INJECTIVE HULLS OF C^* ALGEBRAS

BY
HARRY GONSHOR

Introduction. Our aim is to apply category concepts to the study of commutative C^* algebras. In particular we shall study injective hulls. This appears to have been overlooked in the literature, although injective Banach spaces have been studied extensively. However, in [2] the dual category of compact Hausdorff spaces is studied. It is shown that the projectives are precisely the extremely disconnected spaces and that projective covers always exist. Hence, injective hulls always exist in the original category.

Injective hulls are very important in ring theory, where they are used to obtain generalized quotient rings. They play a critical role in the proof of the Mitchell embedding theorem for abelian categories. They lead also to a novel way of obtaining the decomposition theorem for commutative noetherian rings. Recently Isbell constructed injective hulls in the category of metric spaces [5]. Injective hulls in Banach spaces are discussed in [1] and [4].

I. **The category of commutative C^* algebras.** Let Ω be the category of compact Hausdorff spaces and continuous maps and let Γ be the category of commutative C^* algebras and homomorphisms. (By a homomorphism is meant a map which preserves addition, scalar multiplication, ring multiplication, and the identity. Such a map preserves the adjoint operation and is necessarily continuous.)

Ω and Γ are dual to each other. Specifically let $A \xrightarrow{\alpha} B$ be a map in Γ. This induces the map $M(B) \xrightarrow{\bar{\alpha}} M(A)$ where $(\bar{\alpha}f)(a)=f[\alpha(a)]$. ($M(A)$ is the maximal ideal space of A or equivalently the space of homomorphisms of A into the complex numbers.) $\bar{\alpha}$ is necessarily continuous. Conversely if $M_1 \xrightarrow{\beta} M_2$ is a map in Ω, this induces a map $C(M_2) \xrightarrow{\hat{\beta}} C(M_1)$ [$C(M_1)$ is the * algebra of continuous complex valued functions on M_1] where $(\hat{\beta}f)(M_1)=f(\beta M_1)$.

We now study various properties of the categories Ω and Γ. First, as is well known, Ω is unusually well behaved for topological categories. For example, one-one is equivalent to monic and onto to epic. Furthermore, monics correspond to subspaces. Finally a map which is both monic and epic is a homeomorphism. Let α be a map in Ω and $\hat{\alpha}$ the corresponding map in Γ. Then, α is onto $\Leftrightarrow$ $\hat{\alpha}$ is one-one. Also, α is one-one $\Leftrightarrow$ $\hat{\alpha}$ is onto. It is amusing that the same theorem, namely Tietze's extension theorem, takes care of the nontrivial part of both equivalences.

It now follows by duality that Γ also has the property that one-one is equivalent to monic and onto to epic. The other properties listed above for Ω are also true

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for Γ. (Note that one-one maps in Γ are necessarily norm-preserving. This is easily seen by working with maps in the form $\hat{\alpha}$.)

II. **Essential extensions and complements.** We now obtain results analogous to those in module theory. However, some of the proofs are of necessity entirely different since we use Ω, which is a "topological" category.

Let $A \xrightarrow{\alpha} B$ be a monomorphism in Γ. Then we define α to be essential and B to be an essential extension of A if for all maps $B \xrightarrow{\delta} C$ in Γ, $\delta\alpha$ is one-one $\Rightarrow$ δ is one-one. As usual we often regard A as a subalgebra of B and α as the inclusion map.

As in module theory there are other characterizations of this property.

THEOREM 1. *Let $A \subset B$. Then the following are equivalent*:

(1) *B is an essential extension of A.*

(2) *If I is a closed nonzero ideal in B, then $I \cap A \neq 0$.*

(3) *If $B = C(X)$ and Y is a proper closed subset of X then A contains a nonzero function which vanishes on Y.*

Proof. Since closed ideals are the same as kernels, $1 \Leftrightarrow 2$ follows from the same elementary argument that is used in module theory. $2 \Leftrightarrow 3$ follows from the well-known characterization of closed ideals in $C(X)$.

Note that since every nonzero ideal contains a nonzero closed ideal the word "closed" in condition (2) may be deleted. Note also that kernels are not generally objects in Γ and thus the term "kernel" is used in the usual "pre-category" sense only.

Dually, we obtain essential covers. Let $X \xrightarrow{f} Y$ be onto. Then f is an essential cover if for all maps $Z \xrightarrow{g} X$, fg is onto $\Rightarrow$ g is onto. This condition can be expressed in the simpler form: f restricted to a proper closed subset of X is not onto, or in the terminology of [2] and [6] that f is minimal onto.

THEOREM 2. *$A \xrightarrow{i} A$ is essential where i is the identical mapping. If $A \xrightarrow{\alpha} B$ and $B \xrightarrow{\beta} C$ are essential then $A \xrightarrow{\beta\alpha} C$ is essential. If $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ is essential, then $A \xrightarrow{\alpha} B$ and $B \xrightarrow{\beta} C$ are essential.*

Proof. All proofs are as in module theory except for the proof that $A \xrightarrow{\alpha} B$ is essential. The trouble is that ideals in B are not necessarily ideals in C. Fortunately, the fact that every closed ideal in B is the restriction of a closed ideal in C resolves the problem. It might be instructive to give an alternative proof using the duals. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be minimal onto with f, g each onto. We want to prove that $Y \xrightarrow{g} Z$ is minimal onto. Suppose $I \to Y \xrightarrow{g} Z$ is onto where I is a proper closed subset of Y. Then $f^{-1}(I)$ is a proper closed subset of X and $f^{-1}(I) \to X \xrightarrow{f} Y \xrightarrow{g} Z$ is onto. This contradicts the minimality of gf.

We next prove the existence of maximal essential extensions of a subalgebra A in an algebra B. By these are meant subalgebras C such that $A \subset C \subset B$ and C is maximal in B with respect to the property of being an essential extension of A.

The standard proof for modules fails since the union of an increasing family of subalgebras is not necessarily a subalgebra.

THEOREM 3. *If $A \subset B$, then a maximal essential extension of A in B exists.*

The theorem follows trivially from the following lemma.

LEMMA. *Let A_α be an increasing family of algebras satisfying $A \subset A_\alpha \subset B$ with A essential in A_α for all α. Then A is essential in $\bigcup \bar{A}_\alpha$.*

Proof. Note first that $\bigcup \bar{A}_\alpha$ has no proper subalgebra C satisfying $A_\alpha \subset C$ for all α. We now consider the dual. Let there be epimorphisms $X \xrightarrow{T_\alpha} X_\alpha$ for all α, $X_\beta \xrightarrow{T_{\alpha\beta}} X_\alpha$ for all pairs (α, β) where $\alpha \leqq \beta$, $X \xrightarrow{U} Y$, $X_\alpha \xrightarrow{U_\alpha} Y$ for all α. Assume that all maps commute when defined and that U_α is minimal onto for all α. Finally, suppose that if $X \xrightarrow{S} W$ is onto and maps $W \xrightarrow{S_\alpha} X_\alpha$ exists such that $S_\alpha S = T_\alpha$ for all α, then S is one-one. We must show that U is minimal onto. To see that this is the dual, first regard the family of subobjects as a commuting family of monomorphisms. Then replace A, A_α, $\bigcup \bar{A}_\alpha$, C by Y, X_α, X, W. We have the following commutative diagrams:

$$\begin{tikzcd} X \arrow[r, "T_\beta"] \arrow[dr, "T_\alpha"'] & X_\beta \arrow[d, "T_{\alpha\beta}"] \\ & X_\alpha \end{tikzcd} \qquad \begin{tikzcd} X_\beta \arrow[d, "T_{\alpha\beta}"'] \arrow[dr, "U_\beta"] & \\ X_\alpha \arrow[r, "U_\alpha"'] & Y \end{tikzcd} \qquad \alpha \leqq \beta, \qquad \begin{tikzcd} X_\gamma \arrow[r, "T_{\beta\gamma}"] \arrow[dr, "T_{\alpha\gamma}"'] & X_\beta \arrow[d, "T_{\alpha\beta}"] \\ & X_\alpha \end{tikzcd} \qquad \alpha \leqq \beta \leqq \gamma$$

$$\begin{tikzcd} X \arrow[r, "T_\alpha"] \arrow[dr, "U"'] & X_\alpha \arrow[d, "U_\alpha"] \\ & Y \end{tikzcd}$$

First note that for all points a, $b \in X$ such that $a \neq b$, there is an α such that $T_\alpha(a) \neq T_\alpha(b)$. Otherwise the last condition would be contradicted by choosing W as the identification space of X with a and b identified.

Now suppose $I \to X \xrightarrow{U} Y$ is onto where I is a proper closed subset of X. Then $I \to X \xrightarrow{T_\alpha} X_\alpha \xrightarrow{U_\alpha} Y$ is onto, and since U_α is minimal onto it follows that $I \to X \xrightarrow{T_\alpha} X_\alpha$ is onto for all α. Fix $x \in X$, $x \notin I$ and choose $y \in I$. $(\exists\alpha) T_\alpha(x) \neq T_\alpha(y)$. By continuity there exists an open set U containing y such that $T_\alpha(x) \notin T_\alpha(U)$. Then $T_\beta(x) \notin T_\beta(U)$ for all $\beta \geqq \alpha$ since $T_{\alpha\beta} T_\beta = T_\alpha$. Since I is compact, a standard argument leads to a γ such that $T_\gamma(x) \notin T_\gamma(I)$ [γ is the maximum of a finite linear ordered set]. Thus $T_\gamma(I)$ is not onto. This contradicts the earlier statement that $I \to X \xrightarrow{T_\alpha} X_\alpha$ is onto.

We continue the analogy with module theory by studying complements. In this case, the complement of a subalgebra is a closed ideal and vice-versa. By definition, a complement of an object A of one type is an object B of the other type maximal with respect to the property that $A \cap B=0$: Note that if C is a subalgebra of D and J a closed ideal, the condition $C \cap J=0$ is equivalent to the condition $C \to D \to D/J$ is one-one. In the dual this corresponds to an epimorphism $X \xrightarrow{f} Y$ and a closed subset Z of X such that $Z \to X \xrightarrow{f} Y$ is onto. Thus in order to prove the existence of complements it suffices to show that a minimal Z exists for fixed f and a maximal epic f exists for fixed Z, such that $Z \to X \xrightarrow{f} Y$ is onto.

The first part is left to the reader in [2]. The proof of the second part is similar to the proof of Theorem 3. By Zorn's lemma it suffices to show the following: let there be maps $X \xrightarrow{T_\alpha} X_\alpha$ for all α, $X_\beta \xrightarrow{T_{\alpha\beta}} X_\alpha$ for all pairs (α, β) where $\alpha \leqq \beta$, $Z \xrightarrow{U} X$, and $Z \xrightarrow{U_\alpha} X_\alpha$ for all α. Suppose that all maps commute when defined and that all maps other than U are epimorphisms. Finally, suppose that if $X \xrightarrow{S} W$ is onto the maps $W \xrightarrow{S_\alpha} X_\alpha$ exist such that $S_\alpha S=T_\alpha$ for all α; then S is one-one. Then U is onto.

If U is not onto, then the image of U is a proper closed subset of X. As in the earlier proof there is a γ such that T_γ (image U) is not onto. Since this is the image of U_γ, we have a contradiction. Thus we have proved

THEOREM 4. *In an algebra, complements of subalgebras and of closed ideals always exist.*

REMARK. The same proof suffices to show that the complement of an algebra A can be chosen so as to contain a given closed ideal J such that $A \cap J=0$, and similarly for complements of closed ideals.

As in module theory the following result holds.

THEOREM 5. *Let $A \subset B$. Suppose J is a complement of A and $\bar{A}$ a complement of J. Then $\bar{A}$ is a maximal essential extension of A in B.*

Proof. Consider the natural monomorphism $A \xrightarrow{i} B \xrightarrow{\alpha} B/J$. Let K be a non-zero closed ideal in B/J. Then $\alpha^{-1}(K)$ is a closed ideal in B properly containing J, and by definition of complement, $A \cap \alpha^{-1}(K) \neq 0$. Hence (image $\alpha i) \cap K \neq 0$. Thus αi is essential. Now $A \xrightarrow{i} B \xrightarrow{\alpha} B/J$ factors as $A \xrightarrow{i_1} \bar{A} \xrightarrow{i_2} B \xrightarrow{\alpha} B/J$. Thus $\alpha i=(\alpha i_2)i_1$, where i_1 and αi_2 are monomorphisms. By Theorem 2, i_1 is essential.

Now suppose C is a subalgebra of B properly containing $\bar{A}$. By definition of complement, $J \cap C \neq 0$. Thus $J \cap C$ is a proper closed ideal of C whose intersection with A is zero. Thus A is not essential in C.

REMARK. Theorem 3 can, of course, be proved by means of Theorems 4 and 5.

III. **Injective hulls.** By dualizing the main theorem in [2] every object in Γ has an injective extension which is simultaneously essential. It also follows from [2] that the injective objects are precisely the AW^* algebras. This is fantastic from the point of view of the history of mathematics and illustrates the importance of

injectivity. In other words this is a striking example of how injective objects arise naturally in mathematics for various reasons, and only later is the connection with injectivity noticed.

As in module theory, we have the following trivial results. Injective algebras have no proper essential extensions. Essential injective extensions are unique up to isomorphism, thus the phrase "injective hull" is justified. Injective hulls are the same as maximal essential extensions and also the same as minimal injective extensions. Every essential extension is naturally contained in every injective extension. Finally, a maximal essential extension in an injective algebra is injective.

We are now ready to stop studying the analogy with module theory and to study the special nature of Γ more explicitly. First we have

THEOREM 6. *Let $A \subset B$ with B injective. Then the following are equivalent*:

(1) *B is an essential extension of A.*

(2) *For every nonzero idempotent e in B there exists $a \in A$ such that $a \neq 0$ and $ea = a$.*

Proof. Let $B = C(X)$ where X is extremely disconnected. Every proper closed subset of X is contained in a proper clopen subset of X. Hence in Theorem 1 the word "closed" in condition (3) may be replaced by "clopen." If Y is clopen and e is the characteristic function on Y' then a vanishes on Y if and only if $ea = a$. This proves the result.

REMARK. Note that the condition $ea = a$ is the same as the condition $a \in eB$. Thus in Theorem 1 we may use ideals of the form eB for condition (2).

IV. L^∞ **algebras and rings of operators in Hilbert space.** We now study important C^* algebras arising in analysis in the light of these concepts.

First, it is easily seen that every L^∞ algebra over a countably finite measure space is an AW^* algebra and hence injective. We have the following as an immediate consequence of Theorem 6.

THEOREM 7. *Let $A \subset B$ where $B = L^\infty(M)$. Then the following are equivalent*:

(1) *B is an essential extension of A.*

(2) *For every set of positive measure N in M there exists an $a \in A$ such that $a \neq 0$ and if $p \in N'$, $a(p) = 0$.*

Commutative weakly closed rings of operators on a Hilbert space are AW^* algebras. (In fact, these are the prime examples from which the concept of AW^* algebra was abstracted.) Adapting Theorem 6 to this situation and using notation more appropriate for operator theory we obtain

THEOREM 8. *Let $M \subset N$ where N is weakly closed (or more generally N is AW^*). Then the following are equivalent*:

(1) *N is an essential extension of M.*

(2) *For every nonzero projection $E \in N$, there exists an $A \in M$ such that $A \neq 0$ and such that the range of A is included in the range of E.*

These results, although trivial, illustrate how the concept of essential extension which is borrowed from algebra leads to natural "density" type properties in functional analysis.

The two systems considered are, of course, not really different. For example, every L^∞ algebra on a finite measure space is a maximal abelian subring of the algebra of operators on the Hilbert space of L^2 functions on the same space. Conversely, given an AW^* algebra on a separable Hilbert space, then the maximal ideal space can be made into a measure space in such a way that the algebra of continuous functions coincides with the algebra of L^∞ functions (to be precise, we are dealing with a natural isomorphism since L^∞ consists of classes of functions rather than functions).

V. $C(A, A^*)$ **and** $R(A, A^*)$. Let A be a normal operator on a separable Hilbert space. Let $C(A, A^*)$ be the uniformly closed ring generated by A and A^*, and let $R(A, A^*)$ be the weakly closed ring generated by A and A^*. Then the injective hull of $C(A, A^*)$ is included in $R(A, A^*)$. Note that although the injective hull is unique as an abstract algebra containing $C(A, A^*)$, it is not guaranteed to be unique as a subalgebra of $R(A, A^*)$. The rest of the paper is devoted to answering the natural questions which arise: When is the injective hull of $C(A, A^*)$ equal to $C(A, A^*)$ and when does it equal $R(A, A^*)$?

THEOREM 9. *The following are equivalent*:

(1) *The injective hull of* $C(A, A^*)$ *is* $C(A, A^*)$.

(2) $C(A, A^*)$ *is injective.*

(3) $A = \sum_{i=1}^n \lambda_i E_i$ *where the* E_i *are mutually orthogonal projections. I.e.,* A *has the form*

$$\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix},$$

where each λ_i *stands for a diagonal matrix each of whose entries are* λ_i.

(4) $C(A, A^*)$ *is weakly closed.*

Proof. $3 \Rightarrow 4$ is a standard fact in Hilbert space theory. It has already been pointed out that $4 \Rightarrow 2$. $1 \Leftrightarrow 2$ by definition of the term "injective hull." It remains to show that $2 \Rightarrow 3$. Now $C(A, A^*)$ has the spectrum of A as its maximal ideal space. Also, by (2) the space is extremely disconnected. However, by a Theorem of [2], an extremely disconnected metric space is discrete. Since the spectrum is compact it is finite and thus (3) follows by Hilbert space theory.

Before stating the next theorem, we recall a strong form of the spectral theorem which is useful for our purpose. Let A be a normal operator on a Hilbert space H. Then there exists a direct integral decomposition of H with respect to a measure

on the spectrum of A which is positive on open sets such that A corresponds to the decomposable operator which is multiplication by λ at the point λ. Furthermore, $R(A, A^*)$ corresponds to the class of all scalar operator valued functions and $C(A, A^*)$ to the subclass of continuous functions. (Note that only L^∞ functions correspond to operators.)

The above form is convenient since it is spatial and in particular "respects" the embedding of $C(A, A^*)$ into $R(A, A^*)$.

We now state our main theorem.

THEOREM 10. *The following are equivalent*:

(1) *The injective hull of* $C(A, A^*)$ *is* $R(A, A^*)$.

(2) *A has the form*

$$\begin{pmatrix} \lambda & & & & \\ & \cdot & & & \\ & & \lambda_n & & \\ & & & \cdot & \\ & & & & \cdot \end{pmatrix},$$

where the set $\{\lambda_i\}$ *is discrete.* (*Repetitions are allowed.*)

The proof will proceed by means of several lemmas. Note first that the spectral theorem reduces the problem to a study of ordinary complex valued functions on a compact set K in the plane by identifying scalar operator valued functions with ordinary functions. Specifically, we have the following problem:

Let K be a compact set in the plane with a measure which is positive on nonempty open sets. When is the algebra of L^∞ functions an essential extension of the algebra of continuous functions? By Theorem 7, the condition can be expressed in the language of classical analysis as follows: for every set of positive measure there is a nonzero continuous function which vanishes outside the set except possibly for a set of measure zero.

The bulk of the work will consist of obtaining consequences of the above condition which we shall call P. We begin with a preliminary lemma.

LEMMA 1. *If U is dense open in K and A is a subset of U with measure zero, then $U-A$ is dense.*

Proof. Let $x \in K$ and suppose $\chi \in V$ where V is open. Then $V \cap U$ is a nonempty open subset of U. Hence $V \cap U$ has positive measure. Thus $V \cap U \cap A' \neq \varnothing$, i.e. $V \cap (U-A) \neq \varnothing$.

LEMMA 2. *$P \Rightarrow$ the complement of a dense open set has measure zero.*

Proof. Let U be dense open and suppose that U' has positive measure. $P \Rightarrow$ there exists a nonzero continuous function which vanishes on U except for a set of measure zero, i.e. it vanishes on a set of the form $U-A$ where A has measure zero. By Lemma 1 this is impossible.

LEMMA 3. *$P \Rightarrow$ the complement of a dense set has measure zero.*

Proof. There is no loss in generality by assuming that the set is countable. Let the set $B=\{x_1 \cdots x_n \cdots\}$. Let S_n be a sphere with center x_n such that the measure of $S_n-(x_n)$ is less than $\varepsilon/2n$. This is always possible since for any point x, the collection $\{S_m-(x)\}_{m=1}^{\infty}$, where S_m is the sphere of radius $1/m$, has empty intersection. By Lemma 2, the complement of $\bigcup S_n$ has measure zero. Hence the complement of $\{x_n\}$ has measure less than ε. Since ε is arbitrary, the result follows.

Thus $P \Rightarrow$ the measure is concentrated on a countable set. Hence the measure is discrete. Again, by Lemma 3, every point of positive measure is isolated. Hence, finally, K is the closure of a countable discrete set on which the measure is concentrated.

The converse is comparatively trivial. If $\{x_1 \cdots x_n \cdots\}$ is a discrete set and K its closure, then each of the x's is isolated in K. Every set of positive measure contains one of the x's by hypothesis. On the other hand, since the x's are isolated, for every x the function which is 1 on x_n and 0 otherwise is continuous. This proves property P.

Theorem 10 follows by translating back to the Hilbert space.

We see from Theorem 10 that there are many rings of the form $C(A, A^*)$ whose injective hulls differ from $R(A, A^*)$. These injective hulls are examples of AW^* algebras which are not weakly closed.

VI. **Epilogue.** Although admittedly not very much has been done, at least the relevance of categorical ideas in functional analysis has been demonstrated.

There exists a pair of algebras $A \subset B$ with B injective such that the injective hull of A in B is not unique. In some cases the injective hulls may be characterized in a nice way which ties in with the concept of a Banach-Mazur functional.

Minimal algebras with a given injective hull have been studied. The duals are precisely one point compactifications of discrete spaces.

Finally, we feel that the ideas in (3) are worth exploiting.

We hope to discuss these ideas further in future papers.

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RUTGERS, THE STATE UNIVERSITY,
NEW BRUNSWICK, NEW JERSEY