ON THE EXISTENCE OF IMMERSIONS AND SUBMERSIONS

BY

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1. Introduction. Let $M$ and $N$ be manifolds (always assumed to be smooth, connected, and without boundary) and let $f: M \to N$ be a smooth map. If at each point of $M$ the Jacobian matrix of $f$ has maximal rank, we call $f$ a map of maximal rank. (If $\dim M < \dim N$, then $f$ is an immersion, while if $\dim M > \dim N$, $f$ is a submersion.)

QUESTION. Which homotopy classes of continuous maps $M \to N$ contain a smooth map of maximal rank?

This question has been reduced to a question purely in homotopy theory by M. Hirsch (for immersions) and Phillips (for submersions). (See [7], [17].) Their results are as follows.

We will use the following notation. For any vector bundle $\xi$ over a complex $X$ we let $(\xi)$ denote the stable equivalence class determined by $\xi$. We will say that a stable bundle $(\xi)$ has geometric dimension $\leq n$ (for some positive integer $n$) if there is an $n$-plane bundle over $X$ which is stably isomorphic to $\xi$. For a smooth manifold $V$ we let $\tau_V$ denote the tangent bundle and $\nu_V$ the stable normal bundle; i.e. $\nu_V = -\tau_V$.

**Theorem of Hirsch.** Let $f: M \to N$ be a continuous map between manifolds, where $\dim M < \dim N$. Then $f$ is homotopic to an immersion if, and only if, the stable bundle

$$f^*(\tau_N) + \nu_M$$

has geometric dimension $\leq \dim N - \dim M$.

A dual result holds for submersions.

**Theorem of Phillips.** Let $M$ be an open manifold and $f: M \to N$ a continuous map, where $\dim M > \dim N$. Then $f$ is homotopic to a submersion if, and only if, the stable bundle

$$(\tau_M) + f^*\nu_N$$

has geometric dimension $\leq \dim M - \dim N$.

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We need here to remark that since $M$ is open, the bundle $\tau_M$ is stable over $M$. (Because $M$ has the homotopy type of an $(m-1)$-complex, $m = \dim M$. See [8, §3.2].)

For a simple application of these theorems, suppose that $M$ and $N$ are $\pi$-manifolds (i.e. each has stably trivial tangent bundle) and that $\dim M \neq \dim N$. Since each also has stably trivial normal bundle, it follows by the above theorems that every continuous map $M \to N$ is homotopic to a smooth map of maximal rank.

In the following two sections we use the theorems of Hirsch and Phillips to study more general manifolds $M$ and $N$, using in part results from [20] and [21].

2. Immersion of manifolds. We will use the following notation. $M^m$ and $N^n$ will denote smooth connected manifolds with respective dimensions $m$ and $n$, $m < n$. We define the codimension of a continuous map $M \to N$ to be the positive integer $n - m$. By the basic theorem of Whitney [23] every map of codimension $m$ (i.e. $n = 2m$) is homotopic to an immersion, and so we consider here the case $n < 2m$.

For any bundle $\xi$ over a complex $X$ we let $w_i(\xi) \in H^i(X; \mathbb{Z}_2)$ denote the $i$th Stiefel-Whitney class of $\xi$, $i \geq 0$. For a manifold $V$, we set

$$w_i(V) = w_i(\tau_V), \quad \bar{w}_i(V) = w_i(\nu_V).$$

Suppose now that $M$ and $N$ are manifolds and $f: M \to N$ a continuous map. Set

$$\nu_f = f^*(\tau_N) + \nu_M.$$

We say that $f$ is orientable if

$$f^*w_1(N) = w_1(M),$$

i.e. the stable bundle $\nu_f$ is orientable.

By Hirsch (see §1), it follows that if $f$ is homotopic to an immersion then

$$w_i(\nu_f) = 0, \quad i > n - m,$$

$$\delta w_{n-m}(\nu_f) = 0, \quad n - m \text{ even, } f \text{ orientable.}$$

(Here $\delta$ denotes the Bockstein coboundary associated with the exact sequence $Z \to Z \to \mathbb{Z}_2$.) Thus, in what follows we will be mainly concerned with sufficient conditions for $f$ to be homotopic to an immersion.

The codimension $f = m - 1$, $m \geq 4$.

**Theorem 2.1.** Let $M^m$ and $N^{2m-1}$ be manifolds, $m \geq 4$, and let $f: M \to N$ be a continuous map. If $m$ is odd, assume that $f$ is orientable. Then $f$ is homotopic to an immersion if, and only if,

$$w_m(\nu_f) = 0, \quad m \text{ even,} \quad \delta w_{m-1}(\nu_f) = 0, \quad m \text{ odd.}$$

The proof of the theorem follows at once from classical obstruction theory [18], as will be shown in §5. (In the case $m$ odd we can omit the hypothesis that $f$ is orientable if we use local coefficients.)
If $M$ and $N$ are orientable manifolds, then every map $f: M \to N$ is orientable. Since $H^m(M; \mathbb{Z}) \cong \mathbb{Z}$, we then obtain from 2.1

**Corollary 2.2.** Let $M^{2q+1}$ and $N^{4q+1}$ be orientable manifolds, $q \geq 2$. Then every map $f: M \to N$ is homotopic to an immersion.

**Codimension** $f=m-2$, $m \geq 5$.

**Theorem 2.3.** (a) Let $M^{4q+1}$ and $N^{8q}$ be manifolds, $q \geq 1$, and $f: M \to N$ a continuous map. Then $f$ is homotopic to an immersion if $w_4(v_r)=0$.

(b) Let $M^{4q+2}$ and $N^{8q+2}$ be manifolds, $q \geq 1$, and let $f: M \to N$ be an orientable map. Suppose that

$$\delta w_{4q}(v_r) = 0 \text{ and } w_{4q+2}(v_r) = 0.$$  

If $M$ is closed, suppose also that $M$ is orientable and that

$$f^*w_2(N) = 0, \quad w_{4q}(v_r) \cdot w_2(M) = 0.$$  

Then $f$ is homotopic to an immersion.

(c) Let $M^{4q+3}$ and $N^{8q+4}$ be manifolds, $q \geq 1$, and let $f: M \to N$ be an orientable map. Suppose that $w_{4q+2}(v_r)=0$. If $M$ is open or if $M$ is closed, orientable, and either $f^*w_2(N) \neq 0$ or

$$f^*w_2(N) = 0 \text{ and } w_{4q+1}(v_r) \cdot w_2(M) = 0,$$

then $f$ is homotopic to an immersion.

The proof uses the results of [20], and will be given in §5.

We say that an orientable manifold $M$ is a spin manifold if $w_2(M)=0$; we say that an orientable map $f: M \to N$ is a spin map iff $f^*w_2N=w_2M$.

**Codimension** $f=m-3$, $m \geq 5$.

**Theorem 2.4.** Let $f: M^m \to N^{2m-3}$ be an orientable map, with $m \geq 5$ and $m \equiv 0 \mod 4$. If $m \equiv 1 \mod 4$, assume that $H^{m-1}(M; \mathbb{Z}_2) = 0$. If $m \equiv 2 \mod 4$, assume that either $M$ is open or that $M$ is a closed spin manifold and $f$ is a spin map. If $m \equiv 3 \mod 4$, assume that $M$ is a closed spin manifold and $f$ is a spin map. Then $f$ is homotopic to an immersion if

$$\delta w_{m-3}(v_r) = 0, \quad m \equiv 1 \mod 4,$$

$$w_{m-2}(v_r) = 0, \quad m \equiv 2 \mod 4,$$

$$\delta w_{m-3}(v_r) = 0, \quad w_{m-1}(v_r) = 0, \quad m \equiv 3 \mod 4.$$  

The proof will be given in §5.

**Codimension** $f=m-4$, $m \geq 11$.

**Theorem 2.5.** Let $M^{8q+3}$ and $N^{10q+2}$ be manifolds, $q \geq 1$, and let $f: M \to N$ be a spin map. Suppose that $M$ is a closed spin manifold. If $w_8(v_r)=0$, then $f$ is homotopic to an immersion.
The proof will be given in §5.

Remark. If one takes the manifold \( N \) to be \( \mathbb{R}^n \), then \( v_l = v_M \) and one can obtain stronger results than those given in 2.1–2.4 by using [14]. Note, for example, [6], [11] and [21].

3. Submersion of manifolds. In this section we assume that \( M^m \) and \( N^n \) are smooth connected manifolds with \( m > n \). Moreover, throughout the section we assume that \( M \) is open. Suppose that \( n = 1 \), i.e. \( N = \mathbb{R}^1 \) or \( S^1 \). Then, as observed by Phillips [17], every map \( M^m \to N^1 \) is homotopic to a submersion (since \( M \) has the homotopy type of an \((m-1)\)-complex). We consider here the case \( n = 2 \). For a map \( f: M \to N \) set \( \sigma_j = (\tau_M) + f^* v_N \). We will prove

**Theorem 3.1.** Let \( f: M^m \to N^2 \) be a continuous map, \( m \geq 5 \), where \( M \) is open. If \( m \) is even assume that \( f \) is orientable. Then \( f \) is homotopic to a submersion if, and only if,

\[
\delta w_{m-2}(\sigma_j) = 0, \quad m \text{ odd}, \quad m \text{ even}.
\]

Suppose that \( N \) is a closed orientable surface. Then the stable normal bundle of \( N \) is trivial, and so \( w_i(\sigma) = w_i(M), i \geq 0 \).

On the other hand suppose that \( M = M' - \partial M' \), where \( M' \) is a compact orientable manifold with nonempty boundary \( \partial M' \). It follows from results of Wu and Massey [24], [12], [13], that

\[
w_{m-1}(M) = 0, \quad m \equiv 3 \mod 4, \quad \delta w_{m-2}(M) = 0, \quad m \text{ even}.
\]
(See [5, §2].) Thus from 3.1 we obtain

**Corollary 3.2.** Let \( M' \) be a compact orientable \( m \)-manifold with nonempty boundary \( \partial M' \), and let \( N \) be a closed orientable surface. Let \( M \) denote the open manifold \( M' - \partial M' \). If \( m \geq 5 \) and \( m \equiv 1 \mod 4 \), then every map \( M \to N \) is homotopic to a submersion.

(Note [3] for conditions on an open manifold that it be expressible as \( M' - \partial M' \).

Our results on submersions are much less extensive than the results in §2 on immersions. If \( f: M^m \to N^n \), with \( n > 2 \), then one can still apply the results of [20], [21] to obtain conditions for \( \sigma_j \) to have codimension \( \leq m - n \). However, the results in general will be expressed in terms of higher order cohomology operations.

4. Examples. Let \( M^m \) and \( N^n \) be manifolds, \( m \neq n \). The problem of determining the set of maps from \( M \) to \( N \) of maximal rank falls into two parts: First, determine the homotopy classes of maps from \( M \) to \( N \), \([M, N]\); and second, for each homotopy class of maps, determine whether it contains a map of maximal rank. If \( M \) and \( N \) fit the hypotheses of one of the theorems in §2 or §3, and if \( f: M \to N \), then the second step above consists simply in computing the characteristic classes.
w_k(v), if m<n, w_k(σ), if m>n. By the Whitney duality formula, these classes are
given as follows:

\[ w_k(v) = \sum_{i+j=k} (-1)^j \bar{w}_i(M) \cup f^*w_j(N), \quad w_k(\sigma) = \sum_{i+j=k} w_i(M) \cup f^*\bar{w}_j(N). \]

For an illustration we take N to be the real projective space \( RP^n \) (of dim n)
and the complex projective space \( CP^n \) (of dim 2n).

**Example A.** \( N=RP^n, \ n>1 \). Since \( RP^n \) is the n-skeleton of the Eilenberg-
MacLane space \( K(Z_2, 1) \), it follows that if \( X \) is a complex of dim \(< n \), then \( [X, RP^n] = H^1(X; Z_2) \). The correspondence here is given by \( [f] \rightarrow f^*x \), where \( x \) generates
\( H^1(RP^n; Z_2) \). Since \( w(RP^n) = (1+x)^{n+1} \), we have

\[ w_k(v) = \sum_{i+j=k} \binom{n+1}{i} \bar{v}^i \cup \bar{w}_j(M), \]

where \( f: M^m \rightarrow RP^n, \ m<n, \) and \( v=f^*x \). The results of §2 can now be used to
determine the immersions of \( M^m \) in \( RP^n \), for appropriate dimensions \( m \) and \( n \). (The
difficulty in studying submersions is that in general we do not know how to deter-
mine the set \( [M^m, RP^n] \), when \( m>n \).)

**Example B.** \( N=CP^n, \ n\geq 1 \). Now \( CP^n \) is the \((2n+1)\)-skeleton of the Eilenberg-
MacLane space \( K(Z_2, 2) \), and so if a complex \( X \) has dimension \( \leq 2n \), then \( [X, CP^n] = H^2(X; Z) \), the correspondence being given by \( [f] \rightarrow f^*y \), where \( y \) generates
\( H^2(CP^n; Z) \). Let \( M^m \) be a manifold and \( f: M^m \rightarrow CP^n \) a map, \( m\leq 2n \). Since
\( w(CP^n) = (1+y)^{n+1} \) mod 2, we have

\[ w_{2n}(v) = \sum_{i+j=k} \binom{n+1}{i} \bar{v}^i \cup \bar{w}_{2j}(M), \]

where \( v=f^*y \). The results of §2 can now be used to determine the immersions of
\( M^m \) in \( CP^n \) for appropriate \( m \) and \( n \). Take \( M \) to be \( CP^q \), for example. Since
\( H^2(CP^q; Z) \approx Z, \) we have \([CP^q, CP^n] = Z, q\leq n, \) and so each homotopy class of
map \( f: CP^q \rightarrow CP^n \) is characterized by an integer, called the degree of the map.
(See Feder [4].) By 2.3(b) one can show:

(4.1) Let \( q \) be a positive integer. Then for each integer \( d \) there is an immersion of
\( CP^q+1 \) in \( CP^{q+1} \) of degree \( d \).

**Remark.** (4.1) suggests the following general problem. Let \( q \) and \( n \) be integers,
\( 0<q<n \). Determine the integers \( d \) for which there is an immersion of \( CP^q \) in \( CP^n \)
of degree \( d \). By Whitney [23], if \( n\geq 2q \) all integers \( d \) can occur. By Feder [4], if
\( n\leq [3q/2]-1 \), only \( d=\pm 1 \) can occur. (In [22] we show that for \( q=2, \ n=3, \) only \( d=\pm 1 \) can occur, while if \( q=3, \ n=4, \) then \( d \) can occur if, and only if, there is an
integer \( e \) such that \( 5d^2 = e^2 + 4 \). Note also [4, Theorem 8.3].)

5. **Proofs of theorems.** For a topological group \( G \) let \( BG \) denote the classifying
space for \( G \) constructed by Milnor [15]. Let \( O(n), n\geq 1, \) denote the orthogonal
group of rank \( n, \) and let \( O \) denote the stable orthogonal group [2]. If \( X \) is a complex
then a stable vector bundle over $X$ can be regarded as a map $X \to BO$. Now the natural inclusion $O(n) \subset O$ induces a map $p_n: BO(n) \to BO$, and a stable bundle $\xi$ over $X$ has geometric dimension $\leq n$ if, and only if, there is a map $\eta: X \to BO(n)$ such $p_n \circ \eta = \xi$. Up to homotopy type the map $p_n$ can be regarded as a fiber map [1], with fiber $V_n = O/O(n)$.

By Stiefel (see [18]), $V_n$ is $(n-1)$-connected and (for $n \geq 3$),

$$
\pi_n(V_n) = \begin{cases} 
Z, & n \text{ even}, \\
Z_2, & n \text{ odd}. 
\end{cases}
$$

Thus by standard obstruction theory (e.g. see [18], [10], or [19]), if $X$ has dim $\leq n+1$ then a stable bundle $\xi$ over $X$ has geometric dim $\leq n$ if, and only if,

$$
(*) \quad w_{n+1}(\xi) = 0, \quad n \text{ odd,} \quad \delta w_n(\xi) = 0, \quad n \text{ even},
$$

assuming that $\xi$ is orientable in the case $n$ even. This proves Theorem 2.1. Furthermore, $(*)$ proves 2.3(a) (since $\pi_{4q}(V_{4q-1}) = 0$, see [16]) and also proves 2.4 in the case $\equiv 1 \pmod{4}$ (since $\pi_{4q+1}(V_{4q-2}) = 0$, [16]). Finally, since an open $m$-manifold has the homotopy type of an $(m-1)$-complex, $(*)$ also proves 3.1, and 2.3–2.4 in the cases $M$ is open.

To prove the remaining theorems in §2 (assuming now that $M$ is a closed manifold) we need some results from [20], and [21]. In [20] we do not deal with stable bundles, and so we will need the following relationship between $n$-plane bundles and stable bundles.

**Lemma 5.1.** Let $X$ be a complex of dim $n$ and let $\xi$ be an oriented stable vector bundle over $X$ such that $w_n(\xi) = 0$. Then there is an oriented $n$-plane bundle $\eta$ over $X$ such that $\eta$ is stably equivalent to $\xi$ and $\chi(\eta) = 0$ (where $\chi(\eta)$ denotes the Euler class of $\eta$). Moreover, $\xi$ has geometric dimension $\leq k$ (where $k < n$) if, and only if, $\eta$ has $n - k$ linearly independent cross-sections.

The proof is standard and is left to the reader.

**Proof of 2.3(b).** Let $\eta$ be an $n$-plane bundle over $M$ corresponding to the stable bundle $\nu$. Thus by 5.1 and by the hypotheses of 2.3(b),

$$
w_2(\eta) = w_2(M), \quad w_4(\eta) \cdot w_2(M) = 0, \quad \delta w_4(\eta) = 0, \quad \chi(\eta) = 0,
$$

and so by Theorem 7.3 of [20], $\eta$ has 2 linearly independent cross sections. Thus, by 5.1, $\nu$ has geometric dimension $\leq 4q$ and so by Hirsch, $f$ is homotopic to an immersion.

Before proving 2.3(c) we need a preliminary result. Let $\xi$ be a vector bundle (stable or otherwise) over a complex $X$. Define a homomorphism

$$
\alpha_\xi: H^i(X; Z_2) \to H^{i+2}(X; Z_2), \quad i \geq 0,
$$

by

$$
x \to Sq^2(x) + x \cdot w_2(\xi).
$$
Suppose that $X$ is a closed manifold $M$ of dim $m$, and let $\xi, \eta$ be two bundles over $M$. Then

$$\alpha_\xi = \alpha_\eta : H^{m-2}(M; \mathbb{Z}_2) \rightarrow H^m(M; \mathbb{Z}_2)$$

if, and only if, $w_2(\xi) = w_2(\eta)$, as may be seen by using Poincaré duality. In particular if we take $\xi$ to be the tangent bundle of $M$, then by Wu [24] $\alpha_\xi H^{m-2}(M; \mathbb{Z}_2) = 0$, provided $M$ is orientable, and so we have:

**Lemma 5.2.** Let $\eta$ be a bundle over a closed orientable $m$-manifold $M$, $m \geq 2$. If $w_2(\eta) \neq w_2(M)$, then

$$\alpha_\eta H^{m-2}(M; \mathbb{Z}_2) = H^m(M; \mathbb{Z}_2).$$

**Proof of 2.3(c).** The first obstruction to $\nu_\eta$ pulling back to $BO(4q+1)$ is the class $w_{4q+2}(\nu_\eta)$, which vanishes by hypothesis. The second (and final) obstruction is a coset in $H^{4q+3}(M; \mathbb{Z}_2)$ of the subgroup $\alpha_{\nu_\eta} H^{4q+1}(M; \mathbb{Z}_2)$. (See [9], [10], [20].)

Now if $f^*w_2(N) \neq 0$ then $w_2(\nu_\eta) \neq w_2(M)$, and so by (5.2), $\alpha_{\nu_\eta} H^{4q+1}(M; \mathbb{Z}_2) = H^{4q+3}(M; \mathbb{Z}_2)$, since $M$ is closed. Thus the second obstruction contains zero and hence vanishes, which completes the proof of 2.3(c) in this case.

Suppose on the other hand that

$$f^*w_2(N) = 0, \quad w_{4q+2}(\nu_\eta) \cdot w_2(M) = 0.$$

Then the theorem follows, as above, by using 5.1 and applying 7.3 of [20]. We omit the details.

**Proof of 2.4.** We have already done the case $m \equiv 1 \mod 4$. If $m \equiv 2 \mod 4$, we use Theorem 1.3 of [21] (applied to the bundle $\nu_\eta$), while if $m \equiv 3 \mod 4$ we use 5.1 above together with Theorem 1.1 of [21]. We leave the details to the reader.

**Proof of 2.5.** This follows at once from [21, Theorem 1.3] applied to the bundle $\nu_\eta$.

**References**


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