ON DIMENSION OF SUPPORT FOR STOCHASTIC PROCESSES WITH INDEPENDENT INCREMENTS[1,2]

BY

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Introduction. Herman Rubin [5] showed that if $v(t)$ is a real valued function which is monotone nondecreasing on the interval $(0, \alpha)$, where α is possibly $+\infty$, and if $0 \leqq v(t) \leqq 1$, then there exists a "pure" stochastic process $\{X_t : t \in [0, \infty)\}$ (i.e. for $t \in (0, \infty)$ the distribution function of X_t, denoted by F_t, is either absolutely continuous or continuous singular) with stationary independent increments such that if $t < \alpha$, then F_t is continuous singular with the dimension of its support, denoted by dim supp F_t, being $v(t)$; and if $t \leqq \alpha$, then F_t is absolutely continuous. This can be interpreted as saying that for a "pure" process with stationary independent increments, the only general properties the dimension of the support of F_t has to have as a function of t, are the obvious ones of being monotone nondecreasing and being bounded between 0 and 1.

One is now led to remove the restriction that the process be "pure," to isolate one's attention to the continuous singular component of F_t, denoted by (F_t)c.s., and to ask if a similar result to that of Rubin's holds for dim supp (F_t)c.s. The answer is yes. In §1 of this paper, it is shown that if $\{X_t : t \in [0, \infty)\}$ is a stochastic process with independent increments then

$$\text{dim supp } (F_t)\text{c.s.} = \liminf_{\tau \to t} \text{dim supp } (F_\tau)\text{c.s.}$$

for all $t \in (0, \infty)$ except at possibly a countable number of points. Conversely, in §2 it is shown that if f maps $(0, \infty)$ into $[0, 1]$ and is a lower semicontinuous function at all but possibly a countable number of points, then there exists a stochastic process $\{X_t : t \in [0, \infty)\}$ with stationary independent increments such that dim supp (F_t)c.s. $= f(t)$.

These two results are obtained while investigating the relationship between the total variation of (F_t)c.s., denoted by T.V.(F_t)c.s., and dim supp (F_t)c.s. for a process with independent increment.

At this point I wish to express my gratitude to Professor Howard G. Tucker for

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I. In this section we establish the following. Let $\{X_t : t \in [0, \infty)\}$ be a stochastic process with independent increments. A theorem is proved concerning how T.V.(F_t)c.s., having the analytic property of being continuous from the right at t_0, affects the function $v(t) \doteq \dim \text{supp } (F_t)$c.s. at t_0. Furthermore it is proved that $\dim \text{supp } (F_t)$c.s. $= \liminf_{\tau \to t} \dim \text{supp } (F_\tau)$c.s. except at possibly a countable number of points. Secondly for processes with stationary independent increments a relation along the lines of the obvious one that if T.V.(F_t)c.s.$\cdot$T.V.(F_d)c.s.$=$ T.V.(F_{t+d})c.s. then $\dim \text{supp } (F_{t+d})$c.s. $\geqq \max (\dim \text{supp } (F_t)$c.s., $\dim \text{supp } (F_d)$c.s.$)$ is proved.

For completeness we give the following definitions:

NOTATION. Let A be a Lebesgue measurable subset of the real line; then $|A|$ shall denote the Lebesgue measure of A.

DEFINITION. Let A be a subset of the real line, then the *Hausdorff dimension of A*, $\dim (A)$ is defined as follows:

Let
$$\mathscr{I}_\xi(A) \doteq \{\{I_i\} : I_i \text{ is an open interval}, \bigcup I_i \supset A, \text{ and } |I_i| < \xi\},$$
let $u_\xi^\alpha(A) = \inf \{\sum_{i=1}^\infty |I_i|^\alpha : \{I_i\}_{i=1}^\infty \in \mathscr{I}_\xi(A)\}$, and define
$$\dim (A) = \inf \left\{\alpha : \lim_{\xi \to 0} u_\xi^\alpha(A) = 0\right\}.$$

NOTATION. (The word "Borels" will always denote in the sequel the sigma algebra of Borel sets.) Let G be a nondecreasing function, and let $A \in$ Borels; then $G(A)$ will denote the value that the measure induced by G on the Borels assigns to A. Also let (G)c.s., (G)a.c. and (G)d. denote the continuous singular, absolutely continuous, and discrete components of G respectively.

DEFINITION. Let G be a nondecreasing function; the *dimension of the support of G*, $\dim \text{supp } G$, is defined as
$$\dim \text{supp } G = \inf \{\dim (A) : A \in \text{Borels}, G(A) = \text{T.V.}(G)\}.$$

NOTATION. Let $t \in [0, \infty)$; then let
$$[t] = \max \{n \in I^+ \cup \{0\} : n \leqq t\}.$$

We shall assume the following six facts which can be proved by straightforward elementary techniques.

A.1. If $\{A_i\}$ is a countable collection of sets, then
$$\dim \left(\bigcup_i A_i\right) = \max_i \dim (A_i).$$
Also if A is a set and $x \in (-\infty, \infty)$, then
$$\dim (A) = \dim (A+x).$$

A.2. Let $\{G_t(x) : t \in \text{Reals}\}$ be a collection of distribution functions in x such that for fixed x, $G_t(x)$ is a Borel measurable function in t. Let m be a bounded positive measure on the Borels and let $V(x) = \int_{-\infty}^{\infty} G_t(x)\, dm(t)$. If A is a Borel set, then $G_t(A)$ is a Borel measurable function in t and

$$V(A) = \int_{-\infty}^{\infty} G_t(A)\, dm(t).$$

Also if $V(A) = \text{T.V.}(m)$, then $G(A) = 1$ a.e.$[m]$.

A.3. If G and F are nondecreasing functions, then dim supp $G * F \geqq$ dim supp F.

A.4. If G and F are nondecreasing functions, and G is discrete, then

$$\text{dim supp } G * F = \text{dim supp } F.$$

A.5. Let $\{G_n\}$ be a collection of nondecreasing functions such that $G(x) = \sum_n G_n(x) < \infty$ for every x. If V is a nondecreasing function then $G * V = \sum_n G_n * V$ and dim supp $G = \max_n$ dim supp G_n.

A.6. Let $\{F_n\}$ be a sequence of real valued bounded nondecreasing functions which are continuous from the right at every point $x \in (-\infty, \infty)$. Then $F_n \overset{(c)}{\longrightarrow} F_0$ if and only if for every Borel set A such that $F_0(\text{boundary }(A)) = 0$, then $F_n(A) \overset{\longrightarrow}{_n} F(A)$ ($F_n \overset{(c)}{\longrightarrow} F_0$ means $F_n(x) \to F_0(x)$ for all continuity points of F_0).

LEMMA 1.1. *Let $G(x)$ be a nondecreasing function in x, and assume* dim supp $G > \alpha$. *Then there exists a $\delta_\alpha > 0$ such that if A is a Borel set and* $\dim(A) \leqq \alpha$, *then* $G(A) < \text{T.V.}G - \delta_\alpha$.

Proof. Suppose not. Then clearly there exists a countable sequence $\{A_i\}$ of Borel sets such that $\dim(A_i) \leqq \alpha$ for each i, and $G(A_i) \overset{\longrightarrow}{_i} \text{T.V.}(G)$. By the continuity of measures we see that $G(\bigcup_i A_i) = \text{T.V.}(G)$. But by A.1. we see that $\dim(\bigcup_i A_i) \leqq \alpha$ which, by the definition of dim supp G, implies that dim supp $G \leqq \alpha$, which is a contradiction.

For the rest of this section we adopt the following notation:

Let $\{X_t : t \in [0, \infty)\}$ be a stochastic process; then

$$m(t) = \text{T.V.}(F_t)\text{c.s.}, \qquad m_a(t) = \text{T.V.}(F_t)\text{a.c.}, \qquad m_d(t) = \text{T.V.}(F_t)\text{d.}$$

and if $s > t$

$$F_{t,s} = F_{X_s - X_t}, \qquad m(t, s) = \text{T.V.}(F_{t,s})\text{c.s.}, \qquad m_a(t, s) = \text{T.V.}(F_{t,s})\text{a.c.},$$

$$m_d(t, s) = \text{T.V.}(F_{t,s})\text{d.}, \qquad v(t) = \text{dim supp }(F_t)\text{c.s.}$$

Let $\{X_t : t \in [0, \infty)\}$ be a stochastic process with independent increments. So if $s, t \in (0, \varepsilon)$ and $s > t$, then

$$F_s = ((F_t)\text{c.s.} + (F_t)\text{a.c.} + (F_t)\text{d.}) * ((F_{t,s})\text{c.s.} + (F_{t,s})\text{a.c.} + (F_{t,s})\text{d.}).$$

From this equality it is easily seen that the following key relations hold.

R.1. (F_s)a.c. $=(F_t)$a.c. $* (F_{t,s})+((F_t)$c.s. $+(F_t)$d.$) * (F_{t,s})$a.c. $+((F_t)$c.s. $* (F_{t,s})$c.s.$)$a.c.,

R.2. (F_s)c.s. $=(F_t)$c.s. $* (F_{t,s})$d. $+(F_{t,s})$c.s. $* (F_t)$d. $+((F_t)$c.s. $* (F_{t,s})$c.s.$)$c.s.,

R.3. (F_s)d. $=(F_t)$d. $* (F_{t,s})$d.

THEOREM. *Let* $\{X_t : t \in [0, \infty)\}$ *be a stochastic process with independent increments. Let* $d_0 =$ l.u.b. $\{x \in [0, \infty) : m_d(x) \neq 0\}$. *Under these assumptions the function* $v(t)$ *has the following properties*:

1. $v(t)$ *is monotone increasing on* $(0, d_0)$;
2. *if* $t \in (d_0, \infty)$ *is such that* $m(t+0)=m(t)$, *then* $v(t) \leq \lim \inf_{s \to t^+} v(s)$.
3. $v(t)=\lim \inf_{s \to t} v(s)$ *for all but a countable number of points.*

Proof. In order to establish property 1, we first show that if $t \in (0, d_0)$ then $m_d(t) \neq 0$. So suppose there exists $t \in (0, d_0)$ such that $m_d(t)=0$. Let $s \in (t, d_0]$. Then by R.3 we see that $m_d(s)=0$. This contradicts the definition of d_0 and establishes the fact that if $t \in (0, d_0)$ then $m_d(t) \neq 0$.

In order to complete the proof of property 1, we let $s_1, s_2 \in (0, d_0)$ with $s_1 > s_2$. By the definition of d_0 we see that $m_d(s_2) \neq 0$ and therefore by R.3 we see that $m_d(s_2, s_1) \neq 0$.

Now recalling R.2, A.4 and A.5 we see that $v(s_2) > v(s_1)$ which shows property 1 holds.

To establish property 2 we let $t \in (d_0, \infty)$ such that $m(t+0)=m(t)$. By the definition of d_0 we see that $m_d(t)=0$. So referring to R.2 we see that for $A \in$ Borels

$$(F_s)\text{c.s.}(A) \leq (F_t)\text{c.s.} * (F_{t,s})\text{d.}(A)+(F_t)\text{c.s.} * (F_{t,s})\text{c.s.}(A)$$

(1.1)

$$\leq \int_{-\infty}^{\infty} (F_t)\text{c.s.}(A-x) \, d((F_{t,s})\text{c.s.} +(F_{t,s})\text{d.})(x).$$

The conclusion of the theorem not holding at t means that there exists a real number β and a sequence of real numbers $\{s_j\}$ such that for every $j \in I^+$ we have $v(s_j) < \beta < v(t)$ and $s_j \downarrow t$. For $j \in I^+$, we see that by the definition of dim supp there exists a set A_j such that

(1.2) (F_{s_j})c.s.$(A_j) = m(s_j)$ and $\dim (A_j) < \beta$.

Let

$$A = \bigcup_{j=1}^{\infty} A_j.$$

By A.1. $\dim (A) \leq \beta$, and so by Lemma 1.1 we see that, for every $x \in (-\infty, \infty)$,

(1.3) (F_t)c.s.$(A-x) < m(t)-\delta_\beta$ for some $\delta_\beta > 0$.

Now by (1.2) (F_{s_j})c.s.$(A)=m(s_j)$ for all $j \in I^+$ and therefore by (1.1), (1.3) and the fact that $m(t, s)+m_d(t, s) \leq 1$, we see that

(1.4) $m(s_j) \leq m(t)-\delta_\beta.$

Letting $j \to \infty$ on both sides of the inequality in (1.4) we see that, by the fact that $m(t) = m(t+0)$, $m(t) \leq m(t) - \delta_\beta$. But $\delta_\beta > 0$, which gives a contradiction and the completion of the proof of 2.

Now to prove 3 of this theorem we note the following. R.3 and the fact that $m_d(t, s) \leq 1$ implies $m_d(t)$ is nonincreasing for increasing t, and therefore $m_d(t)$ is continuous from the right at all except possible countably many points t. R.1 clearly implies that $m_a(t)$ is a monotone increasing function and therefore is continuous from the right for all except possibly countably many points t. But $1 - m_a(t) - m_d(t) = m(t)$ and therefore m is continuous from the right for all except possibly countably many points t. So by 2 of this theorem

$$v(t) \leq \liminf_{s \to t+} v(s)$$

holds for all except possibly countably many points t.

Saks ([6], page 261) shows that any real valued function F has the property that the set $\{x : \liminf_{t \to x} F(t) < \lim_{t \to x+} F(t)\}$ is at most denumerable. Therefore we see that $v(t) \leq \liminf_{s \to t} v(s)$ for all but a countable number of points t. Showing that this in turn implies that $v(t) = \lim_{s \to t} v(s)$ at all but a countable number of points t is a standard argument and is therefore omitted. This then completes the proof of the theorem.

II. In this section we establish the following theorem:

THEOREM. *Let a function $m(t)$ which maps, $[0, \infty)$ into $[0, 1]$, be such that*
1. *$m(t+d) \geq m(t)m(d)$ for $t, d \in [0, \infty)$,*
2. *there exists $t^* \in (0, \infty)$ such that $m(t^*) = 0$ or 1, and*
3. *on the interval $\{x \in (0, \infty) : m(x) \neq 0$ and $\ln m(t)/t \neq m'(0)\}$, $\ln m(t)/t$ is a strictly increasing function.*

Let a function $v(t)$ which maps $(0, \infty)$ into $[0, 1]$ be such that
1. *$v(t) = 0$ on the interval $B = \{t \in (0, \infty) : m(t) = 0\}$,*
2. *$v(t)$ is monotone increasing on the interval $A = \{t \in (0, \infty) : m(t)/t = m(0)\}$, and*
3. *$v(t) \leq \lim_{s \to t+} v(s)$ if $t \in (0, \infty) - B$ and $m(t) = m(t+0)$.*

Then there exists a stochastic process, $\{X_t : t \in [0, \infty)\}$, with stationary independent increments such that for $t \in (0, \infty)$: T.V.(F_t)c.s. $= m(t)$ and $\dim \operatorname{supp} (F_t)$c.s. $= v(t)$.

As a corollary to the above result we find that if α is a real extended number and if f is a function on $(0, \alpha)$ (or $(0, \alpha]$) which is lower semicontinuous at all but a countable number of points and is bounded between 0 and 1, then there exists a stochastic process $\{X_t : t \in [0, \infty)\}$, with stationary independent increments such that $f(t) = \dim \operatorname{supp} (F_t)$c.s. for $t \in (0, \alpha)$ (or $(0, \alpha]$) and F_t is absolutely continuous for $t \geq \alpha$ (or $t > \alpha$).

In the first two of the following eight lemmas, let $\{M_n\}$ be a sequence of bounded negative functions such that $M_n(s)$ increases with increasing x, $\sum_n M_n(x) = M(x) > -\infty$ for every $x \in (0, \infty)$, and $\int_0^1 x \, dM(x) < \infty$. Let $\lambda_n = $ T.V.(M_n) and let

$F_{n,m}(x)$ be the distribution function corresponding to the characteristic function

$$\exp\left(\int_{0^+}^{\infty}(\exp(i\mu x)-1)\,d\left\{\sum_{j=n+1}^{m}M_j(x)\right\}\right)$$

where $m > n$, $n \in \{0\} \cup I^+$, and $m \in I^+ \cup \{+\infty\}$.

LEMMA 2.1. *Let* $k \in I^+$; *then there exists a nondecreasing function* $F_k'(x)$ *such that if* $n > k$

$$F_k^n(x) \doteq \left\{\exp\left(-\sum_{i=0}^{n}\lambda_i\right)\right\}\sum_{i=0}^{\infty}\left(\sum_{j=k}^{n}M_j\right)^{*i}(x)/i! \xrightarrow{(c)} F_k'(x),$$

$$\text{T.V.}(F_k') = \exp\left(-\sum_{i=1}^{k-1}\lambda_i\right), \quad and \quad F_k^n * F_{n,\infty}(x) = F_k'(x).$$

Proof. Let us denote the characteristic function of F by ch.fcn.(F). Clearly

$$\text{ch.fcn.}(F_k^n) = \exp\left(-\sum_{i=0}^{n}\lambda_i\right)\exp\left(\int_{0^+}^{\infty}(\exp(iux))\,d\left(\sum_{j=k}^{n}M_j(x)\right)\right).$$

But T.V.$(\sum_{j=k}^{n}M_j) = \sum_{j=k}^{n}\lambda_j$ which implies

$$\text{ch.fcn.}(F_k^n) = \exp\left(-\sum_{i=1}^{k-1}\lambda_i\right)\text{ch.fcn.}(F_{k-1,n}).$$

By definition $\sum_{j=k}^{n}M_j(x) \xrightarrow[n]{} \sum_{j=k}^{\infty}M_j(x)$ and therefore by [3]

$$\text{ch.fcn.}(F_k^n) \xrightarrow[n]{} \text{ch.fcn.}(F_{k-1,\infty})\exp\left(-\sum_{j=0}^{k-1}\lambda_j\right).$$

So let

$$F_k'(x) = \left\{\exp\left(-\sum_{i=0}^{k-1}\lambda_i\right)\right\}F_{k-1,\infty}(x).$$

Therefore

$$F_k^n(x) \xrightarrow{(c)} F_k'(x).$$

But T.V.$(F_k^n) = \exp(-\sum_{i=0}^{k-1}\lambda_i)$ and so by the definition of complete convergence T.V.$(F_k') \doteq \exp(-\sum_{i=0}^{k-1}\lambda_i)$.

The form of the characteristic function of $F_{k-1,\infty}$ clearly implies that for $m > k$, $F_{k-1,\infty} = F_{k-1,m} * F_{m,\infty}$. Now multiplying both sides of this equation by $\exp(-\sum_{j=0}^{k-1}\lambda_j)$ we get the last statement of the lemma and the completion of the proof.

LEMMA 2.2. *Let* V *be a continuous distribution function and* $k \in I^+$. *For each* $n \in I^+$, *let* $\mathscr{I}_n' = \{I_{j,n}' : j \in I^+\}$ *and* $\mathscr{I}_n = \{I_{j,n} : j \in I^+\}$ *be two collections of open intervals where the collection of endpoints from intervals in* $\mathscr{I}_n$ *has no limit points, and* $V[\bigcup_{n=N}^{\infty}\bigcup_{j=1}^{\infty}I_{j,n}'] = 1$ *for every* $N \in I^+$. *If*

1. *there exists an increasing sequence* $\{j_n\}_{n=1}^{\infty} \subset I^+$ *such that* $\sum_{v=1}^{\infty}(\sum_{n=1}^{v}\lambda)^{j_v}/j_v! < \infty$, *and if*

2. *there exists a sequence of real numbers* $\{\xi_n\}$ *where* $\xi_n \downarrow 0$ *and an unbounded sequence* $\{n_i\} \subset I^+$ *such that for* $n, N, m \in I^+$ *and* $m \geq N$:

$$\underset{s=k}{\overset{n}{\ast}} M_s^{\ast e_s} \ast V\left[\bigcup_{n=N}^{m} \bigcup_{j=1}^{\infty} I_{j,n}\right] \geq (1-\xi_N) \prod_{s=k}^{n} \lambda_s^{e_s} V\left[\bigcup_{n=N}^{m} \bigcup_{j=1}^{\infty} I'_{j,n}\right]$$

whenever $\sum_{s=k}^{l} e_s < j_l$ *for each* $l = n_N, \ldots, n$, *then, for every* $r \in I^+$,

$$F'_k \ast V\left[\bigcup_{n=r}^{\infty} \bigcup_{j=1}^{\infty} I_{j,n}\right] = \exp\left(-\sum_{i=1}^{k-1} \lambda_i\right) = \text{T.V.}[F'_k \ast V].$$

Proof. Let $U_{N,m} = \bigcup_{n=N}^{m} \bigcup_{j=1}^{\infty} I_{n,j}$, and let $U'_{N,m} = \bigcup_{n=N}^{m} \bigcup_{j=1}^{\infty} I'_{n,j}$ where $m \geq N$ and m is possibly $+\infty$. Now in order to prove this lemma, it is clearly sufficient to show that for $\varepsilon > 0$, there exists $N(\varepsilon) \in I^+$ such that $N \geq N(\varepsilon)$ implies

1. $F'_k \ast V(U_{N,\infty}) \geq \exp\left(-\sum_{i=1}^{k-1} \lambda_i\right) - \varepsilon$.

By hypothesis V is a continuous distribution function and therefore $F'_k \ast V$ is a continuous distribution function. So if $m \neq +\infty$, then

$$F'_k \ast V(\text{Boundary } (U_{N,m})) = F'_k \ast V(\text{countable set}) = 0$$

and therefore by A.6

$$F_n^k \ast V(U_{N,m}) \xrightarrow[k]{} F'_k \ast V(U_{N,m}).$$

But from this and the hypothesis that $V(U'_{N,\infty}) = 1$ for every $N \in I^+$, we see that in order to prove 1 it is sufficient to verify the following:

If $\varepsilon > 0$, then there exists $N(\varepsilon)$ such that $N \geq N(\varepsilon)$ and $n \geq \max (n_N, k)$ imply

$$F_n^k \ast V(U_{N,m}) \geq \exp\left(-\sum_{i=1}^{k-1} \lambda_i\right) V(U'_{N,m}) - \varepsilon.$$

So let $\varepsilon > 0$. Since $\{n_i\}$ is an unbounded sequence of integers we see from hypothesis 1 that there exists $N' \in I^+$ such that if $N \geq N'$ then

$$\exp\left(\sum_{s=1}^{k-1} \lambda_s\right) \sum_{v=n_N}^{\infty} \left(\sum_{n=1}^{v} \lambda_n\right)^{j_v} \Big/ j_v! < \varepsilon/2.$$

Since $\varepsilon_N \downarrow 0$ there exists $N_1 \in I^+$ such that if $N \geq N_1$ then

$$\exp\left(-\sum_{i=1}^{k-1} \lambda_i\right)\varepsilon_N < \varepsilon/2.$$

Let $N(\varepsilon) = \max (N_1, N')$ and let $N \geq N_\varepsilon$. Finally let $n \geq \max (n_N, k)$. Let

$$R_i = \left\{e \in \underset{s=k}{\overset{n}{\times}} \{1, \ldots, i\}_s : \sum_{s=k}^{n} e_s = i\right\},$$

$$R'_{i,r} = \left\{e \in R_i : \text{for each } l = n_N, \ldots, r; \sum_{s=k}^{l} e_s < j_l\right\},$$

$$R_{i,r} = \left\{e \in R_i : \text{for some } l, n_N \leq l \leq r, \sum_{s=k}^{l} e_s \geq j_l\right\}.$$

Now clearly $R_{i,r} \cap R'_{i,r} = \varnothing$ and $R_{i,r} \cup R'_{i,r} = R_i$. So

$$F_n^k * V(U_{N,m}) = \exp\left(-\sum_{i=1}^{n} \lambda_i\right)\left(\sum_{i=0}^{\infty}\left(\sum_{j=k}^{n} M_j\right)^{*i} * V/i!\right)(U_{N,m})$$

$$\geqq \exp\left(-\sum_{i=1}^{n} \lambda_i\right)\left(\sum_{i=1}^{j_{n_N}-1}\left(\sum_{j=k}^{n} M_j\right)^{*i} * V/i!\right.$$

$$+ \sum_{r=n_N}^{n-1} \sum_{i=j_r}^{j_{r+1}-1} (V/i!) * \sum_{e\in R'_{i,r}} \underset{s=k}{\overset{n}{\LARGE *}} M_s^{*e_s}\left(\frac{1-\sum_{t=k}^{s-1} e_t}{e_s}\right)(U_{N,m})$$

$$\geqq (1-\varepsilon_N)V(U'_{N,m})$$

$$\times \left\{\exp\left(-\sum_{i=k}^{n} \lambda_i\right) - \exp\left(-\sum_{i=1}^{n} \lambda_i\right)\right.$$

$$\times\left(\sum_{i=j_n}^{\infty}\left(\sum_{s=k}^{n} \lambda_s\right)^i\Big/i! - \sum_{r=n_N}^{n-1} \sum_{i=j_r}^{j_{r+1}-1} (1/i!)\sum_{e\in R_{i,r}}\prod_{s=k}^{n} \lambda_s^{e_s}\left(\frac{1-\sum_{t=k}^{s-1} e_t}{e_s}\right)\right)\right\}.$$

First let us note that by Taylor's theorem

$$\sum_{i=j_n}^{\infty}\left(\sum_{s=k}^{n} \lambda_s\right)^i\Big/i! \leqq \exp\left(\sum_{s=k}^{n} \lambda_s\right)\left(\sum_{s=k}^{n} \lambda_s\right)^{j_n}\Big/j_n!.$$

But recall that $n \geqq n_N$, and therefore

$$\exp\left(-\sum_{s=1}^{n} \lambda_s\right)\sum_{i=j_n}^{\infty}\left(\sum_{s=k}^{n} \lambda_s\right)^i\Big/i! < \varepsilon/2.$$

Now let $n-1 \geqq r \geqq l$ and $P \in I^+$ be such that $j_l \leqq P < j_{l+1}$, and let

$$R_{i,r}^l = \left\{e \in R_{i,r} : j_{l+1} > \sum_{s=k}^{l} e_s \geqq j_l\right\},$$

$$R_{i,r}^l(P) = \left\{e \in R_{i,r}^l : \sum_{s=k}^{l} e_s = P\right\}.$$

Clearly if $j_r \leqq i \leqq j_{r+1}$, then

$$\bigcup_{l=n_N}^{r} R_{i,r}^l = R_{i,r} \quad \text{and} \quad \bigcup_{P=j_l}^{j_{l+1}-1} R_{i,r}^l(P) = R_{i,r}^l$$

and therefore

$$\sum_{e\in R_{i,r}}\prod_{s=k}^{n} \lambda_s^{e_s}\left(\frac{i-\sum_{t=k}^{s-1} e_t}{e_s}\right) \leqq \sum_{l=n_N}^{r} \sum_{P=j_l}^{j_{l+1}-1} \sum_{e\in R_{i,r}^l(P)}\prod_{s=k}^{n} \lambda_s^{e_s}\left(\frac{i-\sum_{t=k}^{s-1} e_t}{e_s}\right).$$

But an easy computation shows that

$$\sum_{e\in R_{i,r}^l(P)}\prod_{e=k}^{n} \lambda_s^{e_s}\left(\frac{i-\sum_{t=k}^{s-1} e_t}{e_s}\right) = \binom{i}{P}\left(\sum_{s=k}^{i} \lambda_s\right)^P\left(\sum_{s=l+1}^{n} \lambda_s\right)^{i-P}.$$

This implies that

$$\sum_{r=n_N}^{n} \sum_{i=j_r}^{j_{r+1}-1} (1/i!) \sum_{e \in R_{1,r}} \prod_{s=k}^{n} \lambda_s^{e_s} \binom{i - \sum_{t=k}^{s-1} e_t}{e_s}$$

$$\leq \sum_{r=n_N}^{n} \sum_{i=j_r}^{j_{r+1}-1} (1/i!) \sum_{l=n_N}^{r} \sum_{P=j_l}^{j_{l+1}-1} \binom{i}{P} \left(\sum_{s=k}^{l} \lambda_s\right)^P \left(\sum_{s=l+1}^{n} \lambda_s\right)^{i-P}$$

$$= \sum_{l=n_N}^{n} \sum_{P=j_l}^{j_{l+1}-1} (1/P!) \left(\sum_{s=k}^{l} \lambda_s\right)^P \sum_{i=P}^{j_{n+1}-1} \left(\sum_{s=l+1}^{n} \lambda_s\right)^{i-P} \Big/ (i-P)!$$

$$\leq \sum_{l=n_N}^{n} \left(\sum_{s=k}^{l} \lambda_s\right)^{j_l} \Big/ j_l! \exp\left(\sum_{s=k}^{l} \lambda_s\right) \exp\left(\sum_{s=l+1}^{n} \lambda_s\right)$$

$$< (\varepsilon/2)$$

which clearly completes the proof of the lemma.

The next three lemmas and the conditions on the b_n's are taken from Herman Rubin's technical report [5]. The statements of lemmas three and four and the methods used in their proofs appear here essentially as they appear in his report, and are included here for two reasons: 1. for completeness, and 2. because the source is relatively obscure. The statement of lemma five is presented here as it is found in his report, except that I found I needed the additional hypothesis that $\limsup_n b_n/b_{n+1} = 0$. His proof appears to be incorrect, and I present here a proof which is probably what Rubin had in mind.

In the next three lemmas, let $\{b_n\} \subset I^+$ be such that for n sufficiently large: (1) $b_{n+3}/b_n > k > 1$, and (2) $b_{n+1} - b_n \geq n$. Also let N be an infinite collection of positive integers.

Let $x \in (-\infty, \infty)$, let x be expanded into its binary form, and let π_i map x onto the zero or the one appearing in the ith place to the right of the decimal. Let $n \in I^+$, and let

$$A_n = \{x \in (-\infty, \infty) : \pi_i x = 0 \text{ for } b_n \leq i \leq b_{n-1} - [n/2]\}$$

$$A'_n = \{x \in (-\infty, \infty) : \pi_i x = 0 \text{ for } b_n \leq i \leq b_{n+1} - [n/2] - 1\}.$$

Now lemmas three and four are stated in terms of A_n, but it is quite clear that they are still true lemmas if A_n is replaced by A'_n.

LEMMA 3. $\left| \bigcap_{m=1}^{\infty} \left(\bigcup_{n=m; n \in N}^{\infty} A_n \right) \right| = 0$, and

$$\dim \left(\bigcap_{m=1}^{\infty} \left(\bigcap_{n=m; n \in N}^{\infty} A_n \right) \right) \leq \limsup_{n \in N} b_n/b_{n+1}.$$

Proof. Let $A = \bigcap_{m=1}^{\infty} \left(\bigcup_{n=m; n \in N}^{\infty} A_n \right)$. By the countable additivity of measures and A.1 in §I all we need show in order to prove this lemma is that if l is an integer, then $|A \cap [l, l+1]| = 0$ and $\dim (A \cap [l, l+1]) \leq \limsup_{n \in N} b_n/b_{n+1}$.

Now $A \cap [l, l+1] \subset \bigcup_{n=m; n \in N}^{\infty} (A_n \cap [l, l+1])$ for every $m \in I^+$ by the definition of A. But note that $A_n \cap [l, l+1]$ can be covered by 2^{b_n} intervals of length $2^{-(b_{n+1} - [n/2])}$. So we first see that for every $m \in I^+$

$$|A \cap [l, l+1]| \leq \sum_{n=m}^{\infty} 2^{b_n - b_{n+1} + [n/2]} \leq \sum_{n=m}^{\infty} 2^{-n/2}$$

by assumption (2) on the b_n's. But $\sum_{n=1}^{\infty} 2^{-n/2}$ is a convergent series and therefore $|A \cap [l, l+1]| = 0$. Secondly, we see, by the definition of Hausdorff dimension, that if α is a real positive number and

$$\sum_{n \in N} 2^{b_n - \alpha(b_{n+1} - [n/2])} < \infty,$$

then dim $(A \cap [l, l+1]) \leqq \alpha$. So we let $\alpha > \limsup_{n \in N} b_n/b_{n+1}$ and shall show this series converges. Let $\varepsilon > 0$ be such that

$$(1 - \varepsilon)\alpha > \limsup_{n \in N} b_n/b_{n+1}.$$

There exists $N_1 \in I^+$ such that $n \geqq N_1$ implies $b_n < b_{n+1}(1 - \varepsilon)\alpha$. By condition (2) on the b_n's

$$b_{n+1} \geqq \sum_{i=1}^{n} i = (n+1)n/2;$$

so there exists $N_2 \in I^+$ such that $n \geqq N_2$ implies

$$b_{n+1} - [n/2] > (1 - \varepsilon/2)b_{n+1}.$$

By an easy computation using assumption (1) on the b_n's, we see there exists $N_3 \in I^+$ such that $n \geqq N_3$ implies $b_n \geqq H k_0^n$ where $H = \max(b_1, b_2, b_3)$ and $k_0 = k^{1/4}$.

So if $n \geqq \max(N_1, N_2, N_3)$ then

$$b_n - \alpha(b_{n+1} - [n/2]) \leqq (1 - \varepsilon)\alpha b_{n+1} - (1 - \varepsilon/2)\alpha b_{n+1}$$

$$= -\varepsilon\alpha b_{n+1}/2 < -\varepsilon\alpha H k_0^{n+1}/2.$$

By condition (1) on the b_n's, $k_0 > 1$, and therefore

$$\sum_{n \in N} 2^{b_n - \alpha(b_{n+1} - [n/2])} < \infty$$

and the proof of lemma three is complete.

DEFINITION. We say a random variable X is of Rubin type with Poissons $\{\lambda_n\}$ and partition $\{c_n\}$ if

$$X = \sum_{n=1}^{\infty} \sum_{i=1}^{N_n} \sum_{j=c_n+1}^{c_{n+1}} A_{ij}/2^j,$$

where $\{N_n, A_{ij}, i, j, n = 1, 2, \ldots\}$ are independent random variables, where the distribution of N_n is Poisson with expectation λ_n, where each A_{ij} takes values 0 and 1 with probability 1/2 each, and where $\{c_n\}$ is an *increasing* sequence of positive

integers. For notational convenience we let N_n and $A_{i,j}$ be denoted by $N_n(X)$ and $A_{i,j}(X)$ respectively.

0.1. Let $X_1, \ldots, X_n$ be a finite collection of independent random variables, with X_i being of Rubin type with Poissons $\{\lambda_{i,k}\}$ and partition $\{c_n\}$ for $i = 1, \ldots, n$. Let $Y = \sum_{i=1}^{n} X_i$ and let F_Y be the distribution function of Y. Then by glancing at the characteristic function of F_Y we see that F_Y is the distribution function of a random variable of Rubin type with Poissons $\{\sum_{i=1}^{n} \lambda_{i,k}\}$ and partition $\{c_n\}$.

We note in passing that, by a result due to Hartman and Wintner [4], X has a pure distribution.

In lemmas four and five we assume X is of Rubin type with partition $\{b_n\}$ and Poissons $\{\lambda_n\}$ where

1. $\sum_{n=1}^{\infty} \lambda_n = \infty$, a condition which by [4] implies X has a continuous distribution, and

2. $\lambda_n \leq n$.

LEMMA 4. *If* $\sum_{n \in N} \exp(-\lambda_n) = \infty$, *then* $\Pr.[X \in \bigcap_{m=1}^{\infty} (\bigcup_{n=m; n \in N} A_n)] = 1$, *and therefore*

$$\dim \operatorname{supp}(X) \leq \limsup_{n \in N} b_n/b_{n+1}.$$

Proof. *Claim* 1. *If* c *is a positive real number, then* $\Pr.[N_{n+1}(X) \geq 2^{[n/2] - c}$ *infinitely often*$] = 0$.

By the Borel-Cantelli lemma we see that in order to prove this claim we only need show that

$$\sum_{n=1}^{\infty} \Pr.(N_{n+1}(X) \geq 2^{[n/2] - c}) < \infty.$$

But for each $n \in I^+$, $N_{n+1}(X)$ is a Poisson distributed random variable and therefore

$$\Pr.(N_{n+1}(X) \geq 2^{[n/2] - c}) = \exp(-\lambda_{n+1}) \sum_{n = 2^{[n/2] - c}}^{\infty} (\lambda_{n+1})^j/j!.$$

So by Taylor's Theorem with remainder we see that

$$\Pr.(N_{n-1}(X) \geq 2^{[n/2] - c}) \leq (\lambda_{n+1})^{2^{[n/2] - c}}/(2^{[n/2] - c})!.$$

Now combining assumption 2 on the λ_n's with the ratio test we see that our claim follows.

Using the Borel-Cantelli lemma again with the condition that $\sum_{n \in N} \exp(-\lambda_n) = \infty$, we see that $\Pr.[N_n(X) = 0$ infinitely often$] = 1$. Therefore there exists a set M such that $\Pr.(M) = 1$, and, if $\omega \in M$, then (1) there exists $N'(\omega) \in I^+$ such that if $n \geq N'(\omega)$ then $N_{n+1}(X)(\omega) > 2^{[n/2] - c}$; (2) there exists a sequence $\{n_m(\omega)\} \subset I^+$ with $N_{n_j(\omega)}(X)(\omega) = 0$ for every $j \in I^+$. In order to prove this lemma it is clearly sufficient to show that if $m \in I^+$ then $\Pr.[X \in \bigcup_{n=m}^{\infty} A_n] = 1$. So let $m \in I^+$, $\omega \in M$, and let

$j \in I^+$ be such that $n_j(\omega) \geqq \max(N'(\omega), m)$; then we will show $X(\omega) \in A_{n_j(\omega)}$ which clearly implies $\Pr.[X \in \bigcup_{n=m}^\infty A_n] = 1$. Now by assumption $N_{n_j(\omega)}(X)(\omega) = 0$ and therefore

$$X_{n_j(\omega)}(\omega) = \sum_{n=1}^{n_j(\omega)-1} \sum_{n=n_j+1}^\infty \left(\sum_{i=0}^{N_n(X)(\omega)} \sum_{j=b_n+1}^{b_n+1} A_{i,j}(\omega)2^{-j} \right).$$

But recall that $n_j(\omega) \geqq N'(\omega)$ and therefore

$$\sum_{n=n_j(\omega)+1}^\infty \left(\sum_{i=0}^{N_n(X)(\omega)} \sum_{j=b_n+1}^{b_n+1} A_{i,j}(\omega)2^{-j} \right) \leqq \sum_{n=n_j(\omega)+1}^\infty 2^{[n/2]-c} \left(\sum_{j=b_n+1}^{b_n+1} 2^{-j} \right)$$

$$\leqq \sum_{n=n_j(\omega)+1}^\infty 2^{[n/2]-b_n-c} < 2^{[(n_j(\omega)+1)/2]-b_{n_j(\omega)+1}}.$$

Therefore $\pi_i X(\omega) = 0$ for $b_{n_j(\omega)} < i \leqq b_{n_j(\omega)+1} - [(n_j(\omega)+1)/2]$ which by definition implies $X(\omega) \in A_{n_j(\omega)}$.

LEMMA 5. *Let* $\sum_{n \notin N} \exp(-\lambda_n) < \infty$, $\limsup_{m,n \in N; m < n} b_m/b_n = 0$, *and*

$$\limsup_{n \in I^+} b_n/b_{n+2} = 0;$$

then

$$\dim \operatorname{supp}(X) \geqq \liminf_{n \in N} b_n/b_{n+1}.$$

Proof. Let

$$X_{i,n} = \sum_{i=b_n+1}^{b_n+1} A_{i,j}(X)2^{-j},$$

$$V_1 = \sum_{n \notin N} X_{1,n} = \sum_{n \notin N; N_n(X) \neq 0} X_{1,n} + \sum_{n \notin N; N_n(X) = 0} X_{1,n},$$

$$Z = \sum_{N_n(X) \neq 0} X_{1,n} = \sum_{n \in N; N_n(X) \neq 0} X_{1,n} + \sum_{n \notin N; N_n(X) \neq 0} X_{1,n},$$

$$W = \sum_{n=1}^\infty \sum_{i=2}^{N_n(X)} X_{i,n},$$

$$U = W + \sum_{n \in N; N_n(X) \neq 0} X_{1,n} + V_1.$$

Now first we note $X = W + Z$. Secondly we note that $W + \sum_{n \in N; N_n(X) \neq 0} X_{1,n}$ and V_1 are independent random variables and therefore by A.3 $\dim \operatorname{supp} V_1 \leqq \dim \operatorname{supp}(U)$. Note that

$$X + \sum_{n \notin N; N_n(X) = 0} X_{1,n} = W + Z + \sum_{n \notin N; N_n(X) = 0} X_{1,n}$$

$$= W + V_1 + \sum_{n \in N; N_n(X) \neq 0} X_{1,n} = U$$

But by hypothesis $\sum_{n \notin N} \exp(-\lambda_n) < \infty$, so by the Borel-Cantelli lemma

$$\Pr.(N_n(X) = 0 \text{ for all } n \notin N, \text{ except for finitely many}) = 1$$

which implies $\sum_{n \notin N; N_n(X) \neq 0} X_{1,n}$ is discrete since $X_{1,n}$ is discrete for every n. So by A.4

$$\dim \operatorname{supp}(X) = \dim \operatorname{supp}(U).$$

So in order to prove this lemma, we see that we only need show that

$$\dim (V_1) \geqq \liminf_{n \in N} b_n/b_{n+1}.$$

A theorem proved by S. D. Chatterji [2] says that if $k \in I^+$, and if $\{X_j : j \in I^+\}$ is a collection of independent random variables such that each X_j assumes values in $\{0, 1, 2, \ldots, k-1\}$, if $P_{ij} = \Pr.[X_j = i]$, and if $Y = \sum_{j=1}^{\infty} X_j/k^j$, then

$$\dim \operatorname{supp}(Y) = \liminf_{n \to \infty} \left\{ (-1/(n \ln k)) \sum_{j=1}^{n} \left(\sum_{i=1}^{k} P_{i,j} \ln P_{i,j} \right) \right\}$$

with the convention that $0 \ln 0 = 0$. Let $j(n) = \max \{i : b_i \leqq n\}$, and for $m \in I^+$ let

$$D_m = (1/m)\left(\sum_{j \notin N; j < j(m)} (b_{j+1} - b_j) + m - b_{j(m)} \right) \quad \text{if } j(m) \notin N,$$

$$= (1/m)\left(\sum_{j \notin N; j < j(m)} (b_{j+1} - b_j) \right) \quad \text{if } j(m) \in N.$$

By Chatterji's theorem we find that, by an easy computation,

$$\dim \operatorname{supp}(V_1) = \liminf_{n \to \infty} D_n.$$

Case 1. There exists $\{n_i\} \subset I^+$ such that $n_i \in N$ and $n_i + 1 \in N$.

By hypothesis $\limsup_{m,n \in N; m < n} b_m/b_n = 0$ and therefore $\limsup_i b_{n_i}/b_{n_i+1} = 0$ which implies $\liminf_{n \in N} b_n/b_{n+1} = 0$ and the conclusion of the lemma holds.

Case 2. There exists $N_0 \in I^+$ such that $n \geqq N_0$ and $n \in N$ implies $n + 1 \notin N$.

Let $\varepsilon > 0$; then there exists $n_\varepsilon \in I^+$ such that $n \geqq n_\varepsilon$ and $n \in N$ imply

$$b_n/b_{n+1} > \liminf_{n \in N} b_n/b_{n+1} - \epsilon/2.$$

By hypothesis $\limsup_n b_n/b_{n+2} = 0$ and therefore there exists $n'(\varepsilon)$ such that $n \geqq n'(\varepsilon)$ implies $b_n/b_{n+2} < \varepsilon/2$. Let $N(\varepsilon) = \max(N_0, n_\varepsilon, n'(\varepsilon))$. Let

$$j'(n) = \max \{i : b_i \leqq n, i \in N\}.$$

Now N is an infinite set by assumption and therefore $j'(n) \underset{n}{\to} \infty$. So there exists $N'(\varepsilon)$ such that $n \geqq N'(\varepsilon)$ implies $j'(n) \geqq N(\varepsilon)$. Let $n \geqq \max(N(\varepsilon), N'(\varepsilon))$.

Case i. $j(n) \notin N$. In this case

$$D_n = (1/n)\left(\sum_{j \notin N; j < j(n)} (b_{j+1} - b_j) + (n - b_{j(n)}) \right)$$

$$= (1/n)\left(n - b_{j'(n)+1} + \sum_{j \notin N; j < j'(n)} (b_{j+1} - b_j) \right).$$

Note that if $R, S \in I^+$, then $R/S < (R+1)/(S+1)$, so applying this inequality $n - b_{j'(n)+1}$ times to D_n we see

$$D_n \geq (1/b_{j'(n)+1}) \sum_{j \notin N; \, j < j'(n)} (b_{j+1} - b_j).$$

Therefore recalling that $j'(n) \geq N(\varepsilon)$ and $j'(n) \, \varepsilon \, N$, we obtain

$$D_n \geq (b_{j'(n)} - b_{j'(n)-1})/b_{j'(n)+1}$$

$$\geq b_{j'(n)}/b_{j'(n)+1} - \varepsilon/2 \geq \liminf_{n \in N} b_n/b_{n+1} - \varepsilon.$$

Case ii. $j(n) \in N$. By hypothesis of Case 2, we see that $j(n) - 1 \notin N$ and therefore

$$D_n = (1/n) \sum_{j \notin N; \, j < j(n)} (b_{j+1} - b_j) \geq (b_{j(n)} - b_{j(n)-1})/b_{j(n)+1}.$$

Now recall that $j(n) \geq N(\varepsilon)$ and $j(n) \in N$, so

$$D_n \geq b_{j(n)}/b_{j(n)+1} - \varepsilon/2 \geq \liminf_{n \in N} b_n/b_{n+1} - \varepsilon.$$

In both Case i and Case ii, ε was arbitrary, and so the proof of the lemma is complete.

LEMMA 6. *Let* $m: [0, \infty) \to [0, 1]$ *such that*

1. $m(0) = 1$,
2. $m(t)m(d) \geq m(t+d)$ *for* $t, d \in [0, \infty)$, *and*
3. *there exists* $\alpha > 0$ *such that* $m(t) \neq 0$ *for* $t \in [0, \alpha]$. *Then the right-hand derivative of* m *exists at* 0, *and we call it* $m'(0)$, *and* $\ln m(t)/t \to m'(0)$ *as* $t \to 0^+$.

Proof. Now clearly, by induction, $m(t)m(d) \geq m(t+d)$ implies that $m^n(x/n) \geq m(x)$, and therefore

$$-\ln m(x/n) \leq -(\ln m(x))/n.$$

Let $g = -\ln m(x)$ for $x \in [0, \alpha]$. By hypothesis, $m(t) \leq 1$ for every $t \geq 0$, so this combined with $m(t)m(d) \geq m(t+d)$ implies that m is a nonincreasing function. Since $m(t) \neq 0$ on $[0, \alpha]$ there exists $c > 0$ such that $1 \geq c$ and $m(t) \to c$ as $t \to 0+$. We will now show that $c = 1$, i.e. that m is continuous from the right at 0. Suppose $c < 1$. Then choose $t \in [0, \alpha]$ and let n be such that $m(t) > c^n$. It was already noted that $m^n(t/n) \geq m(t)$. But m is nondecreasing and therefore $c \geq m(t/n)$ which implies $c^n \geq m(t)$; this is a contradiction.

In Bruckner and Ostrow [1, Section 4], one finds the following theorem:

THEOREM. *Let* f *be a continuous nonnegative function on* $[0, c]$ *such that* $f(0) = 0$ *and* $f(x/n) \leq (1/n)f(x)$ *for* $x \in [0, c]$. *Then* f *is differentiable at* 0.

Now if one examines the proof that Bruckner and Ostrow give for this theorem, one sees that they really only use the continuity of f at 0 and on some countable sequence of points which converges to 0. We have already shown that our function

m is continuous at 0, and since m is monotone nonincreasing it is continuous at all but a countable set of points and therefore certainly continuous on a sequence of points converging to 0. But note that the function g is clearly continuous if m is, g is nonnegative since $m(x) \leq 1$, $g(0) = -\ln 1 = 0$, and as was seen at the beginning of the proof, g satisfies the inequality in the theorem of Bruckner and Ostrow. So g has a derivative at 0, which in turn implies that m has a derivative at 0. Now

$$\ln m(x)/x = \left(\int_1^{m(x)} dt/t \right) \Big/ x.$$

Therefore, by the mean value theorem for integrals, there exists t_0, where $1 \leq t_0 \leq m(x)$, such that

$$\ln m(x)/x = (m(x)-1)/(t_0 x).$$

So

$$(m(x)-1)/x \leq \ln m(x)/x \leq (m(x)-1)/(xm(x)).$$

But m is continuous at 0 and $m(0) = 1$, so taking limits across the above inequalities, we see that

$$\lim_{x \to 0+} \ln m(x)/x = m'(0)$$

which completes the proof of this lemma.

LEMMA 7. *Let a function $f(t)$ map $(-\infty, \infty)$ into $(-\infty, \infty)$. Then there exists a countable set $D \subset (-\infty, \infty)$ such that if $x \in (-\infty, \infty)$, then there exists a sequence $\{x_j\} \subset D$ where $x_j \to x+$ and $f(x_j) \to f(x)$ as $j \to \infty$.*

Proof. Let r and s be two rational numbers, and let

$$D_{r,s} = \{t : r < f(t) < s\}.$$

Let $T_{r,s}$ be the set of all points of $D_{r,s}$ which are isolated from the right relative to $D_{r,s}$. Clearly this is a countable set. Now $D_{r,s}$ is a subset of the real numbers, and therefore there exists a countable dense subset of $D_{r,s}$, call it $D'_{r,s}$. Let $E_{r,s} = T_{r,s} \cup D'_{r,s}$ and let

$$D = \bigcup_{r,s \in \mathrm{Ra}} E_{r,s}$$

where Ra is the set of rational numbers.

Now we will show that this set D is the set desired. Let $x \in (-\infty, \infty)$. If $x \in D$, then let $x_j = x$ for every $j \in I^+$ and the conclusion to this lemma is proved. If $x \in (-\infty, \infty) - D$, then let (s_i, r_i) be a sequence of intervals such that $s_i, r_i \in \mathrm{Ra}$ for every $i \in I^+$, $(s_{i+1}, r_{i+1}) \subset (s_i, r_i)$, $\{f(x)\} = \bigcap_i (s_i, r_i)$ and $r_i - s_i < 1/i$. Now clearly by the definition of D_{r_i, s_i}, $x \in D_{r_i, s_i}$ for every $i \in I^+$. Now $x \in (-\infty, \infty) - D$, so x certainly is not isolated from the right relative to D_{r_i, s_i} by the definition of D. So for each i, let $\{x_{i,j}\} \subset D_{r_i, s_i}$ such that $x_{i,j} \underset{j}{\to} x+$ for each $i \in I^+$. But there exists, for each i, an integer $j(i)$ such that $x_{i,j(i)} \in (x, x+1/i)$, and let $x_i = x_{i,j(i)}$. By the definition of D_{r_i, s_i}

$$|f(x) - f(x_i)| < 1/i$$

and therefore $f(x_i) \underset{i}{\to} f(x)$ which completes the proof of this lemma.

LEMMA 8. *Let $\{G_t(x) : t \in (0, \infty)\}$ be a collection of distribution functions such that for fixed x, $G_t(x)$ decreases for increasing t. Let F be a distribution function and λ a bounded nonincreasing function on $(0, \infty)$; then for $i \in I^+$*

$$\left(\int_0^\infty G_\tau \, d\lambda(\tau)\right)^{*i} * F(x) = \int_0^\infty \cdots \int_0^\infty \mathop{\text{\Large\textasteriskcentered}}_{j=1}^{i} G_{\tau_j} * F(x) \, d\lambda(\tau_1) \cdots d\lambda(\tau_n).$$

Let $G(t, x) \doteq G_t(x)$. Note that $G(t, x)$ is a distribution function in x for fixed t, and for fixed x it decreases with increasing t by assumption. Standard arguments imply then that $G(t, x)$ is jointly measurable. Induction and Fubini's theorem quickly yield the lemma.

We now prove the theorem stated at the beginning of this section.

Proof. We shall first prove this theorem for functions $m(t)$ such that A and B are both empty, and then we shall indicate how to modify the definitions and arguments for the general case. The reason for this approach is that all the main ideas and arguments are used in this case, and yet we are not burdened down with superfluous cases and lengthy expressions which only tend to obscure what is really going on.

So let $m(t)$ and $v(t)$ be functions as assumed in the hypothesis with the added condition that $A = B = \varnothing$. Let D_1 be a countable set contained in $(0, \infty)$ such that if $t \in (0, \infty)$ then there exists $\{t_i\} \subset D_1$ such that $t_i \to t^+$ and $v(t_i) \to v(t)$. The existence of D_1 is of course guaranteed to us by Lemma 7. Let

$$D_2 = \{t \in (0, \infty) : m(t) \neq m(t^+)\}.$$

By the monotonicity of m, we see that D_2 is countable. Let

$$D_3 = \{t \in (0, \infty) : \liminf_{s \to t} v(s) \neq v(t)\}.$$

By Lemma 1.2 and by hypothesis, we see that D_3 is countable. Let

$$D = D_1 \cup D_2 \cup D_3.$$

Let $\{c_i\}$ be a sequence of positive integers such that $c_i \downarrow 1$. Now clearly D is countable, and D is enumerated as follows:

Let d_1 be any element of D. Suppose d_n has been defined, and let $d_{n+1} \in D$ be such that for $j = 1, \ldots, n$ either $d_{n+1}/d_j > c_{n+1}$ or $d_j/d_{n+1} > c_{n+1}$. Denote

$$l_1 = \min\left\{j : \sum_{i=1}^{j} 1/4i \geq 1\right\}.$$

Suppose l_n has been defined; then define

$$l_{n+1} = \min\left\{j : \sum_{i=l_n+1}^{j} 1/4i \geq 1\right\}.$$

Define

$$I_n = \{4i : l_{n-1} \leq i < l_n\}.$$

Let $P_1, P_2, \ldots$ be an enumeration of the primes. Let $i \in I^+$, and let $\{l(i,j)\}_{j=1}^\infty$ be a sequence of increasing integers such that

$$\sum_{j=1}^\infty \sum_{n \in P_i l(i,j)} 1/n^{c_j + i} < 1/2^i.$$

Define

$$\bar{P}_i = \bigcup_j I_{P_i l(i,j)}, \qquad P = \bigcup_i \bar{P}_i.$$

Clearly $\bar{P}_i \cap \bar{P}_j = \varnothing$ if $i \neq j$, and if $n, m \in P$, then $|n - m| \geq 4$. If $t \in [0, \infty)$ we denote

$$\lambda(t) = -\ln m(t)/t \quad \text{if } t \neq 0,$$

$$= -m'(0) \qquad \text{if } t = 0.$$

By Lemma 6 and the hypothesis we see that λ is a bounded nondecreasing function on all finite intervals of $[0, \infty)$ and is continuous at 0.

Let $\{v_n\}$ be a sequence of real numbers such that $v_n \uparrow +\infty$, and for convenience let $v_0 = 0$. If $t \in (0, \infty)$, then let us denote

$$[t]^* = \max \{n \in I^+ \cup \{0\} : v_n \leq t\}.$$

If $n \in I^+$ and $t \in (0, \infty)$ we define

$$t(n) = v_{n-1} \quad \text{if } n \leq [t]^* + 1,$$

$$= t \qquad \text{if } n = [t]^* + 2,$$

$$= v_{n-2} \quad \text{if } n \geq [t]^* + 3,$$

$$t'(n) = v_n \qquad \text{if } n \leq [t]^*,$$

$$= t \qquad \text{if } n = [t]^* + 1,$$

$$= v_{n-1} \quad \text{if } n \geq [t]^* + 2,$$

$$\lambda_n^t(x) = \lambda(t(n)) \quad \text{if } x \leq t(n),$$

$$= \lambda(x) \qquad \text{if } t(n) \leq x \leq t'(n),$$

$$= \lambda(t'(n)) \quad \text{if } x \geq t'(n).$$

Let $\lambda_n^t = \text{T.V.}(\lambda_n^t(x))$, and therefore clearly $\lambda_n^t = \lambda(t'(n)) - \lambda(t(n))$. Let $\{k(n)\}$ be a sequence of positive integers such that $k(n) \uparrow \infty$, and such that

$$\sum_n \lambda(v_{k(n)})/2^n < \infty.$$

Let $\{j_n\}$ be a sequence of positive integers such that $\min(j_n, j_{k(n)}) > n$ and

$$\sum_{n=1}^\infty (\lambda(v_n))^{j_n}/j_n! < \infty.$$

Define $j(l) = j_r$ where $l \in \bar{P}_j$ and $d_j \in [v_{r-1}, v_r]$. Let $j'(l) = \max (j(l), j_l)$. Let $\{a_n\}$ be a sequence of positive integers such that if $k \in I^+$, then

$$\sum_{l \in P; l \geq k} (2 j'(l) \ln l)^{2^{a_l}} / 2^{a_l}! < \min \{1/2^k \lambda(v_{k+1}), 1/2^k\}.$$

Finally we define a sequence of positive integers $\{b_n\}$ as follows:

Case 1. $n \notin P$.

Subcase (i). If $n - 1 \notin P$, then $b_n = (b_{n-1} + 1)n$.

Subcase (ii). If $n - 1 \in \bar{P}_j$

$$b_n = [b_{n-1}/v(d_j)] + n \quad \text{when } v(d_j) \neq 0,$$

$$= [b_{n-1}/(1/n)] + n \quad \text{when } v(d_j) = 0.$$

Case 2. If $n \in P_j$, let $b_n = (b_{n-1} + a_n + 1)n$.

By a straightforward computation we see that these b_n's satisfy the conditions imposed on the b_n's in Lemmas 3, 4 and 5.

Claim 1. *Let X be a random variable of Rubin type with Poissons $\{\lambda_n\}$ and partition $\{b_n\}$. If $k \in I^+$ and $\omega \in \bigcap_{n \geq k+1; n \in P} [N_n(X) < 2^{a_n}]$, then*

$$X_k(\omega) = \sum_{n \geq k+1; n \in P} \sum_{l=0}^{N_n(X)(\omega)} \sum_{j=b_n+1}^{b_{n+1}} A_{l,j}(X)(\omega) 2^{-j} < 2^{-b_k}.$$

Proof. If $\omega \in \bigcap_{n \geq k+1; n \in P} (N_n(X) \geq 2^{a_n})$ then by the definition of the $A_{i,j}(X)$'s we see that:

$$X_k(\omega) = \sum_{n \geq k+1; n \in P} 2^{a_n} \left(\sum_{j=b_n+1}^{b_{n+1}} 2^{-j} \right) \leq \sum_{n \geq k+1; n \in P} 2^{a_n - b_n}.$$

By the definition of the b_n's we see that $a_n - b_n < -b_{n-1} - 1$ which clearly completes the proof of Claim 1.

Let G_τ and V_t be the distribution functions corresponding to the random variables X_τ and Y_t which are of Rubin type with Poissons $\{\lambda_{n,\tau}\}$, $\{\lambda'_{n,t}\}$ respectively and common partition $\{b_n\}$, where

$$\lambda_{n,\tau} = 2 \ln n \quad \text{if } n \in \bar{P}_i, d_i > \tau, n > j_{[d_i]^*},$$

$$= 0 \quad \text{otherwise,}$$

$$\lambda'_{n,t} = (t/d_j) \ln n \quad \text{if } n \in \bar{P}_j, n > j_{[d_j]^*},$$

$$= 2 \ln n \quad \text{otherwise.}$$

We assume that $\{N_n(X_t), N_n(Y_t), A_{i,j}(X_t), A_{i,j}(Y_t) : i, j, n \in I^+, t \in (0, \infty)\}$ is a collection of independent random variables.

Claim 2. *For fixed x, $G_t(x)$ decreases as t increases.*

Proof. If $t_2 > t_1$, then clearly by considering the characteristic function of G_{t_1}

and G_{t_2} we see that $G_{t_2} = G_{t_1} * F$ where F is the distribution function of a random variable of Rubin type with Poissons $\{\lambda_n\}$ and partition $\{b_n\}$, where

$$\lambda_n = 2 \ln n \quad \text{if } n \in \bar{P}_i, n \geq j_{[d_i]}, t_2 > d_i \geq t_1,$$

$$= 0 \qquad \text{otherwise.}$$

Therefore $G_{t_2}(x) = \int_0^x G_{t_1}(x-y) \, dF(y) \leq G_{t_1}(x)$ which establishes Claim 2.

Claim 2 shows that for fixed x, $G_t(x)$ is a Borel measurable function of t. So, for $x > 0$ and $t \in [0, \infty)$, let

$$M_n^t(x) = \int_{t(n)}^{t'(n)} G_\tau(x) \, d\lambda_n^t(\tau) - \lambda_n^t \quad \text{if } n \in I^+,$$

$$= -m'(0)V(x) + m'(0) \qquad \text{if } n = 0,$$

where $V(x)$ is any absolutely continuous distribution function. Note that $M_n^t(x)$ increases as x increases and $M_n^t(x) \leq 0$ for every $x \in (0, \infty)$.

Claim 3. *If $m \in I^+$, if $t \geq v_m$ and if $x \geq 1/2^m$, then*

$$G_t(x) > 1 - 1/(2^{[t]^*}\lambda([t]^*+1)).$$

Proof. Recalling the definition of $\lambda_{n,t}$ and noting that the condition $j_{n+1} > j_n$ implies $j_{[t]^*} \leq j_{[d_j]^*}$ for $d_j \geq t$, we see that for all ω,

$$X_t(\omega) = \sum_{n \geq j_{[t]^*}+1; n \in P} \sum_{i=1}^{N_n(X_t)(\omega)} \sum_{j=b_n+1}^{b_n+1} A_{i,j}(Xt)(\omega)2^{-j}.$$

Therefore by Claim 1, if $\omega \in \bigcap_{n \geq j_{[t]^*}+1; n \in P} (N_n(X_t) \leq 2^{a_n})$ then $X_t(\omega) < 1/2^{b_{j[t]^*}}$.

Since b_n and j_n are both greater than or equal to n, and $x \geq 1/2^m$, we see that

$$x \geq 1/2^{b_{j[t]^*}}.$$

Therefore

$$1 - G_t(x) = \Pr.(X_t \in (x, \infty)) \leq \Pr.(X_t \in (1/2^{b_{j[t]^*}}, \infty))$$

$$\leq \sum_{n \geq j_{[t]^*}+1; n \in P} \Pr.(N_n(X_t) > 2^{a_n}).$$

But the $N_n(X_t)$'s are Poisson random variables and therefore

$$1 - G_t(x) \leq \sum_{n \geq j_{[t]^*}+1; n \in P} (2 \ln n)^{2^{a_n}}/2^{a_n}!.$$

By this last inequality and our assumptions on the a_n's we see the proof of Claim 3 is complete.

Claim 4. $\sum_{n=0}^\infty M_n^0(x) > -\infty$ *for every $x \in (0, \infty)$.*

Proof. Fix $x \in (0, \infty)$. Now there exists $l \in I^+$ such that $x \geq 1/2^l$.

So by Claim 3 we see that for $n \geq l+1$

$$M_n^0(x) = \int_{v_{n-1}}^{v_n} G_\tau(x) \, d\lambda_n^0(\tau) - \lambda_n^0 \geq \lambda_n/(2^{n-1}\lambda(v_n))$$

$$\geq -1/2^{n-1}.$$

But for $n \leq l$, $M_n^0(x)$ is a bounded function on $(0, \infty)$ and therefore the proof of Claim 4 follows.

Let

$$M(x) = \sum_{n=0}^{\infty} M_n^0(x),$$

and by Claim 4, $M(x) > -\infty$ for every $x \in (0, \infty)$.

Claim 5. $\int_{0+}^{1} x \, dM(x) < \infty$.

Proof. Now clearly $\int_{0+}^{1} x \, dM_0^0(x) < \infty$ since $M_0^0(x)$ is a bounded function on $(0, \infty)$. So in order to prove Claim 5 we only need show that $\int_{0+}^{1} x \, dM'(x) < \infty$ where $M'(x) = \sum_{n=1}^{\infty} M_n^0(x)$. Now

$$\int_{0+}^{1} x \, dM'(x) = \sum_{l=0}^{\infty} \int_{(1/2^{l+1})+0}^{1/2^l} x \, dM'(x)$$

$$\leq \sum_{l=0}^{\infty} (M'(1/2^l + 0) - M'(1/2^{l+1}))/2^l.$$

But note that

$$M'(x) = \lim_{n} \left(\int_0^{v_n} G_\tau(x) \, d\left(\sum_{j=1}^{n} \lambda_j(\tau) \right) - (\lambda(v_n) - \lambda(0)) \right)$$

which along with the fact that M' increases for increasing x implies that if $\varepsilon > 0$, then

$$\int_{0+}^{1} x \, dM(x) \leq \sum_{l=0}^{\infty} \left(\int_0^{\infty} (G_\tau(1/2^l + \varepsilon) - G_\tau(1/2^{l+1})) \right) d\left(\sum_{n=1}^{\infty} \lambda_n^0(\tau)/2^l \right)$$

$$\leq \sum_{l=0}^{\infty} \left(\lambda(v_{k(l)}) + \int_{v_{k(l)}}^{\infty} (1 - G_\tau(1/2^{l+1})) \right) d\left(\sum_{n=k(l)+1}^{\infty} \lambda_n^0(\tau)/2^l \right).$$

Recall that in the definition of j_n we required that $j_{k(l)} \geq 1$, which along with Claim 3 implies

$$\int_{v_{k(l)}}^{\infty} (1 - G_\tau(1/2^{l+1})) \, d\left(\sum_{n=k(l)+1}^{\infty} \lambda_n^0(\tau) \right)$$

$$\leq \int_{n=k_l+1}^{\infty} \int_{v_{n-1}}^{v_n} (1 - (1 - 1/(2^{n-1}\lambda(v_n)))) \, d\lambda_n^0(\tau)$$

$$\leq 1/2^{k_l - 2}.$$

This estimate along with the assumption on the $k(l)$'s that $\sum_l \lambda(v_{k(l)})/2^l < \infty$ completes the proof of Claim 5.

By Claims 4 and 5 we see that M is a possible M-function in the Lévy representation of infinitely divisible distribution functions. Let $t \in (0, \infty)$ be fixed. Now we let F_t be the distribution function corresponding to the characteristic function:

$$\text{ch.fcn.} \, V_t \cdot \exp \left(t \int_{0+}^{\infty} (\exp(iux) - 1) \, dM(x) \right).$$

Now we let

$$F_{t,n}(x) = \exp \left(-\sum_{i=0}^{n} \lambda_i t \right) \left(\sum_{i=0}^{\infty} \left(\sum_{j=0}^{n} M_j^0 \right)^{*i} * V_t t^i/i! \right)(x).$$

Recalling that $\sum_{i=0}^{n} M_i(x) \xrightarrow[n]{} M(x)$, then a glance at the characteristic function for $F_{t,n}$ shows us that by [3], $F_{t,n} \xrightarrow{(c)} F_t$. Note that since M_0 is absolutely continuous,

$$(F_{t,n})\text{c.s.} = \exp\left(-t\sum_{i=0}^{n}\lambda_i\right)\sum_{i=0}^{\infty}\left(\left(\sum_{j=1}^{n}M_j\right)^{*i} * V_t\right)\text{c.s. } t^i/i!.$$

Claim 6. Let $\tau_1, \ldots, \tau_i$ be i elements of $(0, \infty)$ and let $\tau_0 = \min\{\tau_j : j \in \{1, \ldots, i\}\}$.
1. *If $\tau_0 < t$, then $\mathop{\text{\Large$*$}}_{j=1}^{i} G_{\tau_j} * V_t$ is absolutely continuous.*
2. *If $t \leq \tau_0$, and $\tau_0 \geq d_i \geq t$, then*

$$\mathop{\text{\Large$*$}}_{j=1}^{i} G_{\tau_j} * V_t\left[\bigcap_{n=1}^{\infty}\bigcup_{n\geq m;\, n\in P_i} A_n\right] = 1.$$

Proof. As noted before, $\mathop{\text{\Large$*$}}_{j=1}^{i} G_{\tau_j} * V_t$ is the distribution function of a random variable of Rubin type with Poissons $\{\lambda'_n\}$ and partition $\{b_n\}$ where $\lambda'_n = \sum_{j=1}^{i}\lambda_{n,\tau_j} + \lambda'_{n,t}$. Now reviewing the definition of the $\lambda_{n,t}$'s and the $\lambda'_{n,t}$'s we see that

$$\lambda'_n = (t/d_j)\ln n \quad \text{if } n \in \bar{P}_j,\, n \geq j_{[d_j]^\bullet},\, d_j \leq \tau_0,$$
$$\geq 2\ln n \qquad \text{otherwise.}$$

So if $d_j \leq \tau_0 < t$, then $t/d_j \geq t/\tau_0 > 1$ and therefore clearly $\sum_n \exp(-\lambda'_n) < \infty$. But a theorem due to Rubin [5] and easily accessible via Tucker [7] says that if F is the distribution function of a random variable of Rubin type with Poissons $\{\lambda_n\}$ such that $\sum_n \exp(-\lambda_n) < \infty$, then F is absolutely continuous. Therefore we see that if $\tau_0 < t$ then indeed $\mathop{\text{\Large$*$}}_{j=1}^{i} G_{\tau_j} * V_t$ is absolutely continuous. Since $\lambda_{n,\tau}$ and $\lambda'_{n,t}$ are at most $2\ln n$ for each $n \in I^+$, we see that $\lambda'_n \leq (j+1)\ln n$ and therefore for large enough n, $\lambda'_n \leq n$. So if $t \leq d_j \leq \tau_0$, then $t/d_j \leq 1$. But by the way in which the $\bar{P}_i$'s were defined we see that

$$\sum_{n\in\bar{P}_j}\exp(-\lambda'_n) \geq \sum_{n\in\bar{P}_j} 1/n = \infty.$$

Therefore by Lemma 4 we see 2 of this claim is proved.

Let $T = \{\tau \in (0, \infty) : \tau > t \text{ or } \tau = t \text{ if } t \in D\}$. By assumption on D, there exists $D_0 = \{d'_i\} \subset D$ such that $d'_i \to t^+$ and $v(d'_i) \to v(t)$, and let us assume that if $t \in D$, then $t \in D_0$ also. Let $\tau \in T$. Let

$$A(t, \tau) = \{n \in I^+ : n \in \bar{P}_i,\, d_i \in D_0,\, t \leq d_i \leq \tau\}.$$

Now by Claim 6

$$\mathop{\text{\Large$*$}}_{j=1}^{i} G_{\tau_j} * V_t\left(\bigcap_{m=1}^{\infty}\bigcup_{n\geq m;\, n\in A(t,\tau)} A_n\right) = 1$$

if $A(t, \tau) \neq \varnothing$.

Claim 7. Let $i \in I^+$; then

$$\left(\left(\sum_{j=1}^{n+2}M_j^t\right)^{*i} * V_t\right)\text{c.s.}(x) = \left(\sum_{j=[t]+2}^{n+2}M_j^t\right)^{*i} * V_t(x).$$

For all $\tau \in T$ such that $A_{(t,\tau)} \neq \varnothing$,

$$\left(\left(\sum_{j=1}^{n+2}M_j^t\right)^{*i} * V_t\right)\text{c.s.}\left(\bigcap_{n=1}^{\infty}\bigcup_{n\geq m;\, n\in A(t,\tau)} A_n\right) = (\lambda(n) - \lambda(t))^i.$$

Proof.

$$\left(\sum_{j=1}^{n+2} M_j^t\right)^{*i} * V_t = \left(\sum_{j=[t]+2}^{n+2} M_j^t\right)^{*i} * V_t + \sum_{l=1}^{i} \binom{i}{l}\left(\sum_{j=1}^{[t]+1} M_j^t\right)^{*l}\left(\sum_{j=[t]+2}^{n+2} M_j^t\right)^{*i-l} * V_t.$$

Now Claim 6, Lemma 8, and A.2 imply that every term in the second sum is absolutely continuous.

Lemma 9 implies

$$\left(\sum_{j=[t]+1}^{n+2} M_j^t\right)^{*i} * V_t = \int_t^n \cdots \int_t^n \underset{j=1}{\overset{i}{\text{\Large $*$}}} G_{\tau_j} * V_t \, d\Lambda_n^t \cdots d\Lambda_n^t$$

where $\Lambda_n^t = \sum_{l=[t]+2}^{n+2} \lambda_l^t$. Now if m is continuous from the right at t, then λ is continuous from the right at t, and we see we might as well be integrating from $t+0$ to n. If m is not continuous from the right at t, then by definition of D we see that $t \in D$. So by the remark just preceding this claim and A.2 we see that

$$\left(\sum_{j=[t]+2}^{n+2} M_j^t\right)^{*i} * V_t\left(\bigcap_{m=1}^{\infty} \bigcup_{n \geq m; \, n \in A(t,\tau)} A_n\right) = (\lambda(n) - \lambda(t))^i$$

if $A(t, \tau) \neq \varnothing$.

Now if $\tau > t$, then $A(t, \tau) \neq \varnothing$ and the above equality holds. But $A(t, \tau) \neq \varnothing$ implies $A(t, \tau)$ is an infinite subset of positive integers since $\bar{P}_t$ is an infinite subset of positive integers. So by Lemma 3

$$\left|\bigcap_{m=1}^{\infty} \bigcup_{n \geq m; \, n \in A(t,\tau)} A_n\right| = 0$$

which completes the proof of Claim 7.

Clearly $M(x) = \sum_{i=0}^{\infty} M_i^t(x)$ and the M_i^t's satisfy the conditions imposed on the M_i's of Lemmas 1 and 2.

For $n > [t]^*$, we see that by Claim 7, the definition of the M_i^t's and the definition of $F_{t,n}$ that

$$0.2 \qquad (F_{t,n})\text{c.s.}(x) = \exp\left(-t\sum_{i=0}^{n+2}\lambda_i^t\right)\left\{\sum_{i=0}^{\infty}\left(\sum_{j=[t]^*+2}^{n+2} M_j^t\right)^{*i} * V_t t^i/i!\right\}(x).$$

So by Lemma 1, there exists a nondecreasing function F' such that

$$(F_{t,n})\text{c.s.} \xrightarrow{(c)} F', \quad \text{and} \quad \text{T.V.}(F') = \exp\left(-t\sum_{i=0}^{[t]^*+1}\lambda_i^t\right).$$

But $\sum_{i=0}^{[t]^*+1} \lambda_i^t = \lambda_0^t + \lambda(t) - \lambda(0) = -m'(0) + \lambda(t) + m'(0) = \lambda(t)$, which implies

$$\text{T.V.}(F') = \exp\left(t \ln m(t)/t\right) = m(t).$$

Claim 8. *There exists a sequence of real numbers $\{\varepsilon_n\}$ where $\varepsilon_n \downarrow 0$, and there exists a sequence of positive integers $\{n_N\}$ such that if*

1. $\{e_j : j = [t]+1, \ldots, n\}$ is a finite set of positive integers with $\sum_{l=[t]^\bullet+2}^v e_l < j_v$ for $n_N \leqq v \leqq n$, and

2. $\{\tau(l, i) : [t]+2 \leqq l \leqq n, n_N \leqq i \leqq e_l\}$ is a collection of positive real numbers such that $\tau(l, i) \in [v_{l-1}, v_l)$, then

$$\operatorname*{\LARGE*}_{l=[t]+2}^{n} \operatorname*{\LARGE*}_{i=1}^{e_l} G_{\tau_{l,i}} * V_t\left(\bigcup_{k=N; k \in A(t,\tau)}^{m} A_k\right) \geqq (1-\varepsilon_N) V_t\left(\bigcup_{k=N; k \in A(t,\tau)}^{m} A_k'\right)$$

for

$$t \leqq \tau \leqq \tau_0 = \min\{\tau(l, i) : [t]+2 \leqq l \leqq n, n_N \leqq i \leqq e_l\}$$

and $A(t, \tau) \cap \{k \in I^+ : k = N, \ldots, m\} \neq \varnothing$.

Proof. Let $\varepsilon_N = 1/2^N$, $n_N = N$, and assume $\tau \in [t, \tau_0]$ such that

$$A(t, \tau) \cap \{k \in I^+ : k = N, \ldots, m\} \neq \varnothing.$$

By 0.1 we see that $\operatorname*{\Large*}_{l=[t]+2}^{n} \operatorname*{\Large*}_{i=1}^{e_l} G_{\tau(l,i)}$ is the distribution function of a random variable Y of Rubin type with Poissons $\{\lambda_k^*\}$ and partition $\{b_n\}$ where $\lambda_k^* = \sum_{l=[t]^\bullet+2}^{n} \sum_{i=1}^{e_l} \lambda_{k,\tau(l,i)}$. Now let

$$\omega \in \bigcap_{n \geqq N; n \in P} (N_n(Y) < 2^{a_n}) \cap \left(Y_t \in \bigcup_{k=N; k \in A(t,\tau)}^{m} A_k'\right).$$

So there exists $r \in A(t, \tau)$ with $m \geqq r \geqq N$ such that $Y_t(\omega) \in A_r'$. By reviewing the definition of $A(t, \tau)$ and the definition of the $\lambda_{k,\tau}$'s we see that $\lambda_r^* = \lambda_{r+1}^* = 0$ and therefore

$$Y(\omega) = \sum_{k=1; k \in P}^{r-1} + \sum_{k=r+2; k \in P}^{\infty} \left(\sum_{i=0}^{N_n(Y)(\omega)} \sum_{j=b_n+1}^{b_{n+1}} A_{i,j}(Y)(\omega) 2^{-j}\right).$$

Claim 1 implies that the second sum is $< 2^{-b_r+1}$. By the definition of A_r' we see that (the first sum) $+ Y_t(\omega) \in A_r'$.

Therefore $Y(\omega) + Y_t(\omega) \in A_r$. So since $\{N_n(Y), Y_t : n \in I^+\}$ are an independent collection of random variables we see that

$$\Pr\left(Y + Y_t \in \bigcup_{k=N; k \in A(t,\tau)}^{m} A_k\right) \geqq \Pr\left(\bigcap_{n=N; n \in P}^{\infty} (N_n(Y) < 2^{a_n})\right) \Pr\left(Y_t \in \bigcup_{k=N; k \in A(t,\tau)}^{m} A_k'\right).$$

Now by the hypothesis of this claim

$$\lambda_k^* \leqq 2 j_l \ln k \quad \text{if } k \in \bar{P}_j, d_j \in [v_{l-1}, v_l) \text{ and } N \leqq l \leqq n,$$
$$\leqq 2 j_n \ln k \quad \text{if } k \in \bar{P}_j, d_j > n,$$
$$\leqq 2 j_N \ln k \quad \text{if } k \in \bar{P}_j, d_j < N.$$

So for $k \geqq N$, we see that $\lambda_k^* \leqq \max(j_k, j(k)) \ln k$. Therefore by the definition of the a_n's, and by the fact that the $N_n(Y)$ are Poisson random variables we see that

$$\Pr\left(\bigcap_{n=N; n \in P}^{\infty} (N_n(Y) < 2^{a_n})\right) > 1 - 1/2^N$$

which completes the proof of the claim.

Claim 9. $F'(\bigcap_{N=1}^{\infty} \bigcup_{k=N; k \in A(t,\tau)}^{\infty} A_k) = m(t)$ for $A(t, \tau) \neq \varnothing$.

Proof. Assume τ is such that $A(t, \tau) \neq \varnothing$.

Case 1. $t \in D$. By the definition of $A(t, t)$ we see that $A(t, t) \neq \varnothing$. Therefore by Lemma 2, Lemma 8, and Claim 8 we see that

$$F'\left(\bigcap_{N=1}^{\infty} \bigcup_{k=N; k \in A(t,t)}^{\infty} A_k\right) = m(t).$$

But $A(t, \tau) \neq \varnothing$ implies $\tau \geq t$ and therefore $A(t, \tau) \supset A(t, t)$ which clearly completes the proof of the claim in this case.

Case 2. $t \notin D$. By the definition of D we see that m is continuous from the right at t. Let $\{t_i\}$ be a sequence of positive real numbers such that $t_i \downarrow t$, and therefore $m(t_i) \underset{i}{\to} m(t)$. Since we are assuming $A(t, \tau) \neq \varnothing$ and $t \notin D$ we see that by the definition of $A(t, \tau)$, $\tau > t$. So let i be large enough to insure that $\tau > t_i$.

For $n > [t_i]^* + 1$, let

$$F_n^{t_i} = \exp\left(-t \sum_{i=0}^{n+2} \lambda_i^{t_i}\right)\left\{\sum_{l=0}^{\infty} \left(\sum_{j=[t_i]^*+2}^{n+2} M_j^{t_i}\right)^{*l} * V_t t^l / l!\right\}.$$

By Lemma 1, there exists a nondecreasing function F^{t_i} such that $F_n^{t_i} \overset{(c)}{\Longrightarrow} F^{t_i}$. By Lemma 2, Lemma 8 and Claim 8:

$$F^{t_i}\left(\bigcap_{N=1}^{\infty} \bigcup_{k=N; k \in A(t,t_i)}^{\infty} A_k\right) = \text{T.V.}(F^{t_i}).$$

But $t_i > t$ implies $A(t, \tau) \supset A(t, t_i)$ and therefore

$$F^{t_i}\left(\bigcap_{N=1}^{\infty} \bigcup_{k=N; k \in A(t,\tau)}^{\infty} A_k\right) = \text{T.V.}(F^{t_i}).$$

Glancing back at the definition of F^{t_i} and recalling 0.2 we see that there exists a nondecreasing function $S_n(x)$ such that

$$(F_{t,n})\text{c.s.}(x) = S_n(x) + F_n^{t_i}(x).$$

But by Lemma 1 $F_n^{t_i} * F_{n,\infty} = F^{t_i}$ and therefore $F' = S_n * F_{n,\infty} + F^{t_i}$, which implies

$$F'\left(\bigcap_{N=1}^{\infty} \bigcup_{k=N; k \in A(t,\tau)}^{\infty} A_k\right) \geq F^{t_i}\left(\bigcap_{N=1}^{\infty} \bigcup_{k=N; k \in A(t,\tau)}^{\infty} A_k\right) = \text{T.V.}(F^{t_i}).$$

Again by Lemma 1

$$\text{T.V.}(F^{t_i}) = \exp\left(-t \sum_{l=0}^{[t_i]^*+1} \lambda_l^{t_i}\right).$$

But

$$\sum_{l=0}^{[t_i]^*+1} \lambda_l^{t_i} = \lambda_0^{t_i} + \lambda(t_i) - \lambda(0) = \lambda(t_i)$$

and therefore

$$\text{T.V.}(F^{t_i}) = \exp\left(-t\lambda(t_i)\right).$$

Now since m is continuous from the right at t, we see that λ is continuous from the right at t. So letting i go to $+\infty$, the claim clearly follows.

Let $\tau \in (0, \infty)$ such that $A(t, \tau) \neq \varnothing$. Now by Lemma 3

$$\left| \bigcap_{n=1}^{\infty} \bigcup_{k=n;\, k \in A(t, \tau)}^{\infty} A_k \right| = 0$$

and therefore by Claim 9 F' is continuous singular. From Lemma 1

$$F_t = (F_{t,n})\text{a.c.} * F_{n,\infty} + F'.$$

But clearly the first term is absolutely continuous, and therefore, since F' is continuous singular, $(F_t)\text{c.s.} = F'$. Using the fact that $b_n \geq n(n+1)/2$, an easy calculation shows that for $j \in I^+$,

0.3. $\lim_{n \in P_j} b_n/b_{n+1} = v(d_j)$.

Claim 10. $v(t) \geq \dim \operatorname{supp}(F_t)\text{c.s.}$

Proof. By Claim 9, Lemma 4, and 0.3 we see that if $A(t, \tau) \neq \varnothing$, then

$$\dim \operatorname{supp}(F_t)\text{c.s.} \leq \sup \{v(d_j) : d_j \in A(t, \tau)\}.$$

If $t \in D$, then $A(t, t) \neq \varnothing$ and by 0.3 Claim 10 is proved. If $t \notin D$, then for each $i \in I^+$ take $\tau = d_i'$ and by the definition of D_0, we see that

$$\dim \operatorname{supp}(F_t)\text{c.s.} \leq v(t)$$

which completes the proof of Claim 10.

Claim 11. $\dim \operatorname{supp}(F_t)\text{c.s.} \geq v(t)$.

Proof. From Lemma 2.1 if $n \geq [t]^* + 1$ then

$$(F_t)\text{c.s.} = (F_{t,n})\text{c.s.} * F_{n,\infty}$$

and therefore by A.3

$$\dim \operatorname{supp}(F_t)\text{c.s.} \geq \dim \operatorname{supp}(F_{t,n})\text{c.s.}$$

By A.5

$$\dim \operatorname{supp}(F_{t,n})\text{c.s.} \geq \dim \operatorname{supp}\left(\sum_{j=[t]^*+2}^{n+2} M_j^t * V_t \right).$$

Clearly now by the definition of dim supp we see that in order to prove this claim it is sufficient to show that if E is a Borel set such that

$$\sum_{j=[t]^*+2}^{n+2} M_j * V_t(E) = \lambda(n) - \lambda(t)$$

then $\dim(E) \geq v(t)$. So let E be such a Borel set. Now by Lemma 8 and A.6 we see that $G_\tau * V_t(E) = 1$ for almost all $\tau \in [t, v_n]$ with respect to $\sum_{j=[t]^*+2}^{n+2} \lambda_j^t(\tau)$.

0.4. Let $\tau_1, \tau_2 \in (0, \infty)$ be such that $\tau \geq t \geq \tau_2$. If

$$\sum_{j=1;\, d_j < \tau_2}^{\infty} \sum_{n \in P_j} 1/n^{t/d_j} < \infty$$

then by 0.1, 0.3, a review of the definitions of $\lambda_{n,\tau}$ and $\lambda'_{n,t}$, and Lemma 5 we see that

$$\text{dim supp } V_t * G_\tau \geqq \inf \{v(d_j) : d_j \in [\tau_2, \tau]\}.$$

Case 1. $t \in D$ implies there exists $l \in I^+$ such that $t = d_l$. Let $d_j < d_l$; then by the way in which we enumerated D we see that:
1. If $j > l$, then $t/d_j > c_j$ and therefore, by the way $\bar{P}_j$ was defined,

$$\sum_{j=l+1;\, d_j < t}^{\infty} \sum_{n \in \bar{P}_j} 1/n^{t/d_j} < \infty.$$

2. If $j < l$, then $t/d_j > c_l > 1$; but there can be only finitely many such j's and therefore

$$\sum_{j=1;\, d_j < t}^{l-1} \sum_{n \in \bar{P}_j} 1/n^{t/d_j} < \infty.$$

So by 0.4 we see that

$$\text{dim supp } (G_\tau * V_t) \geqq \inf_{d_j \in [t,\tau]} v(d_j),$$

for all $\tau \in [t, v_n]$ such that $D \cap [t, \tau] \neq \varnothing$.

Subcase i. m is not continuous from the right at t. Then, clearly, λ is also not continuous from the right at t, and therefore $\sum_{j=[t]^*+2}^{n+2} \lambda_j^t(\tau)$ puts positive mass on $\{t\}$. So $G_t * V_t(E) = 1$, which implies dim $(E) \geqq$ dim supp $G_t * V_t$. But by the above

$$\text{dim supp } (G_t * V_t) \geqq v(t)$$

and therefore the claim follows for this subcase.

Subcase ii. m is continuous from the right at t. Now by hypothesis if $\varepsilon > 0$, then there exists δ_ε such that $t' \in (t, t + \delta_\varepsilon)$ implies

$$v(t') \geqq v(t) - \varepsilon.$$

Also by hypothesis, λ is strictly decreasing and therefore $\sum_{j=[t]^*+2}^{n+2} \lambda_j^t(\tau)$ puts positive mass on $(t, t + \delta_\varepsilon)$. So there exists $t'' \in (t, t + \delta_\varepsilon)$ such that $V_t * G_{t''}(E) = 1$ and therefore dim $(E) \geqq$ dim supp $V_t * G_{t''}$. But

$$\text{dim supp } V_t * G_{t''} \geqq \inf \{v(d_j) : d_j \in [t, t'']\} \geqq v(t) - \varepsilon$$

and therefore dim $(E) \geqq v(t) - \varepsilon$. But $\varepsilon > 0$ was arbitrarily selected and therefore the claim follows in this subcase.

Case 2. $t \notin D$. By the definition of D we see that if $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that if $t' \in (t - \delta_\varepsilon, t + \delta_\varepsilon)$ then $v(t') > v(t) - \varepsilon$.

By hypothesis λ is strictly decreasing and therefore $\sum_{n=[t]^*+2}^{n+2} \lambda_n^t(\tau)$ puts positive

mass on $[t, t+\delta_\varepsilon)$. So therefore there exists $t'' \in [t, t+\delta_\varepsilon)$ such that $G_{t''} * V_t(E) = 1$, and so

$$\dim (E) \geqq \dim \operatorname{supp} G_{t''} * V_t.$$

Choose $\tau^* \in (t-\delta_\varepsilon, t)$; then $t/d_j > t/\tau^* > 1$ for all $d_j < \tau^*$ and therefore

$$\sum_{d_j < \tau^*} \sum_{n \in P_j} 1/n^{t/d_j} < \infty.$$

So by 0.4

$$\dim \operatorname{supp} V_t * G_{t''} \geqq \inf \{v(d_j) : d_j \in [\tau^*, t'']\} \geqq v(t) - \varepsilon$$

and therefore $\dim (E) \geqq v(t) - \varepsilon$. But ε was just some number greater than zero, and therefore the claim is established in this case, exhausting all possibilities.

Our theorem is now established if $m(t) \neq 0$ or $m'(0)$ for $t > 0$. To prove this theorem in general just repeat the above arguments with the following modifications.

Let

$$\alpha = \text{l.u.b. } A \qquad \beta = \text{g.l.b. } B.$$

Now let $\{v_n\}$ be a sequence of positive real numbers such that $v_n \geqq \alpha$ and $v_n \uparrow \beta$. For notational convenience we let $v_0 = \alpha$. Now define D_1, D_2 and D_3 as before but now restrict them to (α, β). Now let

$$D_4 = \{t \in A : v \text{ is not continuous at } t\}.$$

Since by hypothesis v is monotone on D_4, we see that D_4 is countable. Now we modify D to be

$$D = \bigcup_{i=1}^{4} D_i \cup \{\alpha, \beta\}.$$

Again D is countable and we enumerate D as before with the restriction that $c_n < 1 + \beta/n$.

The only other modifications we need make is that in the definition of M_n^t we restrict $t \in (\alpha, \beta)$ and

$$M_0^t(x) = \{-m'(0)V(x) + \lambda^\alpha G^\alpha(x) + \lambda^\beta U(x) - (\lambda(\alpha) + \lambda^\beta)\}$$

where $\lambda^\alpha = m'(0) + \lambda(\alpha)$,

$$\lambda^\beta = \lambda(\beta) - \lambda(\beta - 0) \quad \text{if } m(\beta) \neq 0,$$
$$= 0 \qquad\qquad\qquad \text{if } m(\beta) = 0.$$

$U(x)$ is the distribution function whose induced measure puts mass one on $\{0\}$, and G^α is the distribution function of a Rubin type random variable with Poissons $\{\lambda_n^\alpha\}$ and partition $\{b_n\}$ where

$$\lambda_n^\alpha = 2 \ln n \quad \text{if } n \in \bar{P}_j, d_j \leqq \alpha,$$
$$= 0 \qquad\quad \text{otherwise.}$$

COROLLARY. *Let* $\alpha \in (0, +\infty]$. *If* $v(t)$ *maps* $(0, \alpha)$ *into* $[0, 1]$ *(or* $(0, \alpha]$ *if* $\alpha < +\infty$) *such that* $v(t) \leqq \liminf_{\tau \to t^+} v(\tau)$ *at all but possibly a countable number of points of* $(0, \alpha)$, *then there exists a stochastic process,* $\{X_t : t \in [0, \infty)\}$ *with stationary independent increments such that*

(1) *if* $t \in (0, \alpha)$ *(or* $t \in (0, \alpha]$), *then* dim supp (F_t)c.s. $= v(t)$, *and*

(2) *if* $t \in [\alpha, \infty)$ *(or* $t \in (\alpha, \infty)$), *then* F_t *is absolutely continuous.*

Proof. We prove the corollary for $(0, \alpha)$; then a clear modification of this proof will establish this corollary for $(0, \alpha]$. Let $X = \{x \in (0, \alpha) : v(x) > \liminf_{\tau \to x^+} v(x)\}$ be countable and let $\{x_i\}$ be an enumeration of X. Let $\{\xi_i\}$ be a sequence of real numbers such that $\sum \xi_i < \infty$. Finally for $t \in (0, \alpha)$ let

$$\lambda(t) = \left(1 + \sum_{d_i < t} \xi_i\right) t.$$

So for $t \in (0, \infty)$, let

$$m(t) = \exp -t\lambda(t) \quad \text{if } t \in (0, \alpha),$$
$$= 0 \quad \quad \quad \quad \text{if } t \notin (0, \alpha).$$

Let $v'(t)$ be defined as

$$v'(t) = v(t) \quad \text{if } t \in (0, \alpha),$$
$$= 0 \quad \quad \text{if } t \notin (0, \alpha).$$

Now $v'(t)$ and $m(t)$ clearly satisfy the hypothesis of our theorem and the corollary follows.

All the conditions imposed on $m(t)$ and $v(t)$, with the exception that $\ln (m(t))/t$ be strictly decreasing on $(0, \infty) - (A \cup B)$ are certainly necessary conditions in order that there exists a stochastic process, $\{X_t, t \in [0, \infty)\}$, with stationary independent increments such that T.V.(F_t)c.s. $= m(t)$, and $v(t) = $ dim supp (F_t)c.s.

The author has been unable to determine what happens if $\ln m(t)/t$ is not strictly decreasing on $(0, \infty) - (A \cup B)$. In fact, the author has not even been successful in finding a process $\{X_t : t \in [0, \infty)\}$ with continuous distribution functions and with stationary independent increments such that $\ln (\text{T.V.}(F_t)\text{c.s.})/t$ increases over some interval (α, β) where $\alpha, \beta \in (0, \infty)$.

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