THE STRUCTURE AND IDEAL THEORY OF THE
PREDUAL OF A BANACH LATTICE

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1. Introduction. If $V$ is an ordered Banach space over the real field, then the
Banach dual $V^*$ has a natural induced partial ordering. In Theorems 3.1 and 3.2
we present necessary and sufficient conditions for $V^*$ to be a Banach lattice,
extending partial results obtained by Choquet [5] and Andô [1]. The theorems
include as special cases the characterisations of the predual of an $M$-space, [8],
[14], and the predual of an $L$-space, [6], called a simplex space in [11]. We show
how the theory is a natural generalisation of Choquet simplex theory.

If $V^*$ is a Banach lattice, we study the set of closed ideals of $V$. For the special
case of simplex spaces, the results provide direct proofs of theorems in [11].

2. Basic theorems on ordered Banach spaces. An ordered normed space $V$ over
the real field is defined as a normed vector space $V$ with a closed cone $C$ which is
proper in the sense that $C \cap (-C) = \{0\}$ and with the partial ordering given by
saying that $x \leq y$ if and only if $y-x \in C$. $V$ is said to be positively generated if
$C - C = V$ and to have a monotone norm if $0 \leq x \leq y$ implies $\|x\| \leq \|y\|$. $V$ is said to be
regular if it has the properties

(i) if $x, y \in V$ and $-x \leq y \leq x$ then $\|y\| \leq \|x\|$;
(ii) if $x \in V$ and $\varepsilon > 0$ then there is some $y \in V$ with $y \leq x$, $-x$ and $\|y\| < \|x\| + \varepsilon$.

A normed lattice is a regular ordered normed space which is a lattice under its
partial ordering, and a Banach lattice is a complete normed lattice. Banach lattices
are discussed in many places, for example [7], [12], and we follow the standard
terminology. A partially ordered vector space $V$ is said to have the Riesz separation
property if when $a, b \leq c, d \in V$ we can find some $x \in V$ with

$$a, b \leq x \leq c, d.$$ 

For alternative formulations of this condition see [1], [9].

Lemma 2.1. Let $V$ be an ordered positively generated Banach space with a mono-
tone norm $\| \|$, and define

$$\|x\|_1 = \inf \{\|y\| : x, -x \leq y \in V\}$$

for all $x \in V$. Then $\| \|_1$ is a regular norm on $V$ which is equivalent to $\| \|$, and for
any $0 \leq x \in V$ we have $\|x\|_1 = \|x\|$.

For a proof of this see [16].

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Let $V$ be a positively generated ordered Banach space with Banach dual $V^*$. The set $C^*$ of continuous positive functionals is a weak*-closed proper cone in $V^*$. By the Hahn-Banach theorem if $x \in V$ then $x \geq 0$ if and only if $(x, \phi) \geq 0$ for all $\phi \in C^*$. If $V$ is a positively generated ordered Banach space with a monotone norm then, by [16], every positive functional on $V$ is continuous; also if $0 \leq x \in V$ then

$$\|x\| = \max \{(x, \phi) : \phi \in C^*, \|\phi\| \leq 1\}.$$

This follows from the fact that $-x$ is not in the open convex set

$$\{v \in V : \exists c \in C, \|v - c\| < \|x\|\}$$

by the use of the Hahn-Banach theorem. If $V$ is a regular ordered Banach space then for any $0 \leq \phi \in V^*$ we have

$$\|\phi\| = \sup \{(x, \phi) : x \in C, \|x\| \leq 1\}.$$

**Lemma 2.2.** If $V$ is a regular ordered normed space then for any $\phi, \psi \in V^*$ with $-\phi \leq \psi \leq \phi$ we have $\|\phi\| \geq \|\psi\|$.

For let $x \in V$ and $\|x\| < 1$. If $z \geq x$, 0 then

$$((\psi, x) = (\psi, z) + (\psi, x - z) \\
\leq (\phi, z) + (\phi, z - x) \\
= (\phi, 2z - x).$$

If $y \in V$ satisfies $y \geq x$, $-x$ and $\|y\| < 1$ and we put $z = (x + y)/2$ then we obtain

$$(\phi, x) \leq \|\phi\| \|y\| < \|\phi\|.$$

The result now follows immediately.

Very much more information can be obtained about $V^*$, but we first need to consider a special case.

**Lemma 2.3.** Let $V$ be a regular ordered normed space with the Riesz separation property. Then $V^*$ is a Banach lattice.

Let $\phi \in V^*$ and $x \in V$, $x \geq 0$. Then we define

$$(\phi^+, x) = \sup \{(\phi, y) : 0 \leq y \leq x\}.$$

It is easy to see that $\phi^+$ can be uniquely extended to a positive linear functional on $V$, and that $\phi^+$ is continuous with $\|\phi^+\| \leq \|\phi\|$. Clearly, $V^*$ is a lattice with $\phi \lor 0 = \phi^+$. Define $|\phi| = 2\phi^+ - \phi$. For any $y \in V$, $y \geq 0$ we can find some $a \in V$ with $0 \leq a \leq y$ and

$$(\phi, a) \leq (\phi^+, y) \leq (\phi, a) + \epsilon/2,$$

which gives

$$|(\phi, y) - (\phi, 2a - y)| < \epsilon.$$

As $-y \leq 2a - y \leq y$ so

$$|(\phi, 2a - y)| \leq \|\phi\| \|y\|$$

from which we conclude that $\|\phi\| \leq \|\phi\|$. The result now follows immediately.
Let \( V \) be an ordered positively generated Banach space and define:
\[
\Delta = C^* \cap \{ \phi \in V^* : \| \phi \| \leq 1 \}.
\]
\( \Delta \) is a weak*-compact convex set with \( 0 \in \Delta \) and there is a natural map \( \lambda : V \to A_0(\Delta) \), the space of all continuous affine functionals \( \phi \) on \( \Delta \) such that \( \phi(0) = 0 \). By our previous remarks \( \lambda \) is an order isomorphism of \( V \) into \( A_0(\Delta) \), and if \( \| \cdot \|_u \) denotes the supremum norm in \( A_0(\Delta) \), we have for all \( f \in V \)
\[
\| \lambda f \|_u \leq \| f \|.
\]
Identifying \( V \) with \( \lambda V \) we define \( S \) as the cone of functions of the form \( f_1 \vee \cdots \vee f_n \) where \( f_i \in V \). Then if \( L = S - S \), \( L \) is a vector lattice of continuous functions on \( \Delta \). See [5]. For \( \phi \in L \) define
\[
\| \phi \| = \inf \{ \| \psi \| : \psi \in V \cdot \psi \geq \phi_+, -\phi \}.
\]
Then, provided \( V \) is positively generated and has a monotone norm, \( L \) becomes a normed lattice and \( V \) is canonically embedded as a Banach subspace of \( L \).

**Lemma 2.4.** If \( V \) is a regular ordered Banach space then \( V^* \) is also regular.

Let \( \phi \in V^* \) and \( \| \phi \| \leq 1 \). We can, by the Hahn-Banach theorem, extend \( \phi \) to a functional \( \xi \in L^* \) with \( \| \xi \| \leq 1 \). As \( L^* \) is a Banach lattice so by defining \( \psi \) as the restriction to \( V \) of \( \| \xi \| \), we see that \( \psi \geq \phi, -\phi \) and \( \| \psi \| \leq 1 \).

**Lemma 2.5.** If \( V \) is an ordered Banach space such that \( V^* \) is regular then the map \( \lambda : V \to A_0(\Delta) \) is a one-one onto order isomorphism such that for all \( f \in V \)
\[
\| \lambda f \|_u \leq \| f \| \leq 2 \| \lambda f \|_u,
\]
and for all \( 0 \leq f \in V \),
\[
\| \lambda f \|_u = \| f \|.
\]
That \( \| \lambda f \|_u \leq \| f \| \) is clear. Suppose \( f \in V \) and \( |(\phi, f)| = \| f \| \) where \( \| \phi \| = 1 \). Then for any \( \varepsilon > 0 \) we find \( \psi \in V^* \) with \( \psi \geq \phi, -\phi, \| \psi \| < 1 + \varepsilon \).
\[
|\phi, f| = |((\psi + \phi)/2, f) - ((\psi - \phi)/2, f)|
\]
so that
\[
\max |((\psi + \phi)/2, f)| \geq \frac{1}{2} \| f \|.
\]
As \( (\psi + \phi)/2 \geq 0 \) and \( \| (\psi + \phi)/2 \| < 1 + \varepsilon \) so
\[
\| f \| \leq 2 \| \lambda f \|_u.
\]
If \( f \geq 0 \) then \( ((\psi + \phi)/2, f) \geq 0 \) so that
\[
|\phi, f| \leq \max |((\psi + \phi)/2, f)|,
\]
from which we see that
\[
\| f \| \leq \| \lambda f \|_u.
\]
We see that $V$ maps onto a uniformly closed subspace of $A_0(\Delta)$ which separates the points of $\Delta$, and so must conclude by Lemma 4.3 of [13] that

$$\lambda V = A_0(\Delta).$$

A partial converse to Lemma 2.4 is contained in the following

**Lemma 2.6.** Let $V$ be an ordered Banach space such that $V^*$ has a proper positive dual cone $C^*$ and is regular. Then $V$ is positively generated and for any $x, y \in V$ with $-x \leq y \leq x$ we have $\|y\| \leq \|x\|$.

For let $\psi \in V^*$ and $\|\psi\| < 1$. Let $\phi \in V^*$ satisfy $\phi \geq \psi$, $-\psi$ and $\|\phi\| < 1$. Then

$$(\psi, y) = ((\psi + \phi)/2, y) - ((\phi - \psi)/2, y) \leq ((\psi + \phi)/2, x) + ((\phi - \psi)/2, x) = (\phi, x) < \|x\|.$$  

Thus $\|y\| \leq \|x\|$. That $V$ is positively generated follows from [1] upon observing that $V^*$ is $\sigma(0)$ complete.

We continue to suppose that $V$ is an ordered Banach space and that $V^*$ is regular. Following [5], we now define a conical measure $\mu$ as a positive functional in $L^*$. Such a functional defines by restriction to $V$ a point of $C^*$ called the barycentre of $\mu$. The set $P$ of conical measures $\mu$ with $\|\mu\| \leq 1$ is compact in the $\sigma(L^*, L)$ topology. We define a closed partial ordering, $<$, on $P$ by

$$\mu < \nu \text{ if } (\mu, f) \leq (\nu, f) \text{ for all } f \in S.$$  

We say $\mu \in P$ is a representing conical measure for $x \in \Delta$ if $f(x) \leq (\mu, f)$ for all $f \in S$, and observe that every $x \in \Delta$ has at least one maximal representing conical measure.

If $x_1, \ldots, x_n \in \Delta$ and $x_1 + \cdots + x_n = x \in \Delta$ then the functional

$$f \mapsto f(x_1) + \cdots + f(x_n)$$  

is called a discrete conical measure, and is obviously a representing conical measure for $x$. We show that such conical measures are dense in $P$ in a very good sense.

**Lemma 2.7.** Suppose $\mu \in P$ is a representing conical measure for $x \in \Delta$ and that

$$f_r \in S \text{ for } r = 1, \ldots, n.$$  

Then there is a discrete representing conical measure $\nu$ for $x$ such that $\mu(f_r) = \nu(f_r)$ for $r = 1, \ldots, n$.

We can write the weak*-closed cone $C^* \subseteq V^*$ as a union of closed cones $D_1, \ldots, D_m$ such that $f_r|D_s$ are equal to the restrictions of functions in $A_0(\Delta)$. If $f \in L$ and $\geq 0$, we define

$$(\mu_{D_s}, f) = \inf \{(\mu, g) : 0 \leq g \in L \cdot g|D_s > f|D_s\}.$$  

Then $\mu_{D_s} \in P$ and $\mu \leq \mu_{D_1} + \cdots + \mu_{D_m}$. Now we choose $\mu_s \in P$ such that $0 \leq \mu_s \leq \mu_{D_s}$ and $\mu = \mu_1 + \cdots + \mu_m$. If $f \in L$ and $f|D_s \geq 0$ then

$$(f, \mu) \geq (f \wedge 0, \mu_s) \geq (f \wedge 0, \mu_{D_s}) = 0.$$
Thus $\mu_0$ is determined by the value of $f$ on $D_0$ alone. The barycentre $x_0$ of $\mu_0$ is in $D_0$. Finally for $r=1,\ldots,n$

$$(f_r, \mu) = \sum_{s=1}^{m} (f_s, \mu_s) = \sum_{s=1}^{m} (f_s(x_s)).$$

**Lemma 2.8.** Suppose $f_r \in V$ and $f_1 \lor \cdots \lor f_n = f \in S$. Then

$$\max \{ (f, \mu) : \epsilon_\mu < \mu \} = \max \left\{ \sum_{r=1}^{n} f(x_r) : x_r \in \Delta, \sum x_r = x \right\}$$

and this quantity is called $\hat{f}(x)$. The map $x \rightarrow \hat{f}(x)$ is upper semicontinuous (u.s.c.).

The set

$$\left\{ \left\{ x_r \in \Delta \right\}_{r=1}^{n} : \sum_{r=1}^{n} x_r = x \in \Delta \right\}$$

is a closed subset of $\Delta^n$ and so by the continuity of the $f$, we see that the supremum of the right-hand side is always attained. The two sides are equal by Lemma 2.7. That $\hat{f}$ is upper semicontinuous follows from the equation

$$\{ x \in \Delta : \hat{f}(x) \geq \alpha \} = \left\{ x \in \sum_{r=1}^{n} x_r : \sum_{r=1}^{n} f_r(x_r) \geq \alpha \right\}.$$

The following extension of the above lemma to semicontinuous functions will be important in our later discussion of closed ideals.

**Lemma 2.9.** Let $f_r : \Delta \rightarrow [-\infty, \infty)$ be upper semicontinuous affine functionals with $f_r(0)=0$. Then for $f = f_1 \lor \cdots \lor f_n$ and $x \in \Delta$ the supremum

$$\sup \left\{ \sum_{r=1}^{n} f_r(x_r) : \sum_{r=1}^{n} x_r = x, x_r \in \Delta \right\}$$

is always attained and is denoted $\hat{f}(x)$. $\hat{f}$ is an upper semicontinuous function with $\hat{f}(0)=0$ and $\hat{f}(\alpha x) = \alpha \hat{f}(x)$ for all $x \in \Delta$.

The proof is as for Lemma 2.8.

3. The main theorems.

**Theorem 3.1.** Let $V$ be an ordered Banach space, such that $V^*$ is regular. Then the following statements are equivalent:

(i) $V$ is regular and has the Riesz separation property;
(ii) $V^*$ is a lattice;
(iii) the map $f \mapsto \hat{f}(x)$ is linear for any $x \in \Delta$;
(iv) every $x \in \Delta$ has a unique maximal representing conical measure;
(v) the map $\mapsto \hat{f}(x)$ is linear for any $f \in S$.

As an immediate corollary we have
Theorem 3.2. Let $V$ be an ordered Banach space. Then $V^*$ is a Banach lattice if and only if $V$ is regular and has the Riesz separation property.

We now prove Theorem 3.1. The implication (i) $\rightarrow$ (ii) was shown in Lemma 2.3. The implications (ii) $\rightarrow$ (iii) $\rightarrow$ (iv) $\rightarrow$ (v) can be found in [5]. The equivalence (i) $\leftrightarrow$ (ii) is closely related to a Theorem in [1], but is more specific. We now show (v) $\rightarrow$ (i).

Suppose $f, g, h \in V$. Then $(f \lor g)^\lor$ is u.s.c. affine and so we can define $((f \lor g)^\lor \lor h)^\lor$ as in Lemma 2.9. It is easy to show that

$((f \lor g)^\lor \lor h)^\lor = (f \lor g \lor h)^\lor$

and so is u.s.c. affine.

Next suppose that $f, g$ are u.s.c. affine on $\Delta$ and $f, g \leq h$ where $h$ is affine. Then $(f \lor g)^\lor \leq h$ without any continuity assumptions for $h$.

For $f \in -F$ we define $f^\lor$ by $f^\lor = -(-f)^\lor$. Now let $f_1, f_2, g_1, g_2 \in V$ and $f_1 \lor f_2 \leq g_1 \land g_2$. Then if $f = (f_1 \lor f_2)^\lor$ and $g = (g_1 \land g_2)^\lor$ we have $f \leq g$. We construct a sequence $h_n \in A_0(\Delta) \equiv V$ such that

$f - 1/2^n < h_n < g + 1/2^n$.

Suppose $h_n$ is given. Let $z \in \Delta$ and $z = x + y$ where $x, y \in \Delta$ and

$(f \lor h_n)^\lor(z) = f(x) + h_n(y) < (h_n(z) + 1/2^n) \land (g(z) + 1/2^n)$.

Now $(f \lor h_n)^\lor$ is u.s.c. affine and so by simple convexity arguments to be found in [3], we can find $k_n \in A(\Delta)$ with

$(f \lor h_n)^\lor - 1/2^{n+1} < k_n < h_n \lor g + 1/2^{n+1},$

and also satisfying $k_n(0) < 0$.

Similarly we can find $l_n \in A(\Delta)$ with

$f \lor h_n - 1/2^{n+1} < l_n < (h_n \land g)^\lor + 1/2^{n+1},$

and also satisfying $h_n(0) > 0$.

Putting $h_{n+1} = \lambda k_n + (1 - \lambda) l_n$ for a suitable $0 < \lambda < 1$ we obtain

$(f \lor h_n) - 1/2^{n+1} < h_{n+1} < (h_n \land g) + 1/2^{n+1},$

and $h_{n+1}(0) = 0$.

Then

$\|h_n - h_{n+1}\| < 1/2^{n+1},$

so $h_n \to h \in A_0(\Delta)$. It is then clear that

$f_1 \lor f_2 \leq f \leq h \leq g \leq g_1 \land g_2,$

so that $V$ has the Riesz separation property.
We now show that $V$ is regular. Let $f \in A_0(\Delta)$ and $z \in \Delta$. Then we can find $x, y \in \Delta$ with $x + y = z$ and satisfying

$$\{f \lor (-f)\}^\lor(z) = f(x) - f(y) = f(x - y).$$

Now $-z \leq x - y \leq z$ so by the regularity of $V^*$

$$\{f \lor (-f)\}^\lor(z) \leq \|f\|.$$

Now suppose $\varepsilon > 0$. We define $h_1 \in A_0(\Delta)$ so that

$$\{f \lor (-f)\}^\lor -\varepsilon/2 < h_1 < \|f\| + \varepsilon/2.$$

Inductively we construct $h_n \in A_0(\Delta)$ with

$$\{f \lor (-f)\}^\lor -\varepsilon/2^n < h_n,$$

and more exactly with

$$\{f \lor (-f)\}^\lor v h_n - \varepsilon/2^{n+1} < h_{n+1} < h_n + \varepsilon/2^{n+1}.$$

This can be done by the same procedure as above. As before $h_n \to h \in A_0(\Delta)$ and $f \lor (-f) \leq h$. Thus $0 \leq h \in V$ and

$$\|h\| = \|h\|_u \leq \sum_{n=1}^{\infty} \|h_{n+1} - h_n\|_u + \|h_1\|_u < \|f\| + \frac{\varepsilon}{2} + \sum_{n=2}^{\infty} \frac{\varepsilon}{2^n} = \|f\| + \varepsilon.$$

Together with Lemma 2.6 this proves the regularity of $V$.

4. Some special cases. We define an $R$-space $V$ as a regular ordered Banach space with the Riesz separation property. If $V$ is an ordered Banach space we say it is of type $M$ if for any $x, y \neq 0$ we can find $z \geq x, y$ with

$$\|z\| \leq \max\{\|x\|, \|y\|\}$$

and of type $L$ if for any $x, y \geq 0$, we have

$$\|x + y\| = \|x\| + \|y\|.$$

**Lemma 4.1.** If $V$ is an $R$-space of type $L$ then it is a Banach lattice.

See [2].

**Lemma 4.2.** If $V$ is an $R$-space of type $M$ then $V^*$ is a Banach lattice of type $L$.

The proof of this is trivial.

We can now give an immediate proof of a theorem of Dixmier and Kakutani [8], [14].

**Theorem 4.3.** Let $V$ be an ordered Banach space. Then $V$ is an $L$-type Banach lattice if and only if $V^*$ is an $M$-type Banach lattice.
If $V$ is an $L$-type Banach lattice, it is an easy corollary of Lemma 2.3 that $V^*$ is an $M$-type Banach lattice. If $V^*$ is an $M$-type Banach lattice then $V$ is an $R$-space and $V^{**}$ is an $L$-type Banach lattice by Lemma 4.2. As $V$ is canonically embedded in $V^{**}$, so $V$ is of type $L$ and is an $L$-type Banach lattice.

**Theorem 4.4.** Let $V$ be an ordered Banach space. Then $V$ is an $R$-space of type $M$ if and only if $V^*$ is a Banach lattice of type $L$.

For the proof see [6]. Such spaces $V$ are referred to as simplex spaces in [11].

We show how this theory is related to Choquet boundary theory, as expounded in [3], [9] and [17]. Let $V$ be an ordered Banach space with a distinguished order unit $e$ such that for all $x \in V$ we have

$$
\|x\| = \inf \{\alpha : -\alpha e \leq x \leq \alpha e\}.
$$

Then $V$ is regular and if

$$
\Omega = \{\phi \in V^* : \phi \geq 0 \cdot \phi(e) = 1\},
$$

then $\Omega$ is a weak*-compact base for the cone $C^*$ in the sense of [10]. $V$ is isometrically and order isomorphic with $A_0(\Delta)$ and with $A(\Omega)$, the space of continuous affine functionals on $\Omega$. The conical measures can be identified with the regular Borel measures on $\Omega$. $\Omega$ is called a simplex [4] if $C^*$ is lattice-ordered, and it is shown in [9], [15] that this occurs if and only if $A(\Omega)$ has the Riesz separation property.

5. **The ideals in an R-space.** If $V$ is an ordered Banach space, an order ideal $I$ in $V$ is a subspace such that if $0 \leq x \leq y \in I$, then $x \in I$. An ideal is defined as a positively generated order ideal. We now investigate the properties of the closed ideals of an $R$-space. These generalise the results on the closed ideals of a simplex space in [11], and provide direct proofs for those theorems.

**Lemma 5.1.** Let $I$ be a closed ideal in an $R$-space $V$. Then with the restriction norm and ordering $I$ is an $R$-space.

The only part not immediate is the second half of the regularity condition. Let $x \in I$ and let $z \in I$, $z \geq x$, $-x$. Let $y \in V$ with $y \geq x$, $-x$ and $\|y\| < \|x\| + \varepsilon$. Then by the Riesz separation property there is some $\omega \in V$ with $x$, $-x \leq \omega \leq y$, $z$. As $0 \leq \omega \leq z \in I$ so $\omega \in I$ and as $0 \leq \omega \leq y$ so $\|\omega\| \leq \|y\| < \|x\| + \varepsilon$. Thus $I$ is indeed regular.

**Lemma 5.2.** Let $I, J$ be closed ideals in an $R$-space $V$. Then $I \cap J$ is a closed ideal.

We need only verify that $I \cap J$ is positively generated. Let $x \in I \cap J$ and let $y \in I$, $z \in J$ be such that $x$, $-x \leq y$ and $x$, $-x \leq z$. Then we can find $\omega \in V$ such that $x$, $-x \leq \omega \leq y$, $z$. As $0 \leq \omega \leq y$ so $\omega \in I$ and as $0 \leq \omega \leq z$ so $\omega \in J$. This proves the lemma.
Theorem 5.3. Let $I, J$ be closed ideals in an $R$-space $V$. Then $I+J$ is a closed ideal.

We first show that $I+J$ is positively generated. Let $x = i + j$ where $i \in I$ and $j \in J$. Then let $i_k \in I$ and $j_k \in J$ be such that $i_k \leq i$ and $j_k \leq j$. If $x = i_k + j_k$, then $x_k \geq 0$ and $x, -x \leq x_k$.

If in particular in the above $x \geq 0$ then by the Riesz separation property as $0 \leq x \leq i_k + j_k$ we can find $0 \leq i \leq i_k \in I$ and $0 \leq j \leq j_k \in J$ such that $x = i + j$. Thus the positive cone of $I+J$ is the sum of the positive cones of $I$ and $J$. It is now immediate that $I+J$ is an order ideal. As in Theorem 5.1 we see that it is regular.

We now have only to show that $I+J$ is a closed subspace of $V$. Suppose $\varepsilon > 0$ is given. Let $z \in I+J$ and let $z$, $-z \leq i + j$ where $0 \leq i \in I$, $0 \leq j \in J$ and $\|i+j\| < \|z\| + \varepsilon$. Now observe that $z - j$, $-i \leq i$, $z + j$ so by the Riesz separation property if we let $x \in V$ satisfy

$z - j$, $-i \leq x \leq i$, $z + j$

and put $y = z - x$ we have

$-i \leq x \leq i$ and $-j \leq y \leq j$

so that $x \in I$ and $y \in J$. We also have

$\|x\| \leq \|i\| \leq \|i + j\| < \|z\| + \varepsilon$ and $\|y\| \leq \|j\| \leq \|j + i\| < \|z\| + \varepsilon$.

Now let $z_n \in I+J$ and $\sum \|z_n\| < \infty$. Let $x_n = x_n + y_n$ where $x_n \in I$, $\|x_n\| < 2\|z_n\|$ and $y_n \in J$, $\|y_n\| < 2\|z_n\|$. Then we see that $\sum x_n \to x \in I$ and $\sum y_n \to y \in J$, so that $\sum z_n \to x + y \in I+J$. This proves that $I+J$ is closed.

Theorem 5.4. Let $I$ be a closed ideal in an $R$-space $V$ and let $V/I$ be given the quotient norm and the positive cone which is the image of the positive cone of $V$. Then $V/I$ is an $R$-space.

Let $C$ be the positive cone in $V/I$ so that $C = \pi C$ where $\pi : V \to V/I$ is the quotient map. Then $C \cap (\pi C) = \{0\}$. For let $x, y \in C$ and $\pi x = -\pi y$. Then $\pi(x+y) = 0$ so that $x + y \in I$. Then $x, y \in I$ so that $\pi x = -\pi y = 0$.

It is clear that $V/I$ is a positively generated partially ordered space, but we do not yet show that its positive cone is closed.

Suppose $0 \leq \tilde{y} \in V/I$. For some $x \in V$ we have $\pi x = \tilde{y}$ and $\|x\| < \|\tilde{y}\| + \varepsilon/2$. There is also some $z \in V$ with $z \geq 0$ and $\pi z = \tilde{y}$. As $I$ is positively generated there is some $v \in V$ such that $v \geq x, z$ and $\pi v = \tilde{y}$. Also as $V$ is regular there is some $w \in V$ such that $w \geq x, 0$ and $\|w\| < \|x\| + \varepsilon/2$. If now $y \in V$ is chosen so that $0, x \leq y \leq w, v$ then $0 \leq y \in V$, $\pi y = \tilde{y}$ and $\|y\| < \|\tilde{y}\| + \varepsilon$.

Suppose $0 \leq \tilde{y} \leq \tilde{z} \in V/I$ and

$0 \leq a \in V$ has $\pi a = \tilde{y}$,

$0 \leq b \in V$ has $\pi b = \tilde{z} - \tilde{y}$,

$0 \leq c \in V$ has $\pi c = \tilde{z}$ and $\|c\| < \|\tilde{z}\| + \varepsilon$.
Then \( \pi(a+b) = \pi c = \tilde{z} \). As \( I \) is an ideal we can find \( z \in V \) such that \( 0 \leq z \leq a+b, c \) and \( \pi z = \tilde{z} \). By the Riesz decomposition property we can find \( y, x \in V \) such that \( 0 \leq y \leq a \) and \( 0 \leq x \leq b \) and \( x+y = z \). Then \( 0 \leq \pi y \leq \tilde{y} \) and \( 0 \leq \pi x \leq \tilde{z} - \tilde{y} \) and \( \pi y + \pi x = \tilde{z} \). Thus \( \pi y = \tilde{y} \). We now have \( 0 \leq y \leq z \), \( \pi y = \tilde{y} \), \( \pi z = \tilde{z} \) and \( \| z \| \leq \| c \| < \| \tilde{z} \| + \varepsilon \). We immediately obtain
\[
\| \tilde{y} \| \leq \| y \| \leq \| z \| < \| \tilde{z} \| + \varepsilon
\]
and, as \( \varepsilon > 0 \) is arbitrary, so \( \| \tilde{y} \| \leq \| \tilde{z} \| \).

Suppose \( -\tilde{z} \leq \tilde{y} \leq \tilde{z} \in V/I \). Then \( 0 \leq \tilde{y} + \tilde{z} \leq 2\tilde{z} \). We can from the previous paragraph find \( 0 \leq u \leq v \in V \) with \( \pi u = \tilde{y} + \tilde{z} \), \( \pi v = 2\tilde{z} \) and \( \| v \| < 2\| \tilde{z} \| + \varepsilon \). Then \( -\varepsilon/2 \leq u - v/2 \leq \varepsilon/2 \) and \( \pi (u - v/2) = \tilde{y} \). Thus \( \| \tilde{y} \| \leq \| u - v/2 \| \leq \| v/2 \| < \| \tilde{z} \| + \varepsilon/2 \). Thus \( \| \tilde{y} \| \leq \| \tilde{z} \| \), and we have shown that \( V/I \) is a regular partially ordered Banach space.

We can now show that the positive cone in \( V/I \) is closed. Let \( x_n \in V/I \) where \( \sum \| x_n \| < \infty \) and \( \sum_{r=1}^{n} x_r \geq 0 \) for all \( n \). Let \( x_n = a_n - b_n \), where \( a_n \geq 0 \), \( b_n \geq 0 \) and \( \sum \| a_r \| < \infty \), \( \sum \| b_r \| < \infty \). We have \( \sum_{r=1}^{n} a_r \geq \sum_{r=1}^{n} b_r \). Now let \( 0 \leq a_n \in V \) where \( \pi a_n = \tilde{a}_n \) and \( \sum \| a_n \| < \infty \). \( \sum a_n \) converges in \( V \) to a limit \( a \geq 0 \). We now construct \( 0 \leq b_n \in V \) such that \( \sum_{r=1}^{n} b_r \geq \sum_{r=1}^{n} a_r \) and \( \| b_n \| < 2\| b_n \| \). Suppose \( b_1, \ldots, b_{n-1} \) have been so constructed. As \( b_n \geq 0 \) so we can find \( 0 \leq c_n \in V \) with \( \pi c_n = \tilde{b}_n \) and \( \| c_n \| < 2\| b_n \| \). As
\[
\tilde{b}_{n+1} = \sum_{r=1}^{n} a_r - \sum_{r=1}^{n-1} b_r,
\]
so we can find \( d_n \in V \) with \( \pi d_n = \tilde{b}_n \) and
\[
d_n \leq \sum_{r=1}^{n} a_r - \sum_{r=1}^{n-1} b_r = k_n.
\]
We can now find \( e_n \in V \) with \( e_n \leq c_n, d_n \) and \( \pi e_n = \tilde{b}_n \). As \( 0 \leq e_n \leq c_n, k_n \), so we can find \( b_n \in V \) with \( 0 \leq e_n \leq b_n \leq c_n, k_n \). Then \( 0 \leq b_n, \| b_n \| \leq \| c_n \| < 2\| b_n \|, \pi b_n = \tilde{b}_n \) and
\[
\sum_{r=1}^{n} b_r \leq \sum_{r=1}^{n} a_r \leq a.
\]
It is now clear that \( \sum_{r=1}^{n} b_r \rightarrow b \leq a \). Then \( \sum_{r=1}^{n} (a_r - b_r) \rightarrow a - b \geq 0 \). Thus \( \sum_{r=1}^{n} x_r \rightarrow x = \pi (a - b) \geq 0 \), and we have shown \( V/I \) has a closed cone.

Finally we show that \( V/I \) has the Riesz separation property. Suppose \( \tilde{a}, \tilde{b} \leq \tilde{c}, \tilde{d} \) and let \( \pi a = \tilde{a}, \pi b = \tilde{b} \). We can find \( c, d \geq a, b \) with \( \pi c = \tilde{c} \) and \( \pi d = \tilde{d} \). By the Riesz separation property for \( V \) we can find \( e \in V \) with \( a, b \leq e \leq c, d \) and if \( \tilde{e} = \pi e \) we have \( \tilde{a}, \tilde{b} \leq \tilde{e} \leq \tilde{c}, \tilde{d} \). This concludes the proof of the theorem.

Another way of investigating closed ideals, pursued for simplex spaces in [11], is to characterise their annihilators in the dual space.

Returning to the notation of §2, we define a conical face in \( \Delta \) as a closed convex set \( F \) with \( 0 \in F \) such that if \( x, y \in \Delta \) and \( \alpha x + \beta y \in \Delta \) for any \( \alpha > 0, \beta > 0 \) then \( x, y \in F \).
Theorem 5.5. Let \( V \) be an \( R \)-space, \( V^* \) its dual and \( \Delta = \{ \phi \in V^* : \phi \geq 0 \text{ and } \|\phi\| \leq 1 \} \). Then the maps \( I \to I^0 \to I^0 \cap \Delta \) determine a one-one correspondence between the set of all closed ideals \( I \) in \( V \), the set of all weak*-closed lattice ideals in \( V^* \), and the set of all conical faces in \( \Delta \). I may naturally be identified with the space of all continuous affine functionals \( f \) on \( \Delta \) with \( f|I^0 \cap \Delta = 0 \).

It is elementary to show that if \( I \) is a closed ideal then \( I^0 \) is a weak*-closed lattice ideal and \( F = I^0 \cap \Delta \) is a conical face. Now let \( F \) be a conical face. Let \( J \) be the subspace of \( V^* \) generated by \( \Delta \). Then \( J \) is obviously an ideal and it meets the unit ball \( B \) in a weak*-closed set. Specifically,

\[
J \cap B = (1 + e)(\Delta - \Delta) \cap B.
\]

By a well-known theorem on Banach spaces [18] it follows that \( J \) is a weak*-closed subspace. \( J \) is a lattice ideal with \( J \cap \Delta = F \). Let \( I_1 \) be defined as the subspace of \( V \) such that \( f \in I \) and \( \phi \in F \) implies \( (f, \phi) = 0 \). Then \( I_1 = \partial J \) is a closed subspace of \( V = \partial A^0(\Delta) \). \( I_1 \) determines and is determined by \( \Delta \), and if \( \Delta \) is derived from a closed ideal \( I \) then \( I_1 = I \).

All we need to show is that \( I_1 \) is always an ideal. It is clearly an order ideal. Suppose \( f \in I_1 \). Let \( g \) be the function \( g: \Delta \to (-\infty, \infty] \) defined by

\[
g(x) = 0 \quad \text{if } x \in F, \\
= \infty \quad \text{if } x \notin F.
\]

Then \( g \) is l.s.c. affine and \( f, -f \leq g \). If we can find \( h \in A_0(\Delta) \) with \( f, -f \leq h \leq g \), then \( h|F = 0 \), so we see that \( I \), is positively generated and so is an ideal. The theorem is completed by the following separation theorem.

Theorem 5.6. Let \( V \) be an \( R \)-space and

\[
\Delta = \{ \phi \in V^* : \phi \geq 0 \cdot \|\phi\| \leq 1 \}.
\]

If \( -f_1, \ldots, -f_n, g_1, \ldots, g_m \) are l.s.c. affine functionals on \( \Delta \) vanishing at the origin such that \( f_i \leq g_j \) for all \( i, j \), then we can find \( h \in A_0(\Delta) \) with \( f_i \leq h \leq g_j \) for all \( i, j \).

First observe that if \( k_1, \ldots, k_p \) are u.s.c. affine functionals vanishing at the origin then \( (k_1 \lor \cdots \lor k_p)^c \) defined as in Lemma 2.7 is easily shown to be u.s.c. affine. The result now follows by a simple application of the technique of forcing convergence developed in the first part of the proof of Theorem 3.1.

References

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