HOMOLOGICAL DIMENSION AND THE CONTINUUM HYPOTHESIS

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1. **Introduction.** In [9], Kaplansky proves:

Let R be a commutative local domain with quotient field Q. Then the following are equivalent:

(i) The homological dimension of Q_R, $\mathrm{hd}_R(Q), = 1$.

(ii) Q is countably generated.

In [14], Small shows that the hypothesis "R local" may be replaced either by "R is domain such that the Jacobson radical $J \neq 0$ and R/J is Noetherian" or by "R is a regular domain which is not Dedekind, such that R contains an uncountable field," and (i) will still be equivalent to (ii). Small also shows that, if Q is generated by $\aleph_n$ elements, n an integer, then $\mathrm{hd}_R(Q) \leqq n+1$. This implies that, for Q_R countably generated, $\mathrm{hd}_R(Q) = 1$. This special case also appears in Matlis [10].

The question naturally arises if, for all commutative domains R, $\mathrm{hd}_R(Q) = n+1$ if any generating set for Q_R has at least $\aleph_n$ elements and the global dimension of $R > n$. By Matlis [10], the answer is no—a Noetherian domain of Krull dimension 1 always has $\mathrm{hd}_R(Q) = 1$. However, for regular domains, a situation of particular interest to Small in [14], we obtain some information. By means of the techniques of Osofsky [13] and Kaplansky [9], we show:

Let R be a regular local ring of dimension m such that cardinality $R =$ cardinality R/J or R is complete. Let Q be generated by a set of $\aleph_k$ but no fewer elements. Then $\mathrm{hd}_R(Q) = \min\{k+1, m\}$. This is then generalized to regular domains finitely generated over some field. As indicated in Small [14], this gives a new statement equivalent to the continuum hypothesis. Indeed, for $n \in \omega$, let F_m be the ring of polynomials in $m \geqq n+3$ variables over a field F with cardinality $2^{\aleph_n}$. Let Q_m be its quotient field. Then $\mathrm{hd}_{F_m}(Q_m) = n+2 \Leftrightarrow 2^{\aleph_n} = \aleph_{n+1}$. In particular, if $\boldsymbol{R}$ is the field of real numbers, $\mathrm{hd}_{\boldsymbol{R}_3}(\boldsymbol{Q}_3) = 2 \Leftrightarrow$ the continuum hypothesis holds. One can actually write down a module over $\boldsymbol{R}_3$ which is free $\Leftrightarrow 2^{\aleph_0} = \aleph_1$. Let F be the free $\boldsymbol{R}_3$-module generated by $\boldsymbol{R}_3 \times \boldsymbol{R}_3$, i.e. $F = \sum_{(x,y) \in \boldsymbol{R}_3 \times \boldsymbol{R}_3} \oplus (x, y)\boldsymbol{R}_3$. Let M be the submodule of F generated by $\{(y, z) - (x, z) + (x, y)y/z \mid y = za,\ x = yb$, where a and b are nonconstant polynomials$\}$. Then M is projective $\Leftrightarrow$ the continuum hypothesis holds. Since M is infinitely generated, by a result of Bass [4], M is free $\Leftrightarrow M$ is projective. In the appendix, we show how to construct a free basis for M if the continuum hypothesis

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holds. If one changes $\boldsymbol{R}_3$ to $\boldsymbol{R}_2$, the corresponding module always has a free basis, so we see no way to show directly from the module M that if $2^{\aleph_0} \neq \aleph_1$, M cannot have a free basis.

Most of the results in Small [14] and Osofsky [13, §2] are special cases of generalizations proved here. One might hope that the techniques employed could be combined with appropriate induction hypotheses to yield dimensions of other modules over other rings.

2. **Definitions and notation.** Throughout this paper, R will denote a ring with 1. All modules will be unital right R-modules.

$|A|$ will denote the cardinality of the set A.

Let A, B, and C be R-modules, $\lambda: A \to C$, $\mu: B \to C$, $\nu: A \to B$ R-homomorphisms. $(\lambda, \nu): A \to C \oplus B$ will denote the homomorphism defined by $(\lambda, \nu)a = \lambda a + \nu a$ for all $a \in A$. $\lambda \oplus \mu: A \oplus B \to C$ is defined by $(\lambda \oplus \mu)(a+b) = \lambda a + \mu b$ for all $a \in A$, $b \in B$.

A right R-module M will be called *directed* if:

(i) M is generated by a set of elements M' such that $xr = 0 \Leftrightarrow r = 0$ for all $x \in M'$.

(ii) For all $x, y \in M'$, there exists a $z \in M'$ such that $zR \supseteq xR + yR$.

M' will be called a set of free generators for M.

If M is a directed module with free generators M', $u: M' \times M' \to M'$ is called an *upper bound function* if $u(x, y)R \supseteq xR + yR$ for all $x, y \in M'$. We extend u to a function from $\bigcup_{n=2}^{\infty} (M')^n$ to M' inductively by

$$u(m_1, \ldots, m_n) = u(m_1, u(m_2, \ldots, m_n)).$$

Then $u(m_1, \ldots, m_n)R \supseteq \sum_{i=1}^{n} m_i R$. If $X \subseteq M'$ and $u(X \times X) \subseteq X$, X will be called *u-closed*. For all $Y \subseteq M'$, define the u-closure of Y, $\mathrm{cl}\,(Y)$, by

$$\mathrm{cl}\,(Y) = \bigcap_{Y \subseteq X \subseteq M';\, X\ u\text{-closed}} X.$$

If we set $Y_0 = Y$, $Y_{n+1} = Y_n \cup u(Y_n \times Y_n)$, then $\mathrm{cl}\,(Y) = \bigcup_{n=0}^{\infty} Y_n$.

We observe:

2.1 If $|Y| \geqq \aleph_0$, then $|\mathrm{cl}\,(Y)| = |Y|$ since $|Y_{n+1}| = |Y_n|$ for all $n \geqq 0$.

2.2 If X is u-closed, the submodule of M generated by X is directed.

2.3 If M is directed, then M is finitely generated $\Leftrightarrow$ M is cyclic.

2.4 If M is directed and X is a set of free generators for M, then X is directed by inclusion of cyclic submodules.

2.5 If M is directed and countably generated, then $M = \bigcup_{i=0}^{\infty} x_i R$ where $x_i R$ is free and $x_i R \subseteq x_j R$ for all $i \leqq j$.

$\mathrm{hd}_R\,(M)$ will denote the homological dimension of M_R. If $x, y \in M$, $x \leqq y$ will mean $xR \subseteq yR$. If M is directed, $x \in M'$, x^{-1} will denote the R-isomorphism:

$xR \to R$ given by $x^{-1}(xr)=r$. x^{-1} exists since xR is free with basis x. Note that $x \cdot x^{-1}(xr)=xr$.

Let $X \subseteq M_R$, $n \geqq 0$. $P_n(X)$ will denote the free R-module

$$P_n(X) = \sum_{\{x_i \mid 0 \leqq i \leqq n\} \subseteq X;\, x_0 > x_1 > \cdots > x_n} \oplus \langle x_0, \ldots, x_n \rangle R$$

where, for all $r \in R$, $\langle x_0, \ldots, x_n \rangle r = 0 \Leftrightarrow r = 0$. Set $P_{-1}(X)$ = the submodule of M generated by X.

Let $x \in M'$. Set $s(x) = \{y \in M' \mid y < x\}$, $\bar{s}(x) = \{y \in M' \mid y \leqq x\}$. We define a map $x^*: P_n(s(x)) \to P_{n+1}(\bar{s}(x))$ for $n \geqq 0$ by

$$x^* \langle x_0, \ldots, x_n \rangle = \langle x, x_0, \ldots, x_n \rangle.$$

If $n = -1$, $x^*: P_{-1}(x) \to P_0(\bar{s}(x))$ is defined by

$$x^*(xr) = \langle x \rangle r = \langle x \rangle x^{-1}(xr).$$

For $n \geqq 0$, define a function $d_n: P_n(X) \to P_{n-1}(X)$ by

$$d_0 \langle x \rangle = x,$$
$$d_n \langle x_0, \ldots, x_n \rangle = \sum_{i=1}^{n-1} \langle x_0, \ldots, \hat{x}_i, \ldots, x_n \rangle (-1)^i + \langle x_0, \ldots, x_{n-1} \rangle (-1)^n x_{n-1}^{-1}(x_n)$$

where $\hat{x}_i$ means delete x_i.

x^* and d_i are analogous to the "adjoin a vertex" and boundary operators of combinatorial topology, and are precisely the functions defined in [13]. They are connected by a basic relation:

$$d_{n+1}(x^* p) = p - x^* d_n p \quad \text{for all } n \geqq 0,\ p \in P_n(s(x)). \tag{2.6}$$

This relation will often be used without explicit reference to it. It is verified by direct computation.

3. **The projective resolution of a directed R-module.** We apply the argument in [13] to get a projective resolution of a directed R-module which we will use to calculate its dimension in special cases.

3.1 PROPOSITION. *Let M be a directed R-module with set of free generators M' and upper bound function u. Let X be a u-closed subset of M'. Then*

$$(\mathscr{P}_X) \quad \cdots \xrightarrow{d_{n+1}} P_n(X) \xrightarrow{d_n} P_{n-1}(X) \longrightarrow \cdots \xrightarrow{d_1} P_0(X) \xrightarrow{d_0} P_{-1}(X) \longrightarrow 0$$

is a projective resolution of $P_{-1}(X)$ = the submodule generated by X.

Proof. (i) $\mathscr{P}_X$ is a complex. This is a straightforward computation, written out in [13].

(ii) $\mathscr{P}_X$ is exact. $\mathscr{P}_X$ is exact at $P_{-1}(X)$ since X generates $P_{-1}(X)$.

Let $p=\sum_{i=1}^{k}\langle x_0^i,\ldots,x_n^i\rangle r_i\in P_n(X)$, $d_np=0$. Let $x=u(x_0^1,\ldots,x_0^k)$. Assume $x_0^1,\ldots,x_0^l<x=x_0^{l+1}=\cdots=x_0^k$, and set $p'=\sum_{i=1}^{l}\langle x_0^i,\ldots,x_n^i\rangle r_i$, $p''=p-p'$. By definition, $p''=x^*q$ for some $q\in P_{n-(s(x))\text{ or }xR}$. By 2.6,

$$p-d_{n+1}(x^*p') = x^*q+x^*d_np'.$$

Since $\mathscr{P}_X$ is a complex

$$\begin{aligned} 0 &= d_n[x^*(q+d_np')] = q+d_np'+x^*(dq) && \text{if } n>0,\\ &= xx^{-1}(q+d_np')=q+d_np' && \text{if } n=0. \end{aligned}$$

Since for $n>0$, no term of $q+d_np'$ involves the symbol x, and every term of x^*dq does, $q+d_np'=0$. Hence $p=d_{n+1}(x^*p')$.

We are also interested in a projective resolution of a quotient of two directed modules.

3.2 Proposition. *Let M be a directed R-module, X and Y u-closed subsets of M', $X\subseteq Y$. Let ν be the natural map from $P_{-1}(Y)\to P_{-1}(Y)/P_{-1}(X)$, I the identity on $P_n(X)$. Then*

$$\begin{aligned} (\mathscr{P}_{X,Y})\ \cdots &\longrightarrow P_n(X)\oplus P_{n+1}(Y)\xrightarrow{(-d_n,\,I)\oplus d_{n+1}} P_{n-1}(X)\oplus P_n(Y)\\ &\longrightarrow\cdots\xrightarrow{(-d_1,\,I)\oplus d_2} P_0(X)\oplus P_1(Y)\xrightarrow{I\oplus d_1} P_0(Y)\\ &\xrightarrow{\nu d_0} P_{-1}(Y)/P_{-1}(X)\longrightarrow 0 \end{aligned}$$

is a projective resolution of $P_{-1}(Y)/P_{-1}(X)$.

Proof. Clearly $\mathscr{P}_{X,Y}$ is exact at $P_{-1}(Y)/P_{-1}(X)$ since d_0 is onto $P_{-1}(Y)$. Also, $\nu d_0(I\oplus d_1)=0$ since $d_0P_0(X)\subseteq P_{-1}(X)$ and $d_0d_1=0$. Let $z\in\operatorname{kernel}\nu d_0$. Then $d_0(z)\in P_{-1}(X)$. Since $d_0\colon P_0(X)\to P_{-1}(X)$ is onto, there is an $x\in P_0(X)$ such that $d_0(x-z)=0$. Since $\mathscr{P}_Y$ is exact, $z\in P_0(X)+d_1(P_1(Y))$ and $\mathscr{P}_{X,Y}$ is exact at $P_0(Y)$.

Moreover,

$$[I\oplus d_1][(-d_1,I)\oplus d_2] = (-d_1+d_1,d_1d_2) = 0.$$

If $(I\oplus d_1)(a,b)=0$, $a\in P_0(X)$, $b\in P_1(Y)$, then $a+d_1b=0$, and by the exactness of $\mathscr{P}_X$ and $\mathscr{P}_Y$, there is a $z\in P_1(X)$ and $w\in P_2(Y)$ such that $d_1z=d_1b=-a$ and $z=b+d_2w$. Then $(a,b)=[(-d_1,I)\oplus d_2](z,-w)$ so $\mathscr{P}_{X,Y}$ is exact at $P_0(X)\oplus P_1(Y)$.

For $n>1$,

$$[(-d_{n-1},I)\oplus d_n][(-d_n,I)\oplus d_{n+1}] = (d_{n-1}d_n,-d_n+d_n)\oplus d_nd_{n+1} = 0.$$

Hence $\mathscr{P}_{X,Y}$ is a complex. Let

$$[(-d_n,I)\oplus d_{n+1}](a,b) = 0.$$

Then

$$0 = -d_na = a+d_{n+1}b.$$

By the exactness of $\mathscr{P}_X$, $d_{n+1}(b)=-a=d_{n+1}(z)$ for some $z\in P_{n+1}(X)$, and $b-z=d_{n+2}(w)$ for some $w\in P_{n+2}(Y)$. Then $[(-d_{n+1}, I)\oplus d_{n+2}](z, w)=(a, b)$. Hence $\mathscr{P}_{X,Y}$ is exact.

Clearly every module in $\mathscr{P}_{X,Y}$ is projective (indeed free).

4. **The Small inequality.** We apply an argument in Small [14] to get one inequality on the dimension of a directed R-module.

4.1 LEMMA (AUSLANDER). *Let M be a right R-module, $\mathscr{I}$ a nonempty well-ordered set, and $\{N_i \mid i\in\mathscr{I}\}$ a family of submodules of M such that if $i, j\in\mathscr{I}$ and $i\leqq j$, then $N_i\subseteq N_j$. If $M=\bigcup_{i\in\mathscr{I}} N_i$ and* $\mathrm{hd}_R\,(N_i/\bigcup_{j<i} N_j)\leqq n$ *for all $i\in\mathscr{I}$, then* $\mathrm{hd}_R\,(M)\leqq n$.

This is Proposition 3 of Auslander [1].

4.2 PROPOSITION. *Let M be a directed R-module possessing a free generating set X of $\aleph_n$ elements for some $n\in\omega$. Then* $\mathrm{hd}_R\,(M)\leqq n+1$.

Proof. If $n=0$, by 2.5, $M=\bigcup_{i=0}^{\infty} x_iR$. Since x_iR has dimension 0 for each i and $0\to x_iR\to x_{i+1}R\to x_{i+1}R/x_iR\to 0$ is exact, $\mathrm{hd}\,(x_{i+1}R/x_iR)\leqq 1$ for each i. By 4.1, $\mathrm{hd}_R\,(M)\leqq 1$.

Now assume $n>0$ and the proposition holds for $n-1$. Index X by $\aleph_n$, i.e. $X=\{x_\alpha \mid \alpha<\aleph_n\}$. By transfinite induction we define a set of u-closed subsets of X, $\{X_\alpha \mid \alpha<\aleph_n\}$ such that $X_0\supseteq\{x_\alpha \mid \alpha\in\omega\}$, $X_\alpha=\bigcup_{\beta<\alpha} X_\beta$ for α a limit ordinal, and $X_{\beta+1}=\mathrm{cl}\,(X_\beta\cup\{x_\beta\})$. We note $|X_0|<\aleph_n$. Assume $|X_\beta|<\aleph_n$ for all $\beta<\alpha$. If α is a limit ordinal, X_α is a union of $|\alpha|<\aleph_n$ sets of cardinality $<\aleph_n$, so $|X_\alpha|<\aleph_n$. If $\alpha=\beta+1$, since $X_\beta\supseteq X_0$ which is infinite, $|X_\beta\cup\{x_\beta\}|=|X_\beta|<\aleph_n$, so by 2.1, $|X_{\beta+1}|=|X_\beta|<\aleph_n$. Hence each X_α has $|X_\alpha|<\aleph_n$ by induction. Then $M=\bigcup_{\alpha<\aleph_n} P_{-1}(X_\alpha)$, and by the induction hypothesis, $\mathrm{hd}\,(P_{-1}(X_\alpha))\leqq n$. Since $\bigcup_{\beta<\alpha} P_{-1}(X_\beta)=P_{-1}(X_\alpha)$ if α is a limit ordinal or $P_{-1}(X_{\alpha-1})$ if α is a successor ordinal, in the exact sequence

$$0\to\bigcup_{\beta<\alpha} P_{-1}(X_\beta)\to P_{-1}(X_\alpha)\to P_{-1}(X_\alpha)/\bigcup_{\beta<\alpha} P_{-1}(X_\beta)\to 0$$

two of the three terms have dimension $\leqq n$. Hence $\mathrm{hd}\,(P_{-1}(X_\alpha)/\bigcup_{\beta<\alpha} P_{-1}(X_\beta))\leqq n+1$. Then $\mathrm{hd}_R\,(M)\leqq n+1$ by 4.1.

5. **Direct summands of a projective d_nP_n.**

5.1 LEMMA (KAPLANSKY). *A projective module over any ring R is a direct sum of countably generated submodules.*

For a proof see Kaplansky [7].

Let $p\in P_n(X)$, $p=\sum_{i=1}^{m}\langle x_0^i,\ldots,x_n^i\rangle r_i$. We say $x\in X$ *appears* in p if $x=x_j^i$ for some i and j, $0\leqq j\leqq n$, $r_i\neq 0$. For each $p\in P_n(X)$, $\{x\in X \mid x \text{ appears in } p\}$ is finite. For $Y\subseteq P_n(X)$, set

$$a(Y)=\{x\in X \mid x \text{ appears in } p \text{ for some } p\in Y\}.$$

5.2 PROPOSITION. *Let M be a directed R-module with free generators M', upper bound function u and projective dimension $\leqq k$ such that no set of cardinality $\leqq\aleph_n$*

generates M for some $n \in \omega$. Let $Z \subseteq M'$ have $|Z| \leqq \aleph_n$. Then there exists a u-closed set $Y \subseteq M'$ such that $Z \subseteq Y$ and

(a) $|Y| = \aleph_n$.

(b) *No set of cardinality $< \aleph_n$ generates $P_{-1}(Y)$.*

(c) *$d_k P_k(Y)$ is a direct summand of $d_k P_k(M')$.*

Proof. Since $\mathrm{hd}_R(M) \leqq k$, $d_k P_k(M')$ is projective. By 5.1, $d_k P_k(M') = \sum_{i \in \mathscr{I}} \oplus Q_i$ where Q_i is countably generated. For $X \subseteq d_k P_k(M')$, set

$$X^\dagger = \bigcap \left\{ \sum_{i \in \mathscr{J}} Q_i \mid \mathscr{J} \subseteq \mathscr{I}, X \subseteq \sum_{i \in \mathscr{J}} Q_i \right\}.$$

Let X be infinite. Since each element in X is in a finite sum of Q_i's, if $X^\dagger = \sum_{i \in \mathscr{K}} Q_i$, $|\mathscr{K}| \leqq |X|$. Since each Q_i is countably generated, $|a(Q_i)| \leqq \aleph_0$. Then

$$|a(X^\dagger)| = \left| a\left(\sum_{i \in \mathscr{K}} Q_i \right) \right| \leqq |\mathscr{K}| \cdot \aleph_0 \leqq |X|. \tag{5.3}$$

Now let Y_0 be the u-closure of Z (hence $|Y_0| \leqq \aleph_n$). We inductively define Y_α for $\alpha < \aleph_n$ with the following properties:

(i) $|Y_\alpha| \leqq \aleph_n$,

(ii) $Y_\alpha = \bigcup_{\beta < \alpha} Y_\beta$ for α a limit ordinal,

(iii) $Y_{\alpha+1} = \mathrm{cl}\,(a\{[d_k P_k(\mathrm{cl}\,(Y_\alpha \cup \{x\}))]^\dagger\})$ where $x \notin P_{-1}(Y_\alpha)$. (I.e., add an extra element to Y_α, close it to get a directed set, apply d_k to the corresponding $(k+1)$-tuples, take all elements appearing in a minimal set of Q's containing this image, close again.)

Since $|Y_\alpha| \leqq \aleph_n$, $P_{-1}(Y_\alpha) \neq M$. Hence (iii) is always possible. Since a union of $< \aleph_n$ sets of cardinality $\leqq \aleph_n$ has cardinality $\leqq \aleph_n$; cl does not increase cardinality by 2.1; $d_k P_k(X)$ is $|X|$-generated; and $|a(X^\dagger)| \leqq |X|$ by (5.3); (i) will be satisfied by Y_α if it is satisfied by all Y_β for $\beta < \alpha$.

Set

$$Y = \bigcup_{\alpha < \aleph_n} Y_\alpha.$$

Since each Y_α is u-closed, so is Y. Since $Y_{\alpha+1} \supseteq a\{[d_k P_k(Y_\alpha)]^\dagger\}$, $d_k P_k(Y_{\alpha+1}) \supseteq d_k P_k(Y_\alpha)^\dagger$. Hence $d_k P_k(Y) = [d_k P_k(Y)]^\dagger$. Moreover, $|Y_\alpha| \leqq \aleph_n$ implies $|Y| \leqq \aleph_n \aleph_n = \aleph_n$.

By (iii), $P_{-1}(Y_{\beta+1}) \supsetneqq P_{-1}(Y_\beta)$. Hence $P_{-1}(Y) = \bigcup_{\alpha < \aleph_n} P_{-1}(Y_\alpha)$ is a strictly ascending union of a chain of submodules with order type $\aleph_n$. Since any set A of ordinals $< \aleph_n$ such that $|A| < \aleph_n$ must have a supremum $< \aleph_n$, no set of cardinality $< \aleph_n$ can generate $P_{-1}(Y)$. Hence Y must satisfy (a), (b), and (c) of the proposition.

6. **Quotient fields of regular local rings.** In this section, R will denote a commutative domain with quotient field Q.

We note that Q_R is a directed R-module since every cyclic submodule of Q is free and if a/b, $c/d \in Q$, $1/bd \geqq a/b$ and c/d. For convenience, we will take as our free generators for Q the set $Q' = \{1/r \mid 0 \neq r \in R\}$ and let $u(1/r, 1/s) = 1/rs$.

6.1 LEMMA. *Let R' be any ring. Then for any projective R'-module P, $PJ(R') \neq P$.*

Proof. See Bass [3, p. 474].

6.2 THEOREM. *Let T be a multiplicatively closed subset of $R-\{0\}$, $T \cap J(R) \neq \varnothing$. Let M be the submodule of Q generated by $T^{-1}=\{1/t \mid t \in T\}$. Then $\mathrm{hd}_R(M)=1 \Leftrightarrow M$ is countably generated.*

Proof. Assume M is countably generated. By 4.2, $\mathrm{hd}_R(M) \leqq 1$. Since M is divisible by some $x \in J(R)$, $\mathrm{hd}_R(M) \neq 0$ by 6.1. Hence $\mathrm{hd}_R(M)=1$.

Assume $\mathrm{hd}_R(M)=1$. Let $x \in T \cap J(R)$. If M is not countably generated, by 5.2, we may find a countable multiplicatively closed subset S of T such that $x \in S$ and $d_1P_1(S^{-1})$ is a direct summand of $d_1P_1(T^{-1})$, say $d_1P_1(T^{-1})=d_1P_1(S^{-1}) \oplus K$. We apply an argument of Kaplansky [9]. There is a $t \in T$ such that $1/t \notin P_{-1}(S^{-1})$. Then $1/t \cdot P_{-1}(S^{-1}) \subseteq M$ and $1/t \cdot P_{-1}(S^{-1}) \approx P_{-1}(S^{-1})$. Hence

$$\mathrm{hd}_R(1/t \cdot P_{-1}(S^{-1})) = 1.$$

Let

$$Z = 1/t \cdot S^{-1} \cup S^{-1}.$$

By 3.2, the sequence

$$P_0(S^{-1}) \oplus P_1(Z) \xrightarrow{I \oplus d_1} P_0(Z) \xrightarrow{\nu d_0} P_{-1}(Z)/P_{-1}(S^{-1}) \longrightarrow 0$$

is exact, so kernel $\nu d_0 = P_0(S^{-1}) + d_1P_1(Z) = P_0(S^{-1}) \oplus K \cap d_1P_1(Z)$ is projective. Hence $\mathrm{hd}_R(P_{-1}(Z)/P_{-1}(S^{-1})) \leqq 1$. Since $P_{-1}(Z)/P_{-1}(S^{-1})$ is an $R/(t)$ module, by Theorem 1.2 of [8], $P_{-1}(Z)/P_{-1}(Y)$ is $R/(t)$-projective. However $P_{-1}(Z)/P_{-1}(S^{-1})$ is a nonzero $R/(t)$ module divisible by $x+(t) \in J(R/(t))$, contradicting 6.1.

6.3 LEMMA. *Let R be a regular local ring of dimension $n \geqq 2$, and let $\{x_1, x_2, \ldots, x_n\}$ be a regular system of parameters for R. If $\{\alpha_s \mid s \in R/J\}$ is a complete set of coset representatives of $(J, +)$ in $(R, +)$, then $\{x_{i+1}+\alpha_s x_i \mid s \in R/J,\ 1 \leqq i \leqq n-1\}$ are distinct primes in R.*

Proof. These elements are prime since they are in $J-J^2$; they generate distinct ideals since they generate distinct submodules of J/J^2.

6.4 THEOREM. *Let R be a regular local ring of dimension n. Let $\{x_1, \ldots, x_n\}$ be a system of parameters for R, $A \subseteq R/J$, $|A|=\aleph_k$. Let M be generated by a u-closed $M' \subseteq Q'$ such that $M' \supseteq \{1/(x_{i+1}-\alpha_s x_i) \mid s \in A,\ 1 \leqq i \leqq n-1\}$ and $|M'|=\aleph_k$. Then $\mathrm{hd}_R(M)=\min\{n, k+1\}$.*

Proof. We note that, if $k=0$ (as it must if $n=1$), $\mathrm{hd}_R(M) \leqq 1$ by 4.2, and since M is divisible by some nonunit of R, $\mathrm{hd}_R(M) \neq 0$. Hence $\mathrm{hd}_R(M)=1=k+1 \leqq n$.

Now assume $k \geqq 1$. Then $\mathrm{hd}_R(M) \leqq k+1$ by 4.2, and $\mathrm{hd}_R(M) \leqq n=$global dimension of R. Hence we need only show both inequalities cannot hold. We use induction on n.

If $n=2$, $\mathrm{hd}_R(M)\neq 1$ by 6.2, and M is not projective by 6.1, so $\mathrm{hd}_R(M)=2=n$.

Now assume $n\geqq 3$. Let $\mathrm{hd}_R(M)=l<n$, $k+1$. Select a set $A'\subseteq A$ with $|A'|=\aleph_{k-1}$ and by 5.2 find a u-closed set $Y\subseteq M'$ such that $|Y|=\aleph_{k-1}$, no set with fewer elements generates $P_{-1}(Y)$, $Y\supseteq\{1/(x_{i+1}-\alpha_s x_i)\mid 1\leqq i\leqq n-1, s\in A'\}$, and $d_lP_l(Y)$ is a direct summand of $d_lP_l(M)$. Now $\{x_n-\alpha_s x_{n-1}\mid s\in A\}$ form a set of primes of cardinality $\aleph_k$, so there exists $s'\in A$ such that $1/q=1/(x_n-\alpha_{s'}x_{n-1})$ is relatively prime to each $1/y\in Y$. Now $\{x_1, x_2, \ldots, x_{n-1}, q\}$ is a regular system of parameters for R so $R^*=R/(q)$ is a regular local ring of dimension $n-1$ with $R^*/J^*\approx R/J$. Also $\{x_{i+1}-\alpha_s x_i\mid 1\leqq i\leqq n-2, s\in A'\}$ have exactly the same properties in R^* that they had in R (here we are identifying an element in R with its image in R^* for convenience).

As above, set $Z=Y\cup\{q^{-1}y\mid y\in Y\}$. Then $P_{-1}(Z)=q^{-1}P_{-1}(Y)$, so $\mathrm{hd}_R(P_{-1}(Z))=\mathrm{hd}_R(P_{-1}(Y))\leqq l$. Let $d_lP_l(Z)=d_lP_l(Y)\oplus K$. By 3.2, there is a projective resolution of $P_{-1}(Z)/P_{-1}(Y)$ whose lth image $=[(-d_{l-1}, I)\oplus d_l][P_{l-1}(Y)\oplus P_l(Z)]\approx(-d_{l-1}, I)P_{l-1}(Y)\oplus K\approx P_{l-1}(Y)\oplus K$. Hence $\mathrm{hd}_R(P_{-1}(Z)/P_{-1}(Y))\leqq l$. As above, $\mathrm{hd}_{R^*}(P_{-1}(Z)/P_{-1}(Y))\leqq l-1$.

Since R is a unique factorization domain, Y consists of reciprocals of a multiplicative semigroup of R, and q is a prime in R, for $a/b\in q^{-1}P_{-1}(Y)-P_{-1}(Y)$, $ax/b\in P_{-1}(Y)\Leftrightarrow q|x$. Therefore $P_{-1}(Z)/P_{-1}(Y)$ is a torsionless R^*-module. Since $P_{-1}(Z)$ is generated by $M^*=\{q^{-1}y^{-1}\mid y^{-1}\in Y\}$ and $\{y\mid y^{-1}\in Y\}$ is a multiplicative semigroup of R, $P_{-1}(Z)/P_{-1}(Y)$ is a directed R^*-module with upper bound function $u^*\colon M^*\times M^*\to M^*$, $u^*(q^{-1}y^{-1}, q^{-1}z^{-1})=q^{-1}y^{-1}z^{-1}$. Let u_1/u_2, $v_1/v_2\in P_{-1}(Z)$, $q\nmid u_1, v_1$; $u_2=qu_3$, $v_2=qv_3$ where $q\nmid v_3, u_3$. Then

$$(u_1/u_2)\cdot u_3v_1 = (v_1/v_2)\cdot v_3u_1\notin P_{-1}(Y).$$

Hence $P_{-1}(Z)/P_{-1}(Y)$ as an R^*-module is an essential extension of every cyclic submodule, so the map $q^{-1}\to 1$ extends to an isomorphism between $P_{-1}(Z)/P_{-1}(Y)$ and an R^*-submodule of $Q^*=$ the injective hull of R^*. Moreover, the image of M^* consists of reciprocals of a multiplicative semigroup in R^*, and if $u\in P_{-1}(Z)$, $y\in Y$, then $uy\in P_{-1}(Z)$. In particular, $u/(x_{i+1}-\alpha_s x_i)\in P_{-1}(Z)$ for all $s\in A'$, $1\leqq i\leqq n-2$. Thus $P_{-1}(Z)/P_{-1}(Y)$ as an R^*-module is divisible by $1/(x_{i+1}-\alpha_s x_i)$ for all $s\in A'$, $1\leqq i\leqq n-2$, so $\{1/(x_{i+1}-\alpha_s x_i)\mid 1\leqq i\leqq n-2, s\in A'\}\subseteq$ the image of $P_{-1}(Z)/P_{-1}(Y)$ in Q^*.

We now have R^* a regular local ring of dimension $n-1$, $|A'|=\aleph_{k-1}$, and a directed submodule $P_{-1}(Z)/P_{-1}(Y)$ containing $\{1/(x_{i+1}-\alpha_s x_i)\mid 1\leqq i\leqq n-2, s\in A'\}$. By the induction hypothesis, $\mathrm{hd}_{R^*}(P_{-1}(Z)/P_{-1}(Y))=\min\{n-1, k\}$. But by [8, Theorem 1.2], $\mathrm{hd}_{R^*}(P_{-1}(Z)/P_{-1}(Y))\leqq l-1$, where $l<\min\{n, k+1\}$, a contradiction.

6.5 Corollary. *Let R be a regular local ring of dimension n such that a minimal set of generators for Q has cardinality $\leqq|R/J|=\aleph_k$. Then* $\mathrm{hd}_R(Q)=\min\{n, k+1\}$.

Proof. If $n=1$, $\mathrm{hd}_R(Q)=1=\min\{n, k+1\}$. If $n\geqq 2$, we observe that $|Q|=|R|=|R/J|$ by 6.3. Now apply Theorem 6.4.

6.6 COROLLARY. *Let $\boldsymbol{R}$ denote the real numbers. Let $\boldsymbol{R}_n$ be the localization of $\boldsymbol{R}[X_1, \ldots, X_n]$ at the origin, and set $\boldsymbol{Q}_n =$ the quotient field of $\boldsymbol{R}_n$. Then*

$$\mathrm{hd}_{\boldsymbol{R}_n}(\boldsymbol{Q}_n) = n \Leftrightarrow 2^{\aleph_0} \geqq \aleph_{n-1}.$$

Proof. By 6.5, $\mathrm{hd}_{\boldsymbol{R}_n}(\boldsymbol{Q}_n) = \min\{n, k+1\}$ where $2^{\aleph_0} = \aleph_k$. Hence $\mathrm{hd}_{\boldsymbol{R}_n}(\boldsymbol{Q}_n) = n \Leftrightarrow k+1 \geqq n$, i.e. $k \geqq n-1$.

6.7 COROLLARY. *Let F be a field, $\mathscr{I}$ a nonempty set, $\{X_i \mid i \in \mathscr{I}\}$ algebraically independent elements over F, $R = F[\{X_i \mid i \in \mathscr{I}\}]$. Let $|\mathscr{I}| = \alpha$, $|F| = \beta$. Then*

$$\begin{aligned} \mathrm{hd}_R(Q) &= \max(n, k)+1 \qquad \aleph_0\beta = \aleph_k, \quad \alpha = \aleph_n, \\ &= \min(\alpha, k+1) \qquad \aleph_0\beta = \aleph_k, \quad \alpha < \aleph_0, \end{aligned}$$

where an infinite ordinal is replaced by ∞.

Proof. Since every element in R is a finite sum of finite products of elements in $\mathscr{I}$, $|R| = \aleph_0\alpha\beta$.

Assume $\alpha = \aleph_n$. By Small's result (4.2) $\mathrm{hd}_R(Q) \leqq \max(k, n)+1$ since $|Q| = \aleph_{\max(k,n)}$. Let R_M be the localization of R at the ideal generated by some set of $m > n+k$ indeterminants. Then R_M is a regular local ring of dimension m and $|R_M/J(R_M)| = \aleph_{\max(k,n)} = |R|$. By 6.5, $\mathrm{hd}_{R_M}(Q) = \max(k, n)+1$. Since $\mathrm{hd}_R(Q) \geqq \mathrm{hd}_{R_M}(Q)$, the first case of the theorem follows. (If $\max(k, n) = \infty$, one takes a localization at m indeterminants to get $\mathrm{hd}_R(Q) \geqq m$ for all m.)

Assume $\alpha < \aleph_0$. Then $|R| = \aleph_0\beta$. If $\beta < \aleph_0$, then $|R| = |Q| = \aleph_0$ and by 4.2, $\mathrm{hd}_R(Q) = 1 = \min(0+1, \alpha)$. So without loss of generality we may assume $\beta \geqq \aleph_0$. Then $|R| = |F|$. Let R^* be the localization of R at the origin. Then by 6.5, $\mathrm{hd}_{R^*}(Q) = \min(k+1, \alpha) \leqq \mathrm{hd}_R(Q)$. But the global dimension of $R = \alpha$, so $\mathrm{hd}_R(Q) \leqq \alpha$ and $|Q| = \aleph_k$ so $\mathrm{hd}_R(Q) \leqq k+1$. Hence $\mathrm{hd}_R(Q) = \min(k+1, n)$.

6.5 and 6.7 can be generalized slightly. Let $R = K[x_1, \ldots, x_m]$ be a finitely generated ring extension of an infinite field K (the x_i not necessarily indeterminants). Assume R has global dimension $n < \infty$. By Auslander and Buchsbaum [2], some localization of R at a maximal ideal, say R_M, has codimension n. Since the global dimension of $R_M \leqq$ the global dimension of R, R_M is a regular local ring of dimension n. (See Kaplansky [8].)

6.8 COROLLARY. *Let $R = K[x_1, \ldots, x_m]$ have global dimension $n < \infty$, and let $|K| = \aleph_k$. Then $\mathrm{hd}_R(Q) = \min\{n, k+1\}$.*

Proof. Clearly $\mathrm{hd}_R(Q) \leqq n$ and by 4.2 $\mathrm{hd}_R(Q) \leqq k+1$. Let R_M be a localization of R of dimension n. Then $\mathrm{hd}_R(Q) \geqq \mathrm{hd}_{R_M}(Q) = \min\{k+1, n\}$ since $|R_M| = |R| = |K| = \aleph_k = |R/J|$. Hence $\mathrm{hd}_R(Q) = \min\{k+1, n\}$.

6.9 COROLLARY. *Let $R = K[x_1, \ldots, x_m]$. Assume for all positive integers l there exists a prime ideal $M \subseteq R$ such that $l \leqq \mathrm{gl.\,d}\,(R_M) < \infty$. If $|K| = \aleph_h$, then $\mathrm{hd}_R(Q) = h+1$ if $h \in \omega$, ∞ otherwise.*

Proof. If $h \in \omega$, there is an M with $h+2 \leqq \text{gl. d}\,(R_M) < \infty$. Then R_M is regular and $\text{hd}_{R_M}(Q) = h+1 \leqq \text{hd}_R(Q)$. By 4.2, $\text{hd}_R(Q) = h+1$.

If $h \notin \omega$, for all $l \in \omega$ there is an M with $l \leqq \text{hd}_{R_M}(Q) \leqq \text{hd}_R(Q)$. Hence $\text{hd}_R(Q) = \infty$.

In proving 6.4, we needed some property which would enable us to use induction from dimension n to dimension $n-1$ without having a collapse in the number of generators of $P_{-1}(Z)/P_{-1}(Y)$. We chose a situation which yielded the desired result when a set of cardinality $|R/J|$ generated Q. On the opposite end of the scale are complete regular local rings. Here too we can calculate $\text{hd}_R(Q)$. In the case that the characteristics of R and R/J are equal, these are just power series over a field (see [15, p. 307]). In the nonequicharacteristic case they are still close enough to power series to keep track of cardinality of a special set of primes.

6.10 THEOREM. *Let R be a complete regular local ring of dimension n. Let Q be generated by $\aleph_k$ but no fewer elements. Then* $\text{hd}_R(Q) = \min(n, k+1)$.

Proof. Let $|R/J| = \alpha$. Since J^i/J^{i+1} is a finitely generated R/J-module, $|J^i/J^{i+1}| = |R/J|$. Hence $|R/J^{i+1}| = |R/J|$. Now the elements of R are limits of Cauchy sequences in R, so each is completely determined by its sequence of projections in R/J^{i+1}. Moreover there are $|R/J|$ different ways two Cauchy sequences agreeing in R/J^i can differ in R/J^{i+1}. Hence $|R| = |R/J|^{\aleph_0}$. Now assume $n \geqq 2$. Let $\{x_1, x_2, \ldots, x_n\}$ be a system of parameters for R. There are $|R/J|^{\aleph_0}$ Cauchy sequences $\{y_i\}$ in x_1. For each of these, let $\alpha_{\{y_i\}}$ be the element it determines. Any element of the form $x_2 - x_1\alpha_{\{y_i\}}$ is prime since it belongs to J but not J^2 (its projection on $J/J^2 = x_2 - x_1 k$ for some $k \in R$). Let $x_2 - x_1\alpha_{\{y_i\}} = (x_2 - x_1\alpha_{\{z_i\}})\gamma$ where γ is a unit. Then $x_2(1-\gamma) = x_1(\alpha_{\{y_i\}} - \alpha_{\{z_i\}}\gamma)$. Since $x_1 \nmid x_2$, $x_1 | 1-\gamma$, so $\gamma = 1 + x_1 r$. Assume $\gamma = 1 + x_1^n r_n$. Then $\alpha_{\{y_i\}} - \alpha_{\{z_i\}}(1 + x_1^n r_n) = \alpha_{\{y_i - z_i\}} - \alpha_{\{z_i\}} x_1^n r_n$. Since $x_2(1-\gamma) \in J^{n+1}$, $(\alpha_{\{y_i - z_i\}} - \alpha_{\{z_i\}} x_1^n r_n) \in J^n$. Hence $\alpha_{\{y_i - z_i\}} \in J^n$. $\alpha_{\{y_i - z_i\}}$ is a Cauchy sequence in x_1 which lies in J^n, hence $x_1^n | \alpha_{\{y_i - z_i\}}$, and so $x_1^n | (\alpha_{\{y_i - z_i\}} - \alpha_{\{z_i\}} x_1^n r_i)$ so $x_1^{n+1} | 1-\gamma$ for all n. This contradicts the unique factorization property of R. We thus have $|R|$ distinct primes in R of the form $x_{i+1} - f(x_i)$ which remain distinct in $R/(x_n)$ for $n \geqq 3$. The proof of the theorem now proceeds exactly as in 6.4 with these $|R|$ primes replacing $\{1/(x_{i+1} - \alpha_s x_i)\}$.

Note that, if $|R/J| = |R/J|^{\aleph_0}$, since any local ring is a subring of its completion, $|R| = |R/J|$ so 6.5 applies to the ring R. Moreover, if R is any regular local ring with $|R/J| \geqq \aleph_\omega$, then $\text{hd}_R(Q) =$ the dimension of R since 5.2 enables us to find a directed module $M \subseteq Q$ containing $\aleph_{\dim R+1}$ elements of the form $1/(x_{i+1} - \alpha x_i)$ such that $\text{hd}_R(M) \leqq \text{hd}_R(Q)$. We then apply 6.4.

7. Directed modules with linearly ordered free generating sets.

In this section we generalize results in Osofsky [13].

7.1. LEMMA. *Let M be a directed module with a free generating set M' and upper bound function u. Assume no set of cardinality $\leqq \aleph_n$ generates M and every set*

$X \subseteq M'$ with $|X| = \aleph_n$ has an upper bound in M'. If $\mathrm{hd}_R(M) \leqq k$, *then there is a u-closed subset $Y \subseteq M'$ such that the smallest cardinality of a generating set for $P_{-1}(Y)$ is $\aleph_n$ and* $\mathrm{hd}_R(P_{-1}(Y)) \leqq k-1$.

Proof. By 5.2, there exists a u-closed set Y of cardinality $\aleph_n$ such that no set of cardinality $< \aleph_n$ generates $P_{-1}(Y)$ and $d_k P_k(Y)$ is a direct summand of $d_k P_k(M')$. Let z be an upper bound for Y. Then

$$P_{k-1}(M') = P_{k-1}(Y) \oplus \sum_{\text{some } x_i \notin Y} \langle x_0, \ldots, x_{k-1} \rangle R.$$

We may subtract any element in the second sum from each free generator of $P_{k-1}(Y)$ and still have a direct sum. In particular,

$$P_{k-1}(M') = d_k[z^* P_{k-1}(Y)] \oplus \sum_{\text{some } x_i \notin Y} \langle x_0 \cdots x_{k-1} \rangle R$$

and

$$d_k P_k(Y) \subseteq d_k[z^* P_{k-1}(Y)] \subseteq d_k P_k(M').$$

Hence $d_k P_k(Y)$ is a direct summand of a direct summand of $P_{k-1}(M')$. We then have $d_k P_k(Y)$ a direct summand of $P_{k-1}(Y)$, so $d_{k-1} P_{k-1}(Y)$ is projective.

7.2 Lemma (Dual basis lemma). *Let R be any ring, P a right R-module. Then P is projective $\Leftrightarrow$ there exists $\{x_i \mid i \in \mathscr{I}\} \subseteq P$ and $\{f_i \mid i \in \mathscr{I}\} \subseteq \mathrm{Hom}_R(P, R)$ such that for all $x \in P$, $f_i(x) = 0$ for all but a finite number of $i \in \mathscr{I}$, and $x = \sum_{i \in \mathscr{I}} x_i f_i(x)$.*

For a proof see Cartan and Eilenberg [2, p. 132].

7.3 Lemma. *Let R be a ring with no zero divisors, $M = \bigcup_{i=1}^{\infty} x_i R$ an R-module, $x_i R \supsetneqq x_j R$ for all $i \gneqq j$. Then M is not projective.*

Proof. Let $f\colon M \to R$, $f \neq 0$. Then there exists i such that $f(x_i) \neq 0$. Let $0 \neq x \in M$. Then there exists $j \geqq i$ such that $x = x_j r \in x_j R$. Since $x_i \in x_j R$, $f(x_j) \neq 0$ and $f(x) = f(x_j) r \neq 0$ since R has no zero divisors. Since M is a union of proper submodules, M cannot be finitely generated. Hence the dual basis property in 7.2 cannot hold for M.

7.4 Theorem. *Let R be a ring with no zero divisors, M a directed R-module with a linearly ordered set of free generators M'. Then* $\mathrm{hd}_R(M) = n+1 \Leftrightarrow$ *the smallest cardinality of a generating set for M is $\aleph_n$.*

Proof. If M is cyclic, $\mathrm{hd}_R(M) = 0$, and if M is countably generated but not cyclic, $\mathrm{hd}_R(M) \geqq 1$ by 7.3. Now assume the theorem for $n-1$. By 4.2, $\mathrm{hd}_R(M) \leqq n+1$. The linear ordering on M' insures upper bounds for all sets of cardinality $< \aleph_n$. By 7.1, there exists a u-closed subset $Y \subseteq M'$ such that the smallest cardinality of a generating set for $P_{-1}(Y)$ is $\aleph_{n-1}$ and $\mathrm{hd}_R(P_{-1}(Y)) \leqq \mathrm{hd}_R(M) - 1$. By the induction

hypothesis, $\mathrm{hd}_R\,(P_{-1}(Y))=n$, so $\mathrm{hd}_R\,(M)\geqq n+1$. If $\Rightarrow$ fails, 7.1 yields an $\aleph_n$-generated $P_{-1}(Y)$ of dimension $\leqq n$, contradicting the induction hypothesis.

7.5 COROLLARY. *Let R be a ring with no zero divisors and linearly ordered right ideals. If every ideal of R is finitely generated, the global dimension of $R=0$ or 1. Otherwise, the global dimension of $R=2+\sup\{n|R$ possesses an ideal generated by $\aleph_n$ but no fewer elements$\}$.*

Proof. This is an immediate consequence of 7.4 and the global dimension theorem (Auslander [1]).

7.6 COROLLARY. *Let R be a ring with no zero divisors possessing a linearly ordered set of left ideals $\{Rx_i \mid i\in\mathscr{I}\}$ such that for all $y\in R$ there is an $i\in\mathscr{I}$ with $Rx_i\subseteq Ry$. Let Q be the left quotient ring of R. Then $\mathrm{hd}_R\,(Q)=n+1$, where $\aleph_n$ is the cardinality of a smallest generating set for Q_R.*

Proof. By hypothesis, R must be a left Ore domain so $Q=\{x^{-1}r \mid 0\neq x\in R,\ r\in R\}$. For $y\in R$, let $ry=x_i$. Then $y^{-1}=x_i^{-1}r$, so $\{x_i^{-1} \mid i\in\mathscr{I}\}$ generate Q_R, and they are linearly ordered by $\leqq$. We now apply 7.4.

8. **Appendix.** Cohen has shown [6] that it is consistent to assume that $2^{\aleph_0}=\kappa$ for any cardinal κ which is not a countable union of smaller cardinals. By Corollary 6.7, if $\boldsymbol{R}=$the real numbers and $R=\boldsymbol{R}[X_1,\ldots,X_n]$ for $n\geqq 3$, then $\mathrm{hd}_R\,(Q)=k<n \Leftrightarrow 2^{\aleph_0}=\aleph_{k-1}$. In particular, $\mathrm{hd}_R\,(Q)=2\Leftrightarrow$ the continuum hypothesis holds. This is true $\Leftrightarrow d_2P_2((R-\{0\})^{-1})$ is projective and by a result of Bass [4], since $d_2P_2((R-\{0\})^{-1})$ is infinitely generated, it is projective $\Leftrightarrow$ it is free. In this appendix, we construct a free basis for $d_2P_2(M')$ for any directed module with $|M'|=\aleph_1$. Thus one can show constructively that $2^{\aleph_0}=\aleph_1\Rightarrow d_2P_2((R-\{0\})^{-1})$ has a free basis. The reverse implication, however, depends on the number of variables in the polynomial ring. If $S=\boldsymbol{R}[X,\ Y]$, $d_2P_2((S-\{0\})^{-1})$ always has a free basis, regardless of the cardinality of $\boldsymbol{R}$.

For the rest of this appendix, M will denote a directed module with free generators M'.

A.1 LEMMA. *Assume M is countably generated. Then $d_1P_1(M')$ has a free basis of the form $\{d_1\langle a,b\rangle\}$.*

Proof. Since M is countably generated, there exist $\{x_i \mid i\in\omega\}\subseteq M'$ such that $x_0<x_1<\cdots$ and $M=\sum_{i=0}^{\infty}x_iR$. For each $y\in M'$, let $x(y)$ denote the x_i with smallest index i such that $y\in x_iR$. We show $\{d_1\langle x(y),y\rangle \mid y\in M'\}$ is a free basis for $d_1P_1(M')$.

Let $\sum_{j=1}^{n}d_1\langle x(y_j),y_j\rangle r_j=0$, all $r_j\neq 0$, and assume $x(y_n)=x_k$ is the largest x_i occurring. Since $r_n\neq 0$, $x(y_n)$ must appear in another pair $\langle x_k,y_j\rangle$, so in at least one of its appearances, the second component $y_j\neq x_{k-1}$. But then $\langle y_j\rangle r_j$ is a term of $d_1\langle x_k,y_j\rangle$ but of no other $d_1\langle x(y_l),y_l\rangle$, so the sum cannot be zero.

Let $\langle a, b\rangle$ be a generator for $P_1(M')$. Then $d_2\langle x(a), a, b\rangle = \langle a, b\rangle - \langle x(a), b\rangle + \langle x(a), a\rangle a^{-1}b$ and since $\mathscr{P}_{M'}$ is a complex,

$$d_1\langle a, b\rangle = d_1\langle x(a), b\rangle - d_1\langle x(a), a\rangle a^{-1}b.$$

Let $x(a) = x_l$, $x(b) = x_k$. Then

$$d_1\langle x(a), b\rangle = d_1\langle x(b), b\rangle + \sum_{i=k}^{l-1} d_1\langle x_{i+1}, x_i\rangle x_i^{-1}b.$$

Thus every generator for $d_1P_1(M')$ may be expressed as a linear combination of elements in the given set.

A.2 PROPOSITION. *If M is $\aleph_1$-generated, then $d_2P_2(M')$ is free.*

Proof. Since M is $\aleph_1$-generated, there exist u-closed subsets $\{T_\alpha \mid \alpha < \aleph_1\} \subseteq M'$ such that $M' \cap P_{-1}(T_\alpha) \subseteq T_\alpha$ for all α, $P_{-1}(T_\alpha)$ is countably generated, $T_\alpha \subset T_\beta$ for $\alpha < \beta$, and $M = \bigcup_{\alpha<\aleph_1} P_{-1}(T_\alpha)$. Set $T_{-1} = \varnothing$. It is sufficient to show

$$d_2P_2(T_\alpha)/d_2P_2(T_{\alpha-1})$$

has a free basis for all $\alpha < \aleph_1$, α a successor ordinal.

By the conditions on the T_α, there exist $\{x_i \mid i \in \omega\} \subseteq T_\alpha - T_{\alpha-1}$ such that $P_{-1}(T_\alpha) = \bigcup_{i=0}^{\infty} x_iR$. For $z \in T_{\alpha-1}$, $y \in T_\alpha$, define $x_{\alpha-1}(z)$ and $x_\alpha(y)$ as in A.1. Then

$$P_1(T_\alpha) = \sum_{y\in T_\alpha} \langle x_\alpha(y), y\rangle R \oplus \sum_{z\in T_{\alpha-1}} \langle x_{\alpha-1}(z), z\rangle R \oplus \sum_{u\in T_{\alpha-1};\, u\neq x_{\alpha-1}(v)} \langle u, v\rangle R$$
$$\oplus \sum_{a\in T_\alpha - T_{\alpha-1};\, a\neq x_\alpha(b)} \langle a, b\rangle R.$$

For each $\langle a, b\rangle$ with $a = x_{\alpha-1}(b)$ or $a \in T_\alpha - T_{\alpha-1}$ and $a \neq x_\alpha(b)$ there exists a unique element $p = \sum \langle x_\alpha(y_i), y_i\rangle r_i$ such that $d_1\langle a, b\rangle = d_1p$. Then $\langle a, b\rangle - p = d_2q_{a,b}$ for some $q_{a,b} \in P_2(T_\alpha) - P_2(T_{\alpha-1})$. Then $\mathfrak{F} = \{d_2q_{a,b} \mid a = x_{\alpha-1}(b)$ or $a \in T_\alpha - T_{\alpha-1}$, $a \neq x_\alpha(b)\}$ is a free basis for $d_2P_2(T_\alpha)/d_2P_2(T_{\alpha-1})$.

$\mathfrak{F}$ is independent since $\{\langle a, b\rangle\}$ is independent in $P_2(T_\alpha)$ modulo the first and third sums, and the image of $d_2q_{a,b}$ = the image of $\langle a, b\rangle$ in that module.

To show $\mathfrak{F}$ spans, we need only show that, for all $u \in T_\alpha - T_{\alpha-1}$, $d_2\langle u, v, w\rangle$ is a linear combination of elements in $\mathfrak{F}$ and an element in $d_2P_2(T_{\alpha-1})$.

$$d_2\langle u, v, w\rangle = \langle v, w\rangle - \langle u, w\rangle + \langle u, v\rangle v^{-1}w.$$

Let

$$\begin{aligned} \bar{q}_{a,b} &= q_{a,b} \qquad a = x_{\alpha-1}(b) \quad \text{or} \quad a \in T_\alpha - T_{\alpha-1},\ a \neq x_\alpha(b), \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

If $q_{v,w}$ is defined or $v = x_\alpha(w)$, consider $d_2(\langle u, v, w\rangle - \bar{q}_{v,w} + \bar{q}_{u,w} - \bar{q}_{u,v}v^{-1}w)$. This is an element of $\sum_{y\in T_\alpha} \langle x_\alpha(y), y\rangle R$ in the kernel of d_1. By A.1, it must be 0, so

$$d_2(\langle u, v, w\rangle) = d_2\bar{q}_{v,w} - d_2\bar{q}_{u,w} + d_2\bar{q}_{u,v}v^{-1}w.$$

If $v \in T_{\alpha-1}$, $v \neq x_{\alpha-1}(w)$, we apply A.1 to $d_1P_1(T_{\alpha-1})$ to express $d_1\langle v, w\rangle$ uniquely as a sum $\sum d_1\langle x_{\alpha-1}(y_i), y_i\rangle r_i$. Then $p = \langle v, w\rangle - \sum \langle x_{\alpha-1}(y_i), y_i\rangle r_i \in d_2P_2(T_{\alpha-1})$ and $d_2\langle u, v, w\rangle = -d_2\bar{q}_{u,w} + d_2\bar{q}_{u,v}v^{-1}w + p + \sum d_2q_{x_{\alpha-1}(y_i),y_i}r_i$.

We note that if M is countably generated, this construction yields a basis for $d_2P_2(M')$ by taking a smaller union (or even setting $M' = T_0$).

In the above construction of a free basis, the ring never appears—just the generators M'. This is not surprising in view of 4.2—the ring is not used in that proof either. By 6.7 or 7.4 it is impossible to find a free basis for $d_2P_2(M')$ expressed only in terms of elements of M' if M is not at most $\aleph_1$-generated. If there is no 1-1 map between $R = \boldsymbol{R}[X, Y]$ and $\aleph_1$, the free basis for $d_2P_2(Q - \{0\})$, which exists since gl. d. $R = 2$ and infinite R-projectives are free, must depend heavily on the ring R. If one adds another variable to get $R' = \boldsymbol{R}[X, Y, Z]$, the description of $d_2P_2(Q' - \{0\})$ is formally the same but no free basis exists.

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