*}
\delta_1 &= -\xi_2 + \xi_3 + \xi_4 - \xi_3 + \xi_4 - \xi_2 + \xi_4, \\
\delta_2 &= -\xi_2 + \xi_3 + \xi_4 - \xi_3 + \xi_4 - \xi_2 + \xi_4.
\end{align*}$$

Then, $\delta_1, \delta_2$ are of degree $1/2q^2(q^2+1)$ and

$$\delta_1 + \delta_2 = \frac{1}{2}X_3(\frac{1}{2}(q-1), \frac{1}{2}(q-1)) + \frac{1}{2}X_3(\frac{1}{2}(q+1), \frac{1}{2}(q+1)) - \frac{1}{2}X_3(\frac{1}{2}(q^2-1)).$$

Thus, $(\delta_1 + \delta_2, \delta_1 + \delta_2) = 2.$ We can verify directly that $(\delta_1, \delta_2) = 0.$ Hence $\delta_1$ and $\delta_2$ are irreducible.

Now define characters $\delta_3, \delta_4$ of degree $1/2(1+q^2)$ by

$$\begin{align*}
\delta_3 + \delta_1 &= \frac{1}{4}T_1 \delta_1(\frac{1}{4}(q-1), \frac{1}{4}(q-1)) + \frac{1}{4}T_1 \delta_1(\frac{1}{4}(q+1), \frac{1}{4}(q+1)) + \frac{1}{4}T_1 \delta_1(\frac{1}{4}(q^2-1)), \\
\delta_4 + \delta_2 &= \frac{1}{4}T_1 \delta_2(\frac{1}{4}(q-1), \frac{1}{4}(q-1)) + \frac{1}{4}T_1 \delta_2(\frac{1}{4}(q+1), \frac{1}{4}(q+1)) + \frac{1}{4}T_1 \delta_2(\frac{1}{4}(q^2-1)),
\end{align*}$$

where

$$\tau_1 = a_{12}, \quad \tau_2 = a_{21} \quad \text{if } q \equiv 1 \pmod{4}, \quad \text{and}$$

$$\tau_1 = a_{11}, \quad \tau_2 = a_{22} \quad \text{if } q \equiv -1 \pmod{4}.$$ 

Then

$$\delta_3 + \delta_4 = \frac{1}{4}X_3(\frac{1}{4}(q-1), \frac{1}{4}(q-1)) + \frac{1}{4}X_3(\frac{1}{4}(q+1), \frac{1}{4}(q+1)) + \frac{1}{4}X_3(\frac{1}{4}(q^2-1)),$$

and again we can show that $\delta_3, \delta_4$ are irreducible.

(7.2) Consider the subgroups $M_1, M_5$ and characters $\xi_1, \xi_5$ of $M_1, M_5$ respectively given by

$$\begin{align*}
\xi_1 &: a_1 \rightarrow -1, \quad x_1 \rightarrow 1, \\
\xi_5 &: a_5 \rightarrow -1, \quad b_5 \rightarrow -1, \quad y_5 \rightarrow 1, \quad z_5 \rightarrow 1.
\end{align*}$$

Define characters $\delta_5, \delta_6$ of $G$ by

$$\begin{align*}
\delta_5 &= -\xi_1^0 + \xi_5^0 - \xi_5^0 + \xi_1^0 + \xi_1^0 + \xi_1^0, \\
\delta_6 &= -\xi_1^0 + \xi_5^0 - \xi_5^0 + \xi_1^0 + \xi_1^0 + \xi_1^0,
\end{align*}$$

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and characters \( \theta_7, \theta_8 \) by
\[
\theta_6 + \theta_7 = [\mu_3 \times (\mu_1 + \mu_2)]^q + \psi_3, \\
\theta_6 + \theta_8 = [\mu_4 \times (\mu_1 + \mu_2)]^q + \psi_4.
\]
Then
\[
\theta_5 + \theta_6 = -x_1(\frac{1}{4}(q^2+1)) + \chi_3(\frac{1}{4}(q+1), \frac{1}{4}(q-1)), \\
\theta_7 + \theta_8 = x_1(\frac{1}{4}(q^2+1)) + \chi_3(\frac{1}{4}(q+1), \frac{1}{4}(q-1)),
\]
and again it can be shown that \( -\theta_5, -\theta_6, -\theta_7, -\theta_8 \) are irreducible characters of degrees \( \frac{1}{4}q^2(q^2-1), \frac{1}{4}q^2(q^2-1), \frac{1}{4}(q^2-1), \frac{1}{4}(q^2-1) \) respectively.

(7.3) We first construct four characters \( K_1, K_2, K_3, K_4 \) of \( G \) of degrees \( q(q^2+1) \), \( q(q^2+1) \), \( q^2 \), \( q^2 \) respectively, which will be used to construct four irreducible characters of \( G \).

Take the subgroups \( M_3, M_4 \) and characters \( \rho_{31}, \rho_{32} \) of \( M_3 \) and \( \rho_{41}, \rho_{42} \) of \( M_4 \) given by
\[
\rho_{31}: a_3 \rightarrow 1, \quad b_3 \rightarrow 1, \quad x_3 \rightarrow 1, \quad y_3 \rightarrow 1, \quad z_3 \rightarrow 1, \\
\rho_{32}: a_3 \rightarrow 1, \quad b_3 \rightarrow 1, \quad x_3 \rightarrow -1, \quad y_3 \rightarrow -1, \quad z_3 \rightarrow -1, \\
\rho_{41}: a_4 \rightarrow 1, \quad b_4 \rightarrow 1, \quad x_4 \rightarrow 1, \quad y_4 \rightarrow 1, \quad z_4 \rightarrow 1, \\
\rho_{42}: a_4 \rightarrow 1, \quad b_4 \rightarrow 1, \quad x_4 \rightarrow -1, \quad y_4 \rightarrow -1, \quad z_4 \rightarrow -1.
\]
Consider the characters of degree \( q(q^2+1) \)
\[
\rho_{31}^q + \rho_{32}^q - \rho_{41}^q - \rho_{42}^q + \alpha_3^q - \alpha_1^q + \delta_2^q - \delta_2^q + \psi_5, \\
\rho_{31}^q + \rho_{32}^q - \rho_{41}^q - \rho_{42}^q + \alpha_3^q - \alpha_1^q + \delta_2^q - \delta_2^q + \psi_5.
\]
(For the definitions of \( \delta_5, \delta_2 \) see (3.15).) These two characters are identical on all classes of \( G \) except \( D_{31}, D_{32}, D_{33}, D_{34} \). On these classes one of them, which we denote by \( K_1 \), takes values \( (q, -q, -q, q) \); the other, which we denote by \( K_2 \), takes values \( (-q, q, q, -q) \). Let \( \Gamma_1, \Gamma_2 \) be class functions on \( G \) which take values \( (q, -q, -q, q) \) and \( (-q, q, q, -q) \) respectively on \( D_{31}, D_{32}, D_{33}, D_{34} \), and vanish on all other classes of \( G \). Then
\[
K_1 = \frac{1}{4}\chi_3(q-1, q-1)-\frac{1}{4}\chi_4(q+1, q+1)+\Gamma_1, \\
K_2 = \frac{1}{4}\chi_3(q-1, q-1)-\frac{1}{4}\chi_4(q+1, q+1)+\Gamma_2.
\]
Thus \( (K_1, K_2) = 0 \) and \( (K_1, K_1) = (K_2, K_2) = 2 \).

Now consider the following characters of \( M_1, M_2, M_3, M_4, M_5 \) respectively:
\[
\sigma_1: a_1 \rightarrow 1, \quad x_1 \rightarrow 1, \\
\sigma_2: a_2 \rightarrow 1, \quad x_2 \rightarrow -1, \quad y_2 \rightarrow -1, \\
\sigma_3: a_3 \rightarrow 1, \quad b_3 \rightarrow 1, \quad x_3 \rightarrow 1, \quad y_3 \rightarrow 1, \quad z_3 \rightarrow 1, \\
\sigma_4: a_4 \rightarrow 1, \quad b_4 \rightarrow 1, \quad x_4 \rightarrow 1, \quad y_4 \rightarrow 1, \quad z_4 \rightarrow 1, \\
\sigma_5: a_5 \rightarrow 1, \quad b_5 \rightarrow 1, \quad y_5 \rightarrow 1, \quad z_5 \rightarrow 1.
\]
Consider the characters of degree $q^2$

$$-\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2 - \sigma_5^2 + \sigma_1 + \sigma_2 + \frac{1}{2}(q-1)\psi_3 + \psi_4 + (q-2)\psi_5,$$

$$-\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2 - \sigma_5^2 + \sigma_1 + \sigma_2 + \frac{1}{2}(q-1)\psi_3 + \psi_4 + (q-2)\psi_5.$$

(Note that, as in (7.1), the contribution to $-\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2 - \sigma_5^2$ from the classes of the $M_i$ outside $H_i$ is zero.)

Again, we get two characters $K_3, K_4$ which differ only on $D_3, D_3, D_3, D_3,$ and

$$K_3 = -\frac{1}{4}X_4(q^2 + 1) + \frac{1}{4}X_2(q^2 - 1) + \frac{1}{4}X_3(q - 1, q - 1) + \Gamma_1,$$

$$K_4 = -\frac{1}{4}X_4(q^2 + 1) + \frac{1}{4}X_2(q^2 - 1) + \frac{1}{4}X_3(q - 1, q - 1) + \Gamma_2,$$

where $\Gamma_1, \Gamma_2$ are the functions defined above.

We have the following table for the scalar products $(K_i, K_j)$.

<table>
<thead>
<tr>
<th></th>
<th>$K_1$</th>
<th>$K_2$</th>
<th>$K_3$</th>
<th>$K_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_1$</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$K_2$</td>
<td>2</td>
<td>-1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$K_3$</td>
<td></td>
<td>2</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$K_4$</td>
<td></td>
<td></td>
<td></td>
<td>2</td>
</tr>
</tbody>
</table>

Since $(K_1, K_3) = 1$ we can assume that $K_1 = \nu_1 + \nu_2, K_3 = \nu_1 - \nu_4$ where $\nu_i$ or $-\nu_i$ is irreducible ($i = 1, 2, 4$). Then $K_2$ must be either $-\nu_1 + \nu_2$ or $\nu_3 + \nu_4$ for some $\nu_3$ which is distinct from $\nu_1, \nu_2, \nu_4$ and is such that $-\nu_3$ or $\nu_3$ is irreducible. But if $K_2 = -\nu_1 + \nu_2$ then either $\frac{1}{4}(K_1 - K_3)$ or its negative is an irreducible character, which is impossible since it is of degree 0. Hence we have $K_1 = \nu_1 + \nu_2, K_2 = \nu_3 + \nu_4, K_3 = \nu_1 - \nu_4, K_4 = \nu_3 - \nu_2$. These equations are not sufficient to compute the $\nu_i$. For this purpose consider

$$K_5 = \xi_3(q-1) - \theta_0.$$

Now

$$\theta_0 = \frac{1}{4}X_4(q^2 + 1) + \frac{1}{4}X_2(q^2 - 1) + \frac{1}{4}X_3(q - 1, q - 1) + \frac{1}{4}X_4(q + 1, q - 1),$$

and

$$\xi_3(q-1) = \frac{1}{4}X_3(q - 1, q - 1) + \frac{1}{4}X_3(q + 1, q - 1),$$

and so $(K_5, K_i) = 0$ ($i = 1, 2, 3, 4$). Thus $K_5 = \nu_1 + \nu_4$ or $K_5 = \nu_2 + \nu_3$.

Suppose $K_5 = \nu_2 + \nu_3$. We consider the restriction of the class function $\frac{1}{4}(K_4 + K_5)$
to the abelian subgroup $V$ of order $4q^2$ consisting of all elements of the form

$$\begin{pmatrix}
\pm 1 & \lambda \\
\lambda & \pm 1 \\
\pm 1 & \mu \\
\mu & \pm 1
\end{pmatrix} \quad (\lambda, \mu \in F).$$

This function has the following values at the classes of $G$ which meet $V$.

<table>
<thead>
<tr>
<th>$A_1, A'_1$</th>
<th>$A_{21}, A'_{21}$</th>
<th>$A_{22}, A'_{22}$</th>
<th>$A_{31}, A'_{31}$</th>
<th>$A_{32}, A'_{32}$</th>
<th>$A_{41}, A'_{41}$</th>
<th>$A_{42}, A'_{42}$</th>
<th>$D_1$</th>
<th>$D_{21}, D_{22}, D_{23}, D_{24}$</th>
<th>$D_{31}, D_{32}, D_{33}$</th>
<th>$D_{32}, D_{33}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{2}q(1+q^2)$</td>
<td>$\frac{1}{2}q(1+q)$</td>
<td>$q$</td>
<td>$0$</td>
<td>$0$</td>
<td>$\frac{1}{2}(1+q)^2$</td>
<td>$\frac{1}{2}(1+q)$</td>
<td>$\frac{1}{2}(1-q)$</td>
<td>$\frac{1}{2}(1+q)$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

[We remark that $\frac{1}{2}(K_3 + K_5)$ has the same values as $\frac{1}{2}(K_4 + K_5)$ on all the classes of $G$ except $D_{31}, D_{32}, D_{33}, D_{34}$. At these classes it has values $\frac{1}{2}(1+q), \frac{1}{2}(1-q), \frac{1}{2}(1+q)$ respectively.]

We consider the scalar product of the restriction of $\frac{1}{2}(K_4 + K_5)$ to $V$ with the character of $V$ defined by

$$\begin{pmatrix}
1 & \lambda \\
\lambda & 1 \\
1 & \mu \\
\mu & 1
\end{pmatrix} \rightarrow \epsilon(\lambda)e(\mu), \quad \begin{pmatrix}
1 & \\
& -1 \\
& -1 \\
& 1
\end{pmatrix} \rightarrow 1,$$

This scalar product is seen to be

$$\frac{1}{2}(q + 1), \quad \text{if } q \equiv 1 \pmod{4},$$

$$\frac{1}{4}(q + 3), \quad \text{if } q \equiv -1 \pmod{4}.$$  

This shows that $\frac{1}{2}(K_4 + K_5)$ cannot be a character. Hence $K_5 = \nu_1 + \nu_4$, and this enables us to compute the characters $\nu_1, \nu_2, \nu_3, \nu_4$. We thus get four irreducible characters of degrees $q(1+q^2)/2, q(1-q^2)/2, q(1+q^2)/2, q(1+q^2)/2$, and we denote them by $\theta_9, \theta_{10}, \theta_{11}, \theta_{12}$.

(7.4) We now show that the functions $\frac{1}{2}X_4(k, k) \pm \frac{1}{2}X_6(k(q-1)), \frac{1}{2}X_6(k, k) \pm \frac{1}{2}X_6(k(q+1))$ are characters of $G$ also for the values of $k$ which were omitted in §4, i.e. for $k$ a multiple of $\frac{1}{2}(q-1)$ or $\frac{1}{2}(q+1)$. We note that, by (4.8), for a fixed $k$ it is enough to show this for one of these four functions.

Case 1. $k$ is an odd multiple of $\frac{1}{2}(q-1)$. Then $X_6(k, k) = X_6(\frac{1}{2}(q-1), \frac{1}{2}(q-1))$, $X_6(k(q+1)) = X_6(\frac{1}{2}(q^2-1))$, and $\frac{1}{2}X_6(k, k) \pm \frac{1}{2}X_6(k(q+1)) = \Phi_6 + \theta_9 + \theta_4$. 

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Case 2. $k$ is an odd multiple of $\frac{1}{4}(q+1)$. In this case
\[ \frac{1}{2}X_4(k, k) + \frac{1}{2}X_2(k(q-1)) = -\Phi_9 + \theta_3 + \theta_4. \]

Case 3. $k$ is an even multiple of $\frac{1}{4}(q-1)$. In this case
\[ \frac{1}{2}X_4(k, k) + \frac{1}{2}X_2(k(q+1)) = \frac{1}{2}X_3(q-1, q-1) + \frac{1}{2}X_4(q^2-1) = \theta_9 + \theta_{11} + \theta_0. \]

Case 4. $k$ is an even multiple of $\frac{1}{4}(q+1)$. Then
\[ \frac{1}{2}X_4(k, k) + \frac{1}{2}X_2(k(q-1)) = -\theta_{10} + \theta_{12} + \theta_0. \]

We can now define characters $x_6(k), x_7(k), x_8(k), x_9(k)$ as in Lemma 4.9 for any integer $k$. We also get one more irreducible character in each of the families $\{-x_6(k)\}$ and $\{x_7(k)\}$.

(7.5) We now give a simple construction for the Steinberg character [6] of $G$, which will be denoted by $\theta_4$. Let $\Delta_2, \Delta_3, \Delta_4, \Delta_5$ be the identity characters of $M_2, M_3, M_4, M_5$ respectively, and let $\Delta_4$ be the character of $M_4$ defined by $a_1 \rightarrow 1, x_1 \rightarrow -1$. Then the character $\Delta_4^2 - \Delta_5^2 + \Delta_6^2 - \Delta_2^2$ is of degree $q^4$, and is irreducible since it is equal to
\[ \frac{1}{4}X_4(q^2+1) - \frac{1}{4}X_2(q^2-1) + \frac{1}{2}X_3(q-1, q-1) + \frac{1}{2}X_4(q+1, q+1) - \frac{1}{4}X_5(q+1, q-1). \]

We have now exhausted all the irreducible characters of $G$, since we have as many as there are conjugacy classes of $G$.

8. Table of characters. The table that follows contains the characters of $G$ that were obtained in the previous sections. In the case of families of characters indexed by parameters, the parameters take rational integral values. The values of the parameters for which the characters are irreducible are indicated in the table. In some cases it is the negative of a character that is irreducible, and this will not be mentioned explicitly in the table.

The values of the characters at the classes $A_2, \ldots, A_{12}$ are omitted, since they can be obtained from the values at $A_2, \ldots, A_{42}, A'$. Similarly the values of the characters at the classes $C_1(i), \ldots, C_4(i), D_{23}, D_{24}$ are omitted.

It is sufficient to give the values of one character from the pair $\{\xi_{21}(k), \xi_{22}(k)\}$, for the values of the other are then got by replacing $\bar{e}$ by $\bar{e}'$ and $\bar{e}'$ by $\bar{e}$. A similar statement holds for the pairs $\{\xi_{21}(k), \xi_{22}(k)\}, \{\xi_{41}(k), \xi_{42}(k)\}, \{\xi_{41}(k), \xi_{42}(k)\}, \{\Phi_1, \Phi_2\}, \{\Phi_3, \Phi_4\}, \{\Phi_5, \Phi_6\}, \{\Phi_7, \Phi_8\}, \{\theta_1, \theta_2\}, \{\theta_3, \theta_4\}, \{\theta_5, \theta_6\}, \{\theta_7, \theta_8\}$.

Finally we omit the values of the characters at the classes $A_{22}, A_{42}, C_{22}(i), C_{42}(l), D_{22}, D_{23}, D_{24}$, as these can be obtained from the classes $A_{21}, A_{41}, C_{21}(i), C_{22}(l), D_{21}, D_{22}, D_{31}$ by replacing $\bar{e}$ by $\bar{e}'$ and $\bar{e}'$ by $\bar{e}$. Again, the absence of an entry in the table indicates that the corresponding value is zero. We also use the abbreviations $t = \frac{1}{2}(q-1), \alpha_j = \bar{e}^j + \bar{e}^{-j}, \beta_j = \bar{e}^j + \bar{e}^{-j}$, and $s(k, l) = (-1)^k + (-1)^l$.  

<table>
<thead>
<tr>
<th>Character</th>
<th>$\chi_1(j)$</th>
<th>$\chi_2(j)$</th>
<th>$\chi_3(k, l)$</th>
<th>$\chi_4(k, l)$</th>
<th>$\chi_5(k, l)$</th>
<th>$\chi_6(k)$</th>
<th>$\chi_7(k)$</th>
<th>$\chi_8(k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>When irreducible</td>
<td>$j \in R_1$</td>
<td>$j \in R_2$</td>
<td>$k, l \in T_1$, $k \neq l$</td>
<td>$k, l \in T_2$, $k \neq l$</td>
<td>$k \in T_2$, $l \in T_1$</td>
<td>$k \in T_2$</td>
<td>$k \in T_2$</td>
<td>$k \in T_1$</td>
</tr>
<tr>
<td>$A_1$</td>
<td>$(1-q^2)^2$</td>
<td>$1-q^4$</td>
<td>$(1+q)^2(1+q^2)$</td>
<td>$(1-q^2)(1+q^2)$</td>
<td>$1-q^4$</td>
<td>$(1-q)(1+q^2)$</td>
<td>$q(q-1)(1+q^2)$</td>
<td>$(1+q)(1+q^2)$</td>
</tr>
<tr>
<td>$A'_1$</td>
<td>$(-1)^2(1-q^2)^2$</td>
<td>$(-1)^2(1-q^2)$</td>
<td>$(-1)^k(1+q^2)^2$</td>
<td>$(-1)^k(1-q)^2$</td>
<td>$(-1)^k(1-q^2)$</td>
<td>$(-1)^k(1+q^2)$</td>
<td>$q(q-1)(1+q^2)$</td>
<td>$(1+q)(1+q^2)$</td>
</tr>
<tr>
<td>$A_{21}$</td>
<td>$1-q^2$</td>
<td>$1-q^2$</td>
<td>$(1+q)^2$</td>
<td>$(1-q)^2$</td>
<td>$1+q^2$</td>
<td>$1-q$</td>
<td>$q(q-1)$</td>
<td>$1+q$</td>
</tr>
<tr>
<td>$A_{31}$</td>
<td>$1-q$</td>
<td>$1+q$</td>
<td>$1+3q$</td>
<td>$1-q$</td>
<td>$1-q$</td>
<td>$1$</td>
<td>$-q$</td>
<td>$1+2q$</td>
</tr>
<tr>
<td>$A_{32}$</td>
<td>$1+q$</td>
<td>$1-q$</td>
<td>$1+q$</td>
<td>$1-3q$</td>
<td>$1+q$</td>
<td>$1-2q$</td>
<td>$-q$</td>
<td>$1$</td>
</tr>
<tr>
<td>$A_{41}$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>$B_1(i)$</td>
<td>$\bar{y}^{-1} + \bar{y}^{-1}$</td>
<td>$\bar{y} + \bar{y}^{1}$</td>
<td>$\beta_{ik}$</td>
<td>$-\beta_{ik}$</td>
<td>$\alpha_{ik}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$B_2(i)$</td>
<td>$\bar{y}^{1} + \bar{y}^{-1}$</td>
<td>$\bar{y}^{1} + \bar{y}^{-1}$</td>
<td>$\beta_{ik}$</td>
<td>$-\beta_{ik}$</td>
<td>$\alpha_{ik}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$B_3(i, j)$</td>
<td>$\alpha_{ik}a_{il} + \alpha_{jk}a_{jl}$</td>
<td>$\beta_{il}\beta_{jk}$</td>
<td>$\beta_{ik}\beta_{jk}$</td>
<td>$\beta_{ik}\beta_{jk}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$B_4(i, j)$</td>
<td>$\beta_{il}\beta_{jk}$</td>
<td>$\beta_{il}\beta_{jk}$</td>
<td>$\beta_{ik}\beta_{jk}$</td>
<td>$\beta_{ik}\beta_{jk}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Character</td>
<td>$x_{1}(j)$</td>
<td>$x_{2}(j)$</td>
<td>$x_{3}(k, l)$</td>
<td>$x_{4}(k, l)$</td>
<td>$x_{5}(k, l)$</td>
<td>$x_{6}(k)$</td>
<td>$x_{7}(k)$</td>
<td>$x_{8}(k)$</td>
</tr>
<tr>
<td>-----------</td>
<td>------------</td>
<td>------------</td>
<td>---------------</td>
<td>---------------</td>
<td>---------------</td>
<td>------------</td>
<td>------------</td>
<td>------------</td>
</tr>
<tr>
<td>When irreducible</td>
<td>$j \in R_1$</td>
<td>$j \in R_2$</td>
<td>$k, l \in T_3, k \neq l$</td>
<td>$k, l \in T_3, l \in T_1$</td>
<td>$k \in T_2$</td>
<td>$k \in T_2$</td>
<td>$k \in T_1$</td>
<td></td>
</tr>
<tr>
<td>$B_6(i, j)$</td>
<td>$1 + q \beta_{ij}$</td>
<td></td>
<td></td>
<td></td>
<td>$\beta_{ik} \alpha_{hl}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$B_6(i)$</td>
<td></td>
<td>$1 + q \beta_{ij}$</td>
<td>$1 - q \beta_{ik} \alpha_{hl}$</td>
<td></td>
<td>$\beta_{ik} + 1 - q$</td>
<td>$- q \beta_{ik} + 1 - q$</td>
<td>$1 + q$</td>
<td></td>
</tr>
<tr>
<td>$B_6(i)$</td>
<td>$\beta_{ij}$</td>
<td>$\beta_{ik} \alpha_{hl}$</td>
<td></td>
<td></td>
<td>$\beta_{ik} + 1$</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$B_6(i)$</td>
<td>$(1 - q) \alpha_{ij}$</td>
<td>$(1 - q) \alpha_{ij}$</td>
<td></td>
<td></td>
<td>$1 - q$</td>
<td>$- (1 - q)$</td>
<td>$\alpha_{ik} + 1 + q$</td>
<td></td>
</tr>
<tr>
<td>$B_6(i)$</td>
<td>$\alpha_{ij}$</td>
<td>$\alpha_{ij} \alpha_{hl}$</td>
<td></td>
<td></td>
<td>1</td>
<td>$- 1$</td>
<td>$\alpha_{2ik} + 1$</td>
<td></td>
</tr>
<tr>
<td>$C_1(i)$</td>
<td></td>
<td></td>
<td>$(1 - q)(\beta_{ik} + \beta_{il})$</td>
<td>$(1 - q) \beta_{ik}$</td>
<td>$(1 - q) \beta_{ik}$</td>
<td>$(1 - q) \beta_{ik}$</td>
<td>$(1 - q) \beta_{ik}$</td>
<td></td>
</tr>
<tr>
<td>$C_21(i)$</td>
<td></td>
<td></td>
<td>$(1 - q) \beta_{ik}$</td>
<td>$\beta_{ik}$</td>
<td>$\beta_{ik}$</td>
<td>$\beta_{ik}$</td>
<td>$\beta_{ik}$</td>
<td></td>
</tr>
<tr>
<td>$C_3(i)$</td>
<td></td>
<td></td>
<td>$(1 - q)(\alpha_{ik} + \alpha_{il})$</td>
<td>$(1 - q) \alpha_{il}$</td>
<td>$(1 + q) \alpha_{ik}$</td>
<td>$(1 + q) \alpha_{ik}$</td>
<td>$(1 + q) \alpha_{ik}$</td>
<td></td>
</tr>
<tr>
<td>$C_{43}(i)$</td>
<td></td>
<td></td>
<td>$\alpha_{ik} + \alpha_{il}$</td>
<td></td>
<td>$\alpha_{il}$</td>
<td>$\alpha_{il}$</td>
<td>$\alpha_{il}$</td>
<td></td>
</tr>
<tr>
<td>$D_{1}$</td>
<td>$(1 + q)^2 s(k, l)$</td>
<td>$(1 - q)^2 s(k, l)$</td>
<td>$(- 1)^{s}(1 - q)^2$</td>
<td>$(- 1)^{s}(1 - q)^2$</td>
<td>$(- 1)^{s}(1 - q)^2$</td>
<td>$(- 1)^{s}(1 + q)^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$D_{21}$</td>
<td>$(1 + q)s(k, l)$</td>
<td>$(1 - q)(- 1)^{s}(k, l)$</td>
<td>$(- 1)^{s}(1 - q)$</td>
<td>$(- 1)^{s}(1 - q)$</td>
<td>$(- 1)^{s}(1 - q)$</td>
<td>$(- 1)^{s}(1 + q)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$D_{22}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Character</td>
<td>$\chi_0(k)$</td>
<td>$\xi_1(k)$</td>
<td>$\xi_1'(k)$</td>
<td>$\xi_2(k)$</td>
<td>$\xi_3(k)$</td>
<td>$\xi_{21}(k)$</td>
<td>$\xi_{21}(k)$</td>
<td></td>
</tr>
<tr>
<td>-----------</td>
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<td>-------------</td>
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<td>-----------</td>
<td>------------</td>
<td>-------------</td>
<td>-------------</td>
<td></td>
</tr>
<tr>
<td>When irreducible</td>
<td>$k \in T_1$</td>
<td>$k \in T_2$</td>
<td>$k \in T_2$</td>
<td>$k \in T_1$</td>
<td>$k \in T_1$</td>
<td>$k \in T_4$</td>
<td>$k \in T_2$</td>
<td></td>
</tr>
<tr>
<td>$A_1$</td>
<td>$q(1+q)(1+q^2)$</td>
<td>$(1-q)(1+q^2)$</td>
<td>$q(1-q)(1+q^2)$</td>
<td>$(1+q)(1+q^2)$</td>
<td>$q(1+q)(1+q^2)$</td>
<td>$\frac{1}{2}(1-q^4)$</td>
<td>$\frac{1}{2}(1-q)^2(1+q^2)$</td>
<td></td>
</tr>
<tr>
<td>$A_1'$</td>
<td>$q(1+q)(1+q^2)$</td>
<td>$(-1)^s(1+q)$</td>
<td>$(1+q^2)$</td>
<td>$(-1)^s(1+q)$</td>
<td>$(1+q^2)$</td>
<td>$(1+q)^2$</td>
<td>$rac{1}{2}(1-q)^2(1-q^4)$</td>
<td></td>
</tr>
<tr>
<td>$A_{21}$</td>
<td>$q(1+q)$</td>
<td>$1+q^2-q$</td>
<td>$q$</td>
<td>$1+q+q^2$</td>
<td>$q$</td>
<td>$\frac{1}{2}(1+q)(1-q)\hat{e}$</td>
<td>$\frac{1}{2}(1-q)(q-1)\hat{e}$</td>
<td></td>
</tr>
<tr>
<td>$A_{21}$</td>
<td>$q$</td>
<td>$1-q$</td>
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REFERENCES


The Ramanujan Institute, University of Madras, Madras, India