CONVERGENT HIGHER DERIVATIONS
ON LOCAL RINGS

BY

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I. Introduction. In this paper we define a quasi-local ring \( R \), or \( (R, M) \), to be a commutative ring with unity having a unique maximal ideal \( M \) such that \( \bigcap_{n=1}^{\infty} M^n = \{0\} \). Thus a Noetherian quasi-local ring is a local ring. A higher derivation \( D = \{D_i\}_{i=1}^{\infty} \) on a quasi-local ring \( R \) is said to be convergent if, for all \( a \) in \( R \), \( \sum_{i=0}^{\infty} D_i(a) \) is a convergent series in the \( M \)-adic topology. \( D_0 \) always denotes the identity mapping. If \( R \) is complete the mapping \( \alpha_D : a \mapsto \sum_{i=0}^{\infty} D_i(a) \) is an endomorphism of \( R \) which induces the identity mapping on the residue field of \( R \) (Lemma 1). With suitable restrictions on \( D \), \( \alpha_D \) is an automorphism and hence an inertial automorphism. A seemingly “natural” additional condition sufficient to insure that \( \alpha_D \) is an automorphism is the condition

\[
D_i(M) \subseteq M^2, \quad i \geq 1.
\]

A convergent higher derivation which satisfies (1) is said to be \( M \)-convergent.

In a number of recent papers [4], [5], [7], Neggers, Wishart, and the author have used convergent higher derivations to study the inertial automorphisms of particular kinds of complete local rings. In particular Neggers [5] used higher derivations to relate properties of the higher ramification groups of a ramified \( n \)-ring to its derivation structure. The author has shown [4, Theorem 3.1] that if \( R \) is an unramified \( n \)-dimensional complete regular local ring then every inertial automorphism of \( R \) is of the form \( \alpha_D \) where \( D = \{D_{i_1, \ldots, i_n}\} \) is a convergent higher derivation on \( n \)-indices. By defining \( H_m \) to be \( \sum_{i_1 + \ldots + i_n = m} D_{i_1, \ldots, i_n} \) one obtains a higher derivation \( H \) on one index such that \( \alpha_H = \alpha_D \), and \( H \) is, in fact, what is called “strongly convergent” in this paper (Definition 3). The representation of inertial automorphisms by higher derivations provides a convenient means for determining the factor groups of the higher ramification groups of \( R \) in this case [4, Theorems 2.1, 2.2, 2.3].

This paper is primarily concerned with convergent higher derivations as such. A bit of calculation with the possibility of defining a composition of higher derivations so that the condition \( \alpha_{D \circ D'} = \alpha_D \alpha_{D'} \) obtains leads to Definition 2. Theorem 1 asserts that the set of all higher derivations \( H(R, R) \) on any (noncommutative) ring \( R \) is a group with respect to this composition. §II is concerned with closure properties of various convergent subsets of \( H(R, R) \) with respect to both the group

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operation and taking inverses, all in the case in which $R$ is quasi-local. Theorem 2 states that the convergent higher derivations form a subsemigroup $H_c(R, R)$ of $H(R, R)$ and Theorem 3 states that the subsets $H_M^c(R, R)$, $H_M^u(R, R)$, and $H_M^s(R, R)$ of $M$-convergent, uniformly $M$-convergent and strongly $M$-convergent higher derivations (see Definition 3) form subgroups of $H(R, R)$. An example following the proof of Theorem 2 illustrates the fact that $D$ may be convergent and $\alpha_D$ may be an automorphism whereas $D^{-1}$ is not convergent.

It is readily seen that if $D$ is $M$-convergent then $\alpha_D$ is in $H_1$, the subgroup of those inertial automorphisms $\alpha$ satisfying the condition $\alpha(a) = a + M^2$ for all $a$ in $M$. Conversely, if $\alpha = \alpha_D$ and $\alpha$ is in $H_1$ then $D$ is $M$-convergent. If $R$ is a $v$-ring (unramified) every inertial automorphism is in $H_1$. If $R$ is an unramified complete regular local ring then the mapping $\Delta: D \to \alpha_D$ is a homomorphism of $H^M_c(R, R)$ onto $H_1$. As a matter of fact $\Delta$ restricted to $H^M(R, R)$ still maps onto $H_1$. It follows from work of Wishart [7, pp. 50, 51] that a ramified ring may have inertial automorphisms represented by $D$ in $H^M_c(R, R)$ but not by $D$ in $H^M(R, R)$.

§II deals with the question of necessary and sufficient conditions on a complete local ring $R$ that every convergent higher derivation be uniformly convergent. Theorem 5 asserts that if the residue field $k$ has characteristic $p$, the condition that $k$ have a finite $p$-basis is sufficient and if $R$ is regular this condition is necessary. If $R$ is regular and $k$ has characteristic zero ($R$ is a power series ring over $k$) then every convergent higher derivation is uniformly convergent if and only if $k$ has finite transcendency degree over its prime field.

II. Closure properties. Initially we assume $S$ to be an arbitrary associative ring and $R$ an over ring of $S$.

**Definition 1.** A higher derivation $D$ of $S$ into $R$ is a set $\{D_i\}_{i=1}^{\infty}$ of mappings of $S$ into $R$ such that for all $i \geq 1$ and all $a, b$ in $S$,

(i) $D_i(a + b) = D_i(a) + D_i(b),$

(ii) $D_i(ab) = \sum_{j=0}^{i} D_j(a)D_{i-j}(b),$

where $D_0$ denotes the identity mapping. The symbol $H(S, R)$ will designate the set of all higher derivations of $S$ into $R$, and $Q$ will represent the higher derivation $\{Q_i\}$ such that $Q_i$ is the zero mapping for all $i \geq 1$.

**Definition 2.** If $H$ and $D$ are in $H(S, S)$ then $K = H \circ D$ is the set of mappings $\{K_i\}_{i=1}^{\infty}$ where

\[
K_i = \sum_{j=0}^{i} H_j D_{i-j}.
\]

**Proposition 1.** The set of mappings $K$ as defined by (2) is a higher derivation.

**Proof.** Proposition 1 and Theorem 1, below, follow immediately from the following fact first observed by Schmidt [6]. Let $G$ represent the group of all automorphisms $\alpha$ on the power series ring $R[[X]]$ satisfying the conditions

(i) $\alpha(X) = X$ and (ii) $\eta \alpha(a) = a$ for $a$ in $R$ where $\eta(\sum a_i X^i) = a_0$. Given $\alpha \in G$,
$D^a = \{D^a_i\}$ is in $H(R, R)$ where $D^a_i(a)$ is the coefficient of $X^i$ in $a(a)$. The mapping $\alpha \mapsto D_\alpha$ is a one-to-one correspondence between $G$ and $H(R, R)$ which then induces a group structure on $H(R, R)$, the induced operation being (2). Thus, we have

**Theorem 1.** Given any ring $R$, $H(R, R)$ is a group with respect to the composition (2).

For later use we exhibit below an explicit description of $D^{-1}$ in terms of $D$. Let $(r, n)$ be a partition of the integer $n$ into $r$ nonnegative summands. If $(r, n) = i_1, \ldots, i_r$ we let $[D]_{i_1, \ldots, i_r}$ be the sum of the formally distinct products of the $r$ maps $D_{i_1}, \ldots, D_{i_r}$. Thus, if $(3, 5) = \{1, 2, 2\}$ then $[D]_{(3, 5)}$ is $D_1 D_3^2 + D_2 D_3 D_2 + D_2^2 D_1$.

Given $D$ in $H(R, R)$ we define $\overline{D}$ by

$$\overline{D}_n = \sum_{(r, n)} (-1)^r[D]_{i_1, \ldots, i_r}, \quad n \geq 1,$$

and contend that $\overline{D} = D^{-1}$.

The expression $\sum_{i_1, \ldots, i_r} D_{i_1} \overline{D}_{i_2 - i_1}$ is a sum of terms of the form $D_{i_1} \cdots D_{i_{r+1}}$ each such terms occurring twice in $D_{i_1} \overline{D}_{i_2 - i_1}$ with coefficient $(-1)^{r'}$ and in $D_{i_1} \overline{D}_{r+1}$ with coefficient $(-1)^{r+1}$. Hence $D \circ \overline{D} = Q$. But this equality uniquely determines the set of maps $\overline{D}$ and thus $\overline{D} = D^{-1}$.

**Lemma 1.** Let $(R, M)$ be a quasi-local ring and let $S$ be a subring of $R$ with the property that every nonunit of $S$ is in $M$. If $D$ in $H(S, R)$ converges then $D \circ (S) \subseteq M$ for $i > 0$.

**Proof.** Let $u$ be a unit in $S$ such that $D_i(u)$ is a unit for some $i > 0$ and let $n$ be the least such integer. Since

$$0 = D_n(1) = D_n(uu^{-1}) = uD_n(u^{-1}) + u^{-1}D_n(u) + \sum_{i=1}^{n-1} D_i(u)D_{n-i}(u^{-1}),$$

it follows that $D_n(u^{-1})$ is also a unit. Since $D$ converges there is a largest integer, say $s$, such that $D_s(u)$ is a unit, and a largest integer $t$ such that $D_t(u^{-1})$ is a unit. Now $0 = D_{s+t}(1) = D_{s+t}(uu^{-1})$ and $D_{s+t}(uu^{-1}) \equiv D_s(u)D_t(u^{-1}) \pmod{M}$, which yields a contradiction. Thus $D_t(u)$ is in $M$ for all units $u$. Next, let $a$ be in $S \cap M$. Then $D_t(1 + a) = D_t(1) + D_t(a)$ is in $M$ and thus $D_t(a)$ is in $M$. This proves Lemma 1.

**Theorem 2.** If $R$ is a quasi-local ring the set $H_c(R, R)$ of convergent higher derivations on $R$ is a subsemigroup of $H(R, R)$.

**Proof.** Let $D$ and $H$ be in $H_c(R, R)$. Given $a$ in $R$ and a positive integer $n$, there is an integer $m$ such that if $i \geq m$ then $H_i(a)$ is in $M^n$ and there exists an integer $t$ such that if $i \geq t$ then $H_t(a)$ is in $M^n$ for $j = 0, 1, \ldots, m - 1$. It is readily seen from (ii) of Definition 1 and from Lemma 1 that $D_i(M^j) \subseteq M^j$ for all $i > 0$, and $t > 0$. Thus, if $s$ is the maximum of $2m$ and $2t$ and if $j > s$ then $\sum_{i=0}^{j-1} D_i H_{i-j}(a)$ is in $M^j$. Thus $D \circ H$ is in $H_c(R, R)$. 
A simple example illustrates the fact that a convergent higher derivation need not have a convergent inverse. Let \( k \) be any field and let \( k[[X]] \) be the power series ring in the indeterminate \( X \) over \( k \). We define \( D \in H_k(k[[X]], k[[X]]) \) by the conditions

(i) \( D_j(a) = 0 \) for \( a \in k \) and all \( j > 0 \);
(ii) \( D_1(X) = X, D_2(X) = 0 \) for \( i \geq 2 \).

These conditions determine an obviously unique higher derivation by [2, Theorem 2] and Proposition 2 which appears later in this paper. We note that:

\[
D_n^{-1}(X) = \sum_{(r,n)} (-1)^r[D]_{(r,n)}(X) = (-1)^n D_1(X) = (-1)^n X.
\]

Since this is true for any \( n > 0 \) it follows that \( D^{-1} \) does not converge. Note, however, that \( a_D \) is an automorphism. As this example suggests a sufficient condition for \( D \in H_c(R, R) \) to have a convergent inverse is that \( D(M) \subseteq M^2 \), by which is meant \( D_i(M) \subseteq M^2 \) for all \( i > 0 \). We shall see (Lemma 5) that this condition is fulfilled if \( R \) is a \( v \)-ring, a one-dimensional complete regular local ring having characteristic zero with residue field having characteristic \( p \neq 0 \).

**Definition 3.** Let \( (R, M) \) be a quasi-local ring and let \( S \) be a subring. \( D \) in \( H_S(S, R) \) is said to be

(a) \( M \)-convergent if \( D(S \cap M) \subseteq M^2 \);
(b) uniformly \( M \) convergent if \( D \) is \( M \)-convergent and converges uniformly;
(c) strongly convergent if \( D_i(S) \subseteq M^i \) for \( i = 1, 2, \ldots \). Strong \( M \)-convergence is defined as in (b).

The symbols \( H_a(S, R) \) and \( H_c(S, R) \) will represent the subsets of \( H(S, R) \) consisting of the uniformly convergent \( D \) and the strongly convergent \( D \) respectively. A superscript \( M \) indicates \( M \) convergence i.e. \( H^M_a(S, R) \) is the set of all uniformly \( M \) convergent \( D \) in \( H(S, R) \).

**Theorem 3.** Let \( R \) be a quasi-local ring. \( H^M_a(R, R) \), \( H^M_c(R, R) \) and \( H^M_t(R, R) \) are all subgroups of \( H(S, R) \). \( H_a(R, R) \) is a subsemigroup of \( H_c(R, R) \).

**Proof.** Obviously the product of \( M \)-convergent higher derivations is \( M \)-convergent. We note that if \( D \) and \( H \) of the proof of Theorem 2 are in \( H_a(R, R) \) then the proof is independent of the choice of \( a \) and hence \( D \circ H \) is in \( H_a(R, R) \).

If \( D \) is in \( H^M_t(R, R) \) then

\[
D_i(M^i) \subseteq M^{i+1}, \quad i \geq 1, \quad u \geq 0.
\]

Relation (4) implies closure in \( H^M_t(R, R) \) and also leads immediately to the conclusion that if \( (r, i) \) is any partition of \( i+1 \) and \( D \in H^M_t(R, R) \) then \( [D]_{(r,0)}(M^i) \subseteq M^{i+1} \). Thus if \( D \) is in \( H^M_t(R, R) \) so is \( D^{-1} \). The example following Theorem 2 is a strongly convergent higher derivation. If \( D \) represents the higher derivation in question and \( H = D \circ D \) then \( H_a(X) = X \), illustrating the fact that \( H_a(R, R) \) is neither closed with respect to product nor with respect to taking inverse.
In order to verify that the inverse of \( D \) in \( H_c^m(R, R) \) is in \( H_c^m(R, R) \) it is sufficient to show that, given \( a \) in \( R \) and \( m \geq 0 \), there is an integer \( n \) such that if \( i_1, \ldots, i_r \) is any partition into positive integers of \( t > n \) then,

\[
D_{i_1} \cdots D_{i_r}(a) \in M^m.
\]

Since for \( D \) in \( H_c^m(R, R) \)

\[
D_i(M^j) \subseteq M^{j+1}, \quad i > 0, \quad j \geq 0
\]

it follows that (5) holds if \( r \leq m \). There is an integer \( n_1 \) such that if \( i > n_1 \) then \( D_i(a) \in M^m \) and an integer \( n_2 \) such that if \( i_2 > n_2 \) then \( D_{i_2}D_{i_1}(a) \in M^m \) for \( i_1 = 1, 2, \ldots, n_1 \).

Iteratively, we define integers \( n_1, n_2, \ldots, n_{m-1} \) such that, if \( 0 < j < m \) and \( i_j > n_j \), then \( D_{i_j}D_{i_{j-1}} \cdots D_{i_1}(a) \in M^m \) if \( 0 < i_t \leq n_t \) for \( t = 1, \ldots, j - 1 \). Let \( n' \) be the maximum of \( n_1, n_2, \ldots, n_{m-1} \) and let \( n = m(n' + 1) \). If \( j_1, \ldots, j_r \) are positive integers such that \( j_1 + \cdots + j_r > n \) then either \( r \leq m \) or \( j_1 > n' \) for some \( i \). In either case \( D_{j_1} \cdots D_{j_r}(a) \in M^m \). It follows then from (3) that if \( D \) is \( M \) convergent so is \( D^{-1} \). If \( D \) is in \( H_c^m(R, R) \) the above argument again applies independently of the choice of \( a \). We conclude that \( D^{-1} \) is in \( H_c^m(R, R) \) if \( D \) is in \( H_c^m(R, R) \).

III. Uniformly convergent higher derivations. We begin with some basic facts about extensions of higher derivations and their convergence properties. Let \( T \) be a commutative overring of a ring \( S \) and let \( a \in S \) be invertible in \( S \). Then if \( D \in H(S, T) \)

\[
D_n(a^{-1}) = \sum_{(r, n)} (-1)^{r+1} a^{-(r+1)}C(r, n)[D(a)]_{r, n},
\]

where \( C(r, n) = r!/n_1! \cdots n_t! \) and \( n_1, \ldots, n_t \) represent the number of times the distinct integers of \( (r, n) \) occur in \( (r, n) \). Also if \( (r, n) = j_1, \ldots, j_r \), then \([D(a)]_{r, n}\) is the sum of all the formally distinct products of the \( r \) quantities \( D_{j_1}(a), \ldots, D_{j_r}(a) \).

For \( n = 1 \) we have \( D_1(a^{-1}) = -a^{-1}D_1(a) \). Proceeding by induction, \( 0 = D_2(aa^{-1}) = \sum D_2(a)D_{n, 2}(a^{-1}) \) or \( D_n(a^{-1}) = -a^{-1} \sum_{i=0}^{n-1} D_{n, i}(a)D_i(a^{-1}) \). Substitution of (6, i) in the right hand side of this equality for \( i = 1, \ldots, n - 1 \) yields (6, n) without difficulty. Let \( T \) and \( S \) be as above and let \( D \in H(S, T) \). The mapping \( \tau_D : S \to T[[X]] \) given by (7) is an isomorphism with the property \( \eta_D \) is the identity on \( S \) where again \( \eta(\sum a_iX^i) = a_0 \). Conversely, if \( \tau : S \to T[[X]] \) is a homomorphism such that \( \eta \tau \) is the identity on \( S \) then \( \tau(a) = a + \sum X^iD_i(a) \) and \( D' = \{D_i \} \) is in \( H(S, T) \). As in the proof of Theorem 1, \( D \rightarrow \tau_D \) is a one-to-one correspondence between \( H(S, T) \) and the set of isomorphisms \( \tau \) of \( S \) into \( T[[X]] \) such that \( \eta \tau \) is the identity map on \( S \).

Let \( M \) be a multiplicatively closed subset of \( S \) each element of which has an inverse in \( T \). Thus \( S_M \) the ring of quotients with respect to \( M \) is a subring of \( T \).

**Lemma 2.** Each \( D \) in \( H(S, T) \) has a unique extension to \( H(S_M, T) \).
Proof. The lemma follows from the existence and uniqueness of the extension of \( \tau_D \) to \( S_M \).

**Lemma 3.** Let \( S \) be a subring of the quasi-local ring \((R, M)\) and let \( B \) be a subset of \( R \). Let \( D \) be in \( H(S[B], R) \).

(i) If \( D \) converges on \( S \) and on \( B \) then \( D \in H_c(S[B], R) \).
(ii) If \( D \) is uniformly convergent on \( S \) and on \( B \) and \( D(S[B]) \subseteq M^n \) then \( D \in H_u(S[B], R) \).
(iii) If \( D \) is strongly convergent on \( S \) and on \( B \) then \( D \in H_t(S[B], R) \).
(iv) If \( D(S \cap M) \subseteq M^2 \) and \( D(B \cap M) \subseteq M^2 \) then \( D(S[B] \cap M) \subseteq M^2 \).

Proof. Each element in \( S[B] \) is a sum of terms of the form \( sb_1 \cdots b_t \) where \( s \in S; b_1, \ldots, b_t \in B \) and \( t \geq 0 \). Now

\[
D(s, b_1, \ldots, b_t) = \sum_{i_0 + \cdots + i_t = n} D_{i_0}(s) D_{i_1}(b_1) \cdots D_{i_t}(b_t).
\]

Clearly, if \( D \) converges at \( s, b_1, \ldots, b_t \) then \( D \) converges at \( sb_1 \cdots b_t \).

Statement (ii) is a consequence of the following lemma which will be useful elsewhere.

**Lemma 4.** Let \( S \) be a subring of a quasi-local ring \((R, M)\) and let \( B \) be a subset of \( S \). If \( D \in H(S, R) \) converges uniformly on \( B \) and \( D(B) \subseteq M \) then given \( \varepsilon > 0 \), there is an \( m > 0 \) such that given any product \( b_1 \cdots b_t \) of \( t \geq 1 \) elements in \( B \), \( D_{i_1}(b_1) \cdots D_{i_t}(b_t) \in M^m \) whenever \( i_1 + \cdots + i_t > m \).

Proof. There is an integer \( r \) such that if \( i > r \), then \( D_i(B) \subseteq M^n \). Let \( m = nr \). Then, if \( i_1 + \cdots + i_t > m \) either \( n \) of the \( i \)'s are different from zero or one of them is greater than \( r \). In either case \( D_{i_1}(b_1) \cdots D_{i_t}(b_t) \) is in \( M^n \).

To prove (iii) of Lemma 3 we simply observe that if \( D(a) \in M^t \) for \( a \) in \( S \) or in \( B \) then (8) is in \( M^n \). Statement (iv) is immediate.

**Corollary 3.1.** If \( D \in H_c(S[B], R) \) converges uniformly on \( S \), where \( B \) is a finite set and \( D(S[B]) \subseteq M^n \), then \( D \in H_u(S[B], R) \).

**Corollary 3.2.** Let \( M \) be a multiplicatively closed subset of \( S \) each element of which has an inverse in \( R \) and let \( \tilde{D} \in H(S_M, R) \) be the extension of \( D \in H(S, R) \). If \( D(S) \subseteq M \), it follows that

(i) if \( D \in H_c(S, R) \) then \( \tilde{D} \in H_c(S_M, R) \);
(ii) if \( D \in H_u(S, R) \) then \( \tilde{D} \in H_u(S_M, R) \);
(iii) if \( D \in H_t(S, R) \) then \( \tilde{D} \in H_t(S_M, R) \).

Proof. Let \( M^{-1} \) denote the set of inverses of the elements of \( M \). Then \( \tilde{D}(M^{-1}) \subseteq M \) in view of (6) and the assumption that \( D(S) \subseteq M \). Also, it follows from Lemma 4 and (6) that if \( D \in H_c(S, R) \) then \( \tilde{D} \) converges on \( M^{-1} \), and if \( \tilde{D} \in H_u(S, R) \) then \( \tilde{D} \) converges uniformly on \( M^{-1} \). If \( D \in H_t(S, R) \) it is apparent from (6) that \( D \) is strongly convergent on \( M^{-1} \). The observation that \( S_M = S[M^{-1}] \) and an appeal to Lemma 3 completes the proof.
The symbol $V$ will represent a valuation ring having characteristic zero with residue field $k$ of characteristic $p \neq 0$. Let $\pi$ be a prime element of $V$ and let $e$ be the ramification of $V$, that is $pV = \pi^eV$, and we write $e = p^r$ where $(p, r) = 1$. Let $(R, M)$ be a regular local ring containing $V$ such that $\pi V = V \cap M$.

**Lemma 5.** Each $D$ in $H_c(V, R)$ has the property $D(\pi V) \subset M^2$ and thus $H_c(V, R) = H^2_c(V, R)$.

**Proof.** For some positive integer $t$, $\pi V \subset M^t \setminus M^{t+1}$, i.e. $\pi V \subset M^t$ but $\pi V \notin M^{t+1}$. Thus $\pi \in M^t \setminus M^{t+1}$. Let $i$ be the least integer such that $D_i(\pi) \notin M^2$. We assume $t > 1$. Now

$$D_{tr}(\pi^i) = [D_t(\pi)]^i + \sum_{i_1 + \cdots + i_r = tr, \text{ some } i_j < i} D_{i_1}(\pi) \cdots D_{i_r}(\pi).$$

Since $[D_t(\pi)]^i \in M^t \setminus M^{t+1}$ and the second term is seen to be in $M^{t+1}$ we have $D_{tr}(\pi^i) \in M^t \setminus M^{t+1}$. Similarly,

$$(9) \quad D_{tr}(\pi^{p^r}) = [D_{tr}(\pi^i)]^{p^r} + \sum_{i_1 + \cdots + i_{ps} = p^tr, \text{ some } j, k} D_{i_1}(\pi^i) \cdots D_{i_{ps}}(\pi^i).$$

Again $[D_{tr}(\pi^j)]^{p^r} \in M^{p^r \setminus M^{p^r+1}}$ and the remaining term on the right of (9) is seen to be in $M^{p^r+1}$ since each summand occurs a multiple of $p$ times. We conclude from (9) that

$$D_{tr}(\pi^{p^r}) \in M^{p^r \setminus M^{p^r+1}}.$$

For some unit $u$ in $V_p = un^{p^r}$ and

$$(10) \quad 0 = D_{tr}(p) = uD_{tr}(\pi^{p^r}) + \sum_{j=1}^{p^r-1} D_j(u)D_{tr-j}(\pi^{p^r}).$$

By an argument like that above applied to the right side of (10) we conclude that $D_{tr}(p) \in M^{p^r \setminus M^{p^r+1}}$ which is the desired contradiction.

If $t = 1$ then we observe as above that $D_1(\pi^i) \in M^1 \setminus M^{2}$ and hence that $D_{tr}(\pi^{p^r}) \in M^{p^r \setminus M^{p^r+1}}$. It follows that $D_{tr}(p) = D_{tr}(un^{p^r}) \notin M^{p^r+1}$; a contradiction. This proves Lemma 5.

**Lemma 6.** If $D$ is in $H(V, R)$ and $a$ is in $V$ then $D_i(a^n) \subset M^i$ for $i < p^n - 1$.

**Proof.** We note that

$$D_i(a^n) = \sum_{i_1 + \cdots + i_{qs} = i} D_{i_1}(a) \cdots D_{i_{qs}}(a)$$

$$(11) \quad = C[p^n; q_1, \ldots, q_t][D_{i_1}(a)]^{q_1}, \ldots, [D_{i_r}(a)]^{q_t}$$

where the set $i_1, \ldots, i_{qs}$ consists of $q_r$ integers $j_r$ for $r = 1, \ldots, t$ and $C[p^n; q_1, \ldots, q_t]$ is the indicated multinomial coefficient. Since $i < p^n - 1$, and hence $q_r < p^n - 1$ for at least one $q_r$, it follows that the maximum integer $t$ such that $p^t|\pi^n$ for all $q_r$ is less than $n - j$. Thus $p^t|C[p^n; q_1, \ldots, q_t]$. (Here we are using the fact that if $s$ is the largest integer such that $p^s|\pi^n$ for all $r$ then $p^n - s|C[p^n; q_1, \ldots, q_t]$.) It follows from (11) that $D_i(a^n) \subset M^i$. 

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We now make an additional assumption on \( V \) and \( R \), namely that \( R \) is complete in the \( M \)-adic topology and \( V \) is a complete subring with \( e = 1 \).

**Theorem 4.** Let \( S \) be a \( p \)-basis for \( k \) the residue field of \( V \) and let \( S \subset V \) be a set of representatives of the elements in \( S \). If \( f \) is a mapping of \( S \times I \) into \( R \) where \( I \) denotes the positive integers then

(a) There is one and only one \( D \in H(V, R) \) with the property \( D(s) = f(s, i) \) for (\( s, i \)) in \( S \times I \).

(b) \( D \) is in (i) \( H^m(V, R) \), (ii) \( H^N(V, R) \), (iii) \( H^N(V, R) \) if and only if \( D(S) \subset M \) and (i) \( D \) converges on \( S \), (ii) \( D \) converges uniformly on \( S \), (iii) \( D_i(S) \subset M^i \) for \( i = 1, 2, \ldots \).

**Proof.** In order to prove part (a) we consider \( V_0 \) the complete subring of \( V \) having residue field \( k_0 \), the maximal perfect subfield of \( k \). Since \( S \) is an algebraically independent set over \( k_0 \), \( S \) is algebraically independent over \( V_0 \). Thus by a standard Zorn's Lemma argument using [2, Theorem 2] we can define \( H \in H(V_0(S), R) \) by the conditions (i) \( H \) restricted to \( V_0 \) is the zero higher derivation and (ii) \( H_i(s) = f(s, i) \) for \( s \in S \) and \( i \in I \).

Let \( U \) be a basis for \( k \) as a linear space over \( k_0(S) \) and let \( U \) be a set of representatives in \( V \) of the elements in \( U \). We assume that \( 1 \) is in \( U \).

The set \( U^{p^n} \) of \( p^n \)-th powers of elements of \( U \) is also a basis for \( k \) over \( k_0(S) \) [3, p. 347]. If \( V_0(S) \) is the ring of rational functions over \( V_0 \) in the elements of \( S \) then \( V_1 = V_0(S) \cap V \) is a valuation ring with residue field \( k_0(S) \). Thus, given \( a \in V \), there are elements \( a_1, \ldots, a_n \) in \( V_1 \) and \( u_1, \ldots, u_n \) in \( U \) such that

\[
a = \sum a_i u_i^{p^n}, \quad \text{mod } p^n.
\]

Moreover, the \( a_i \) are uniquely determined, \( \text{mod } p^n \), by the condition (12).

For \( i = 1, \ldots, m \) and \( a \in V \), let

\[
D^{(m)}(a + p^n V) = \sum H_i(a) u_i^{p^n} + M^m,
\]

where \( a = \sum a_i u_i^{p^n} \), \( \text{mod } p^n \), according to (12). The fact that the \( a_i \) are determined, \( \text{mod } p^n \), assures that \( D^{(m)} \) is a well defined map of \( V/p^n V \) into \( R/M^m \). We define the desired \( D \in H(V, R) \) by the coset intersection

\[
D_i(a) = \bigcap_{n > i} D_i^{(n)}(a + p^n V).
\]

The following equalities which will be verified in turn, permit us to conclude that \( D \) is a higher derivation. For \( A \) and \( B \) in \( V/p^n V \)

\[
D^{(m)}(A + B) = D^{(m)}(A) + D^{(m)}(B) \quad \text{for } i = 1, \ldots, m,
\]

\[
D^{(m)}(AB) = \sum_{f = 0}^{1} D^{(m)}(A) D^{(m)}(B)
\]

and for \( a \in V \) the following coset inclusion holds.

\[
D^{(m)}(a + p^n V) \supset D^{(m+1)}(a + p^{n+1} V).
\]
Statement (15) is clear from the definition. In order to establish (16) we let 
\[ A = \sum a_k u_k^{2^m} + p^mV \quad \text{and} \quad B = \sum b_k u_k^{2^m} + p^mV, \]
using (12). Thus
\[ AB = \sum a_k b_k u_k^{2^m} + p^mV. \]

Using (12) we have 
\[ u_k u_j = \sum d_{kj} \mod pV. \]
Thus [2, Lemma 1],
\[ u_k^{2^m} u_j^{2^m} = \sum_{t=0}^{3m-1} p^t \sum s_{k,t,i} c_{k,t,i}^{2^m-i} u_i^{2^m}, \mod p^{2m}V, \]
where \( s_{k,t,i} \) is a rational integer and \( c \in V_1 \). Substituting (19) into (19) we have
\[ D^{(m)}(AB) = \sum H_i(\sum a_k b_k p^t s_{k,t,i} c_{k,t,i}^{2^m-i}) u_i^{2^m} + M^m. \]
Since \( p \) and \( s_{k,t,i} \) are rational integers \( H_i(p^t) = H_i(s_{k,t,i}) = 0 \), for all \( i \). Also, by Lemma 6, \( H_i(c_{k,t,i}^{2^m-i}) \) is in \( M^m \) if \( t < m \), since \( i \leq m \). Thus, mod \( M^m \), we have
\[ H_i(\sum a_k b_k p^t s_{k,t,i} c_{k,t,i}^{2^m-i}) = \sum H_i(a_k b_k) p^t s_{k,t,i} c_{k,t,i}^{2^m-i} = \sum_{t=0}^{m-1} H_i(a_k) H_i(b_k) p^t s_{k,t,i} c_{k,t,i}^{2^m-i}. \]
Thus, substituting this last expression into (20) and then using (19) we find that (20) reduces to \( \sum H_i(a_k) H_i(b_k) u_i^{2^m} + M^m \) from which (16) follows.

Relation (17) can be verified as follows. Using (12) for \( n = 1 \) we have 
\[ u_k^{2^m} = \sum a_k u_i, \mod p, \] the \( a_i \) being in \( V_1 \). Thus [2, Lemma 1]
\[ u_k^{2^{m+1}} = \left[ \sum a_k u_i \right]^{2^m} = \sum_{t=0}^{3m-1} p^t \sum s_{k,t,n} c_{k,t,n}^{2^m-i} u_n^{2^m}, \mod p^{2m}V, \]
Again, \( s_{k,t,n} \) is a rational integer and \( c_{k,t,n} \in V_1 \). Thus if \( a + p^m + 1 \)
\[ D^{(m+1)}(a + p^m + 1) = \sum H_i(a_k) u_i^{2^m} + M^{m+1} \]
(22)
\[ D_i^{(m)}(a + p^m V) = \sum H_i(a_k) \sum_{t=0}^{3m-1} p^t \sum s_{k,t,n} c_{k,t,n}^{2^m-i} u_n^{2^m} + M^{m+1}. \]
But, \( a + p^m V = \sum (a_k \sum p^t s_{k,t,n} c_{k,t,n}^{2^m-i} u_n^{2^m} + p^m V \)

Relation (17) then follows in view of (22).

Since \( \bigcap_{m=1} M^m = 0 \), \( D_i \) as defined by (14) is a uniquely determined element of \( R \). Properties (15) and (16) assure that conditions (i) and (ii) of Definition 1 hold mod \( M^m \) for all \( m \). Thus \( D \) is a higher derivation.

In order to show that \( D \) is an extension of \( H \) we note that if \( a \in V_1 \) then 
\[ D_i^{(m)}(a) = H_i(a) + M^m \] since \( 1 \in U_m \). Thus \( D_i(a) = \bigcap_m D_i^{(m)}(a + p^m V) = H_i(a). \)
It remains to show that $D$ is determined by $W=\{D_i(s)\}_{i=1,\ldots,n}$. Certainly, the restriction of $D$ to $V_1 \subset V_0(S)$ is completely determined by $W$ since $D_i(a) = 0$ for $i > 0$ and $a \in V_0$ by Lemma 6 and the fact that $V_0$ is for each $n > 0$ the completion of the subring generated by the $p^n$th powers of elements in $V_0$. Let $a$ be any element in $V$. By (12) $a = \sum a_i u_i^{p^m}$, mod $p^{3m+1}$, where the $a_i$ are in $V_1$. If $j < m$,

$$D_i(\sum a_i u_i^{p^m}) = \sum D_i(a_i) u_i^{p^m}, \quad \text{mod } M^m,$$

by Lemma 6. Hence $D_i(a) = \sum D_i(a_i) u_i^{p^m}$, mod $M^m$. Thus $D$ is determined, mod $M^m$ by its restriction to $S$. But $m$ is arbitrary. It follows that $D$ is uniquely determined by its action on $S$. This proves (a) of Theorem 4.

If $D$ in $H(V, R)$ converges then $D(V) \subset M$. Hence the condition $D(S) \subset M$ is necessary for $D$ to be in $H^*_0(V, R)$, $H^*_0(V, R)$ or $H^*_M(V, R)$. The remaining condition is clearly necessary in each case.

Let $D$ in $H(V, R)$ be such that $D(S) \subset M$ and $\sum D_i(s)$ converges for all $s \in S$. To show that $D$ is in $H^*_M(V, R)$ it is only necessary to show that $D$ converges in view of Lemma 5. Given $n > 0$. By Lemma 6 $D_i(V^{p^n+1}) \subset M$ for $j \leq n$. But $V = V^{p^n+1} [S] + pV$ and hence $D_i(V) \subset M$ or

$$D(V) \subset M.$$  

**Lemma 7.** Let $(T, M)$ be a quasi-local ring with residue field having characteristic $p \neq 0$. Let $S$ be a subring of $T$. If $D \in H(S, T)$ maps $S$ into $M$ then

$$D(S^{p^n}) \subset M^{n+1}, \quad \text{for } n = 1, 2, \ldots.$$  

**Proof.** We argue by induction on $n$ using

$$D_i(a^p) = p a^{p-1} D_i(a) + \sum_{i_1 + \cdots + i_p = i} D_i_1(a_{i_1} \cdots a_{i_p}).$$

Since at least two of the integers $i_1, \ldots, i_p$ are different from zero $D_i(a^p)$ is in $M^2$. If in (25) $a = b^p$ then, by induction, $D_i(b^p) \in M^{n+1}$ and hence $D_i(b^{p^n+1}) \in M^{n+2}$.

By relation (23) and Lemma 7 then $D(V^{p^n}) \subset M^{n+1}$. Given $a$ in $V$ and $t > 0$, $a = f(s_1, \ldots, s_t)$, mod $p^tV$, where $f \in V^{p^t}[X_1, \ldots, X_q]$ has degree $< p^t$ in each $X_i$, and $\{s_1, \ldots, s_t\} \subset S$. We choose $n$ so that if $i > n/qp$ then $D_i(s_i) \in M^t$ for $j = 1, \ldots, q$.

$$D_i(b s_1^{p^n} \cdots s_t^{p^n}) = \sum_{i_0 + \cdots + i_q = i} D_{i_0}(b) D_i(s_1) \cdots D_{i_q}(s_q).$$

If $i > n$ in (26) either $i_0 > 0$ or $i_j > n/qp$ for some $j > 0$. Thus, since $b \in V^{p^t}$, $D_0(b) D_i(s_1) \cdots D_{i_1} + \cdots + D_{i_q}(s_q)$ is in $M^t$. Since every term in $V^{p^t}[s_1, \ldots, s_t] \subset M^t$ and of the type treated in (26) it follows that, if $i > n$, $D_i(a) \in M^t$. Thus $D$ converges.

If $D$ converges uniformly on $S$ then the $n$ of the previous paragraph can be chosen so that if $i > n/qp^t$ then $D_i(S) \subset M^t$, from which it follows that $D_i(V) = D_i(V^{p^t}[S] + p^tV) \subset M^t$.

Thus $D \in H^*_M(V, R)$. Similarly, if $D_i(S) \subset M^t$ a like argument leads to the conclusion that $D_i(V) \subset M^t$. 

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Theorem 5. If \((R, M)\) is a complete local ring with residue field \(k\) having characteristic \(p \neq 0\) then \(H_c(R, R) = H_d(R, R)\) if \(k\) has a finite \(p\)-basis. If \(R\) is regular \(H_c(R, R) = H_d(R, R)\) only if \(k\) has a finite \(p\)-basis.

Proof. As in Theorem 4 we let \(S\) be a set of units in \(R\) which map biuniquely onto a \(p\)-basis \(\bar{S}\) for \(k\) under the canonical map of \(R\) onto \(k\). It is assumed that \(\bar{S}\) is finite. Let \(\mathcal{M}\) be the set of multiplicative representatives of the element in \(k_0\), the maximal perfect subfield of \(k\). We choose an arbitrary \(D\) in \(H_c(R, R)\) and observe first that \(D(\mathcal{M}) = \{0\}\), by Lemma 7 since each \(a\) in \(\mathcal{M}\) is a \(p^m\)th power for all \(m\). Thus if \(T\) is the subring of \(R\) generated by \(\mathcal{M}\) then \(D|_T\), the restriction of \(D\) to \(T\) is the zero higher derivation. By Corollary 3.1 \(D|_T\) is uniformly convergent.

Let \(U\) be a subset of \(R\) which maps biuniquely onto \(\bar{U}\) a basis for \(k\) as a linear space over \(k_0(S)\). As we have observed before the set \(U_n\) of \(p^n\)th powers of the elements in \(U\) maps onto a basis for \(k\) over \(k_0(\bar{S})\).

Let \(t > 0\) be fixed. If \(M = \sum_{i=1}^t w_iR\), then \(a \in R \Rightarrow a = \sum_i f_i u_i^t + \mu, \quad \mu \in M^t, \quad f_i \in T[S][w_1, \ldots, w_s]\).

Hence applying Corollary 3.1 to obtain \(D|T[S][w_1, \ldots, w_s]\) uniformly convergent, we pick an \(n\) such that \(j > n\) implies

\[D_j(T[S][w_1, \ldots, w_s]) \subset M^t.\]

Thus since \(D(M^t) \subset M^t, D_j(a) = D_j(\sum_i f_i u_i^t) + D_j(\mu) \in M^t.\)

Since the choice of \(n\) depends only on \(t, S, \) and \(\{w_1, \ldots, w_s\}\) it follows that \(D\) converges uniformly on \(R\). Inclusion the other way is obvious so the first part of Theorem 5 is proved.

In proving the rest of Theorem 5 we will have use for the following proposition whose proof is standard and will be omitted.

Proposition 2. Let \(S\) be a subring of a complete local ring \((R, M)\) and let \(D\) be in \(H(S, R)\). If \(D\) is continuous in the induced topology then \(D\) extends and in only one way to a higher derivation \(D^*\) on \(S^*\) the completion of \(S\) in \(R\). If \(D\) is convergent so is \(D^*\). If \(D\) is uniformly convergent so is \(D^*\). If \(D(S) \subset M\) then \(D^*(S^*) \subset M\).

Assuming \(R\) to be regular we consider the converse. If \(R\) has characteristic \(p\) then \(R\) is a power series ring \(k[[X_1, \ldots, X_n]]\) in a finite number of indeterminates \(X_1, \ldots, X_n\) over its residue field \(k\). We assume that \(k\) possesses a \(p\)-basis \(S\) with infinite cardinal. Let \(\{s_i\}_{i=1}^\infty\) be a countable sequence of elements in \(S\). A higher derivation \(D^{(i)}\) in \(H(k, k[[X_1, \ldots, X_n]])\) is uniquely determined by the conditions (i) \(D^{(i)}(s_i) = \delta_{i,j}\) (ii) \(D^{(i)}(s_j) = 0\) for \(j \geq 1\) and \(s \in S, s \neq s_i\) [2, Theorem 1]. The theorem referred to here applies to \(D \in H(k, k)\) but the proof applies to the case in which...
the range of $D$ is a ring containing $k$. Let $H^{(i)}$ be defined by $H^{(i)}_{nt} = X_n^i D^{(i)}$; $n \geq 1$, and $H^{(0)}_{nt} = \theta$, for $m$ not a multiple of $i$, $\theta$ being the zero map. $H^{(i)}$ so defined is a convergent higher derivation. $H^{(i)}$ is extended to $H^{(i)}$ on $k[X_1, \ldots, X_n]$ by the condition $H^{(i)}_{nt}(X_t) = 0$ for $j \geq t$, and $t = 1, \ldots, n$. $H^{(i)}$ extended is again, by Lemma 3, a convergent higher derivation. Finally, let $E = H^{(1)} \circ H^{(2)} \circ \cdots \circ H^{(n)}$. Thus $E_n = (H^{(1)} \circ \cdots \circ H^{(n)})_n$ since $H^{(m)}_{nt} = \theta$ for $m < i$. It follows readily that $E$ is a well-defined higher derivation, and is clearly convergent. Let $E^*$ represent the extension of $E$ to $k[[X_1, \ldots, X_n]]$. Again by Proposition 2, $E^*$ is a convergent higher derivation. It follows immediately from the definition of $E^*$ that $E^*(s_t)$ is in $M$ and not in $M^2$. Hence, $E^*$ is not uniformly convergent.

Assume now that $R$ has characteristic zero. Then $R = R_s[\pi]$ where

$$R_1 = V[[X_1, \ldots, X_n]]$$

is a power series ring in $n$ indeterminates over an unramified $v$-ring $V$ and $\pi$ is a root of an Eisenstein polynomial $f$ over $R$ [1, Theorem 1].

The following facts will be useful. Let $K$ be the quotient field of $R_1$.

(A) A given higher derivation on $R_1$ has a unique extension to a higher derivation on $K$. This follows from Lemma 2.

(B) A higher derivation $D$ on $K$ has a unique extension $\overline{D}$ to $K[\pi]$ [2, Theorem 3]. If $D$ is convergent on $K$, $\overline{D}$ will be convergent if and only if $\sum \overline{D}(\pi)$ converges. If $D(R_1) \subset R$ then $\overline{D}(R) \subset R$ if and only if $\overline{D}(\pi) \in R$.

Let the minimal polynomial of $\pi$ over $R$, be $f = X^e + f_{e-1} X^{e-1} + \cdots + f_0$ and let $f'$ denote the ordinary derivative of $f$.

**Lemma 8.** If $f'(\pi) \in M^fM^{s+1}$ and $D \in H_e(R_1, R_1)$ is such that $D(f_j) \in M^{st+j}$ for $j = 0, \ldots, e-1$ then the extension of $D$ to $R$ will be convergent and will map $R$ into $R$.

**Proof.** We choose the same symbol $D$ for the extension of the given higher derivation. Application of the defining properties of a higher derivation to $D(f(\pi)) = 0$ yields

$$f'(\pi) D(\pi) = -f^{D_1}(\pi) - \sum_{j_1 + \cdots + j_e = t; 0 \leq j_e < 1} D_{j_1}(\pi) \cdots D_{j_e}(\pi)$$

$$- \sum_{i=1}^{n-1} \sum_{j_0 + \cdots + j_i = t; 0 \leq j_i < 1} D_{j_0}(f_0) D_{j_1}(\pi) \cdots D_{j_i}(\pi)$$

(27)

where $f^{D_1} = D((X_{e-1})^e + \cdots + D(f_0))$. For $i = 1$ we have the familiar formula $D_1(\pi) = f^{D_1}(\pi)|f'(\pi)$ and hence, since $D_1(f_j) \in [f'(\pi)]^2 M^{t+j}$ for $j = 0, \ldots, e-1$ we have $D_1(\pi) \in f'(\pi) M^t$. If, for $i < r$, $D_i(\pi) \in f'(\pi) M^t$ then by (27) $D_i(\pi) \in f'(\pi) M^t$. Thus $D(R) \subset R$. In order to show that $\sum D_i(\pi)$ converges we assume that for any integer $s$, $1 < s < r$, there is an integer $N_s > eN_{s-1}$ such that if $i > N_s$ then $D_i(f_j) \in M^{st}$ for $j = 0, \ldots, n$ and $D_i(\pi) \in M^{(s+1)t}$. Then since $D$ converges on $R$, there is an $N$ such that if $i > N$ then $D_i(f_j) \in M^{st}$ for all $j$. Let $N_r$ be the larger of $eN$ and $eN_{r-1}$. It follows then from (27) that for $i > N_r$, $D_i(\pi) \in M^{(r+1)t}$ and the lemma is proved.
If \( S \) is a set of representatives in \( V \) of a \( p \)-basis for its residue field \( k \) then
\[ V = V^{p^m}[S] + p^mV \text{ for any } m > 0. \]
Thus there is a finite subset \( S_1 \) of \( S \) such that \( f_j \in V^{p^m}[S_i] + p^mV \). Assuming \( S \) to be an infinite set we enumerate a countable subset \( \{s_i\}_{i=1}^{\infty} \) of \( S - S_1 \) and we define a higher derivation \( D \in H_c^M(V, R) \) by
\[ D(s_j) = \delta_{ij}[f'(\pi)]^i, \]
for \( i, j > 0 \), and \( D(s) = 0 \) for \( s \in S - \{s_i\}_{i=1}^{\infty} \). By Theorem 4, \( D \) is in \( H_c^M(V, R) \) and is not in \( H_c(V, R) \) since \( D \) does not converge uniformly on \( S \). We extend \( D \) to \( V[X_1, \ldots, X_n] \) and hence, by Proposition 2, to \( R_1 \) by the requirement \( D(X_i) = 0 \) for \( i = 1, \ldots, n \), using the same symbol for the extended map. Since \( \sum D_i(X_j) \) converges for \( j = 1, \ldots, n \), \( D \in H_c^M(R_1, R) \), \( D \notin H_n(R_1, R) \). By construction of \( D \) the conditions of Lemma 8 are met and hence \( D \) extends to a higher derivation in \( H_c(R, R) \) which is not in \( H_n(R, R) \).

The following lemma is needed in order to obtain an analogue to Theorem 5 in case the residue field \( R \) has characteristic zero.

**Lemma 9.** Let \( k_0, k_1, \) and \( k \) be fields such that \( k_0 \subseteq k_1 \subseteq k \). Let
\[ D \in H_c(k_1, k[[X_1, \ldots, X_n]]) \]
and assume \( k_1 \) separable algebraic over \( k_0 \). If \( D \) restricted to \( k_0 \) is uniformly convergent then \( D \) is also uniformly convergent. If \( D \in H(k_1, k[[X_1, \ldots, X_n]]) \) is convergent \( (M \) convergent) on \( k \) then
\[ D \in H_c(k_1, k[[X_1, \ldots, X_n]]) \quad (D \in H_c^M(k_1, k[[X_1, \ldots, X_n]])). \]

**Proof.** Let \( u \) be in \( k_1 \) and let \( f \) be its minimal polynomial over \( k_0 \). If
\[ f = X^n + \sum_{i=0}^{n-1} f_i X^i \]
then, as in the proof of Lemma 8,
\[ f'(u) D_i(u) = -f D_i(u) - \sum_{j_1 + \cdots + j_n = i; \ 0 \leq j_l \leq i} D_{j_1}(u) \cdots D_{j_n}(u) \]
\[ - \sum_{l=0}^{n-1} D_{j_0}(f_l) D_{j_1}(u) \cdots D_{j_l}(u) \]
for \( i = 1, 2, \ldots \).

Using (28) and induction on \( i \) we observe below that \( D_i(u) \) is a sum of terms of the form
\[ b D_{i_1}(a_1) \cdots D_{i_r}(a_r), \quad i_1 + \cdots + i_r = i. \]

Relation (28, i) exhibits a representation of \( D_{i_1}(u) \) as a sum of terms of the form (29, 1). Assuming that, for \( i < j \), \( D_i(u) \) is a sum of the form (29, i) we substitute these sums in (28, j) and conclude that \( D_j(u) \) is of the same form. The first assertion of Lemma 9 now follows from Lemma 4.

Let \( D \in H(k_1, k[[X_1, \ldots, X_n]]) \) be convergent on \( k_0 \) and let \( u \) be as above. Now \( f'(u) D_i(u) \) was observed to be a sum of terms of the form (29, i) from which fact
one concludes that \( \sum D_i(u) \) converges if \( D \) converges on \( k_0 \). The remaining statement is obvious.

**Theorem 6.** If \( (R, M) \) is a complete regular local ring having residue field \( k \) with characteristic zero then \( H_c(R, R) = H_u(R, R) \) if and only if \( k \) has finite transcendency degree over its prime field.

**Proof.** In this case \( R \) is a power series ring \( k[[X_1, \ldots, X_n]] \) in \( n \)-indeterminates over \( k \). Let \( k_0 \) be the prime field of \( k \) and let \( B \) be a transcendency basis of \( k \). Then, by Proposition 2 and Lemma 9, it is sufficient to show that \( H_c(k_0(B), R) = H_u(k_0(B), R) \) if and only if \( B \) is finite. Since the first nonzero mapping of a higher derivation is a derivation and there are no nonzero derivations with domain \( k_0 \) it follows that every higher derivation on \( k \) is trivial on \( k_0 \). Hence if \( D \in H_c(k_0(B), R) \) then \( D \) is uniformly convergent on \( k_0 \) and, if \( B \) is finite, \( D \) is uniformly convergent on \( k_0[B] \) by Lemma 1 and Corollary 3.1, and hence is uniformly convergent on \( k_0(B) \) by Corollary 3.2.

If \( B \) is infinite we choose a countable subset \( \{b_i\}_{i=1}^\infty = B' \) in \( B \) and define a \( D \in H_M(k_0(B), R) \) by the conditions \( D_i(b_j) = \delta_{ij}X_j \), for \( i, j \geq 1 \), and \( D_i(b_j) = 0 \) for \( i \geq 1 \) and \( b \) in \( B \), \( b \) not in \( B' \). \( D \) is \( M \)-convergent on \( k_0 \) and on \( B \) and hence \( D \) is in \( H^M(k_0(B), R) \) by Lemma 3. Since \( D_i(b_j) \notin M^2 \) for all \( j \), \( D \) is not uniformly convergent.

**References**


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