ON QUASI-ELLIPTIC BOUNDARY PROBLEMS

BY

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1. Introduction. In this paper we consider the boundary problem

(1.1) \( P(x, D)u = f \) in \( x_n > 0, \)

(1.2) \( Q_j(x, D)u = g_j \) on \( x_n = 0, 1 \leq j \leq r, \)

where \( P(x, D) \) is a quasi-elliptic operator and \( \{Q_j(x, D)\}_{j=1}^r \) is a system of boundary operators satisfying the complementing condition.

First we derive the a priori estimate of the form

(1.3) \[ \|v\|_m \leq C(\|P(x, D)v\|_0 + \sum_{j=1}^r |Q_j(x, D)v|_{k_j} + \|v\|_0) \]

for suitable boundary norms \( |Q_j(x, D)v|_{k_j}, j = 1, \ldots, r. \)

Secondly we shall consider about the hypo-analyticity at the boundary \( x_n = 0 \) for the above problem with the simple boundary operators.

In §2, a definition of a quasi-elliptic operator is given. In §3, a definition of a quasi-elliptic boundary problem is given. §4 and §5 are devoted to derive an a priori estimate (coerciveness estimate) for the case of constant coefficients. In §6 and §7, we shall prove the coerciveness estimate for the case of variable coefficients. In §8 we prove the regularity at the boundary of the solutions of quasi-elliptic boundary problem.

In §§9, 10 and 11 we consider hypo-analyticity at the boundary for quasi-elliptic boundary problems. Theorem 9.2 was suggested by Professor Lions.

The author is deeply indebted to the authors [2], [3] and [12] to make this paper. The author expresses his hearty thanks to Professors Mizohata, Lions, and Kuroda for their valuable suggestions.

2. Preliminaries. Let \( \mathbb{R}^n \) be the \( n \)-dimensional Euclidean space whose point is denoted by \( (x_1, \ldots, x_n) \). For convenience, set \( x = (x_1, \ldots, x_{n-1}) \), \( y = x_n \) and denote by \( (x, y) \) a point of \( \mathbb{R}^n \). The half spaces \( y > 0 \) and \( y \geq 0 \) are denoted by \( \mathbb{R}_+ \) and \( \mathbb{R}_n \) respectively.

Let \( \alpha = (\alpha_1, \ldots, \alpha_n) \) be a multi-index of nonnegative integers with length \( |\alpha| = \alpha_1 + \cdots + \alpha_n \). We put \( D_j = i^{-1}(\partial/\partial x_j), 1 \leq j \leq n, (i^2 = -1) \), and

\[ D_x = (D_1, \ldots, D_{n-1}), \quad D_y = D_n, \quad D = (D_1, \ldots, D_n). \]

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Then the linear differential operators of order $m$ with constant coefficients can be written as

\begin{equation}
P(D) = P(D_x, D_y) = \sum_{|\alpha| \leq m} a_\alpha D_1^\alpha D_2^\alpha,
\end{equation}

where the coefficients $a_\alpha$ are complex constants. The polynomial corresponding to $P(D_x, D_y)$ is

\begin{equation}
P(\xi, \eta) = \sum_{|\alpha| \leq m} a_\alpha \xi_1^{\alpha_1} \cdots \xi_{n-1}^{\alpha_{n-1}} \eta^n,
\end{equation}

where $\xi = (\xi_1, \ldots, \xi_{n-1})$.

**Definition 2.1.** A quasi-elliptic operator of weight $q$ is defined as an operator (2.1) satisfying the following conditions:

\begin{align*}
m_i &> 0, \quad 1 \leq i \leq n, \quad \text{are given integers}, \quad m = \max_{1 \leq j \leq n} m_j, \\
q_i &> 0, \quad 1 \leq i \leq n, \quad \text{given integers}, \quad m = \max_{1 \leq j \leq n} m_j, \\
\langle \alpha, q \rangle &> 0, \quad \text{for any } (\alpha, q) \in \mathbb{Z}^n.
\end{align*}

The principal part of $P$ with respect to the weight $q$ is $P_0(\xi, \eta) = \sum_{\langle \alpha, q \rangle \leq m} a_\alpha D_\alpha$.

(2.4) \[ \sum_{1 \leq j \leq n-1} |\xi_j|^{m_j} + |\eta|^{m_n} \leq C |P_0(\xi, \eta)| \quad \text{for any } (\xi, \eta) \in \mathbb{R}^n. \]

We see that the quasi-elliptic operators are hypo-elliptic. We note that when $m_j = m$ for every $j$, the quasi-elliptic operators are just the elliptic operators of order $m$. If $m_n = 1$ and $m_j = 2$ for $j < n$, we find that the heat equation is quasi-elliptic. Also, the $p$-parabolic equations in the sense of Petrowsky are quasi-elliptic (cf. Friberg [3]).

Let $\tau_1(\xi), \ldots, \tau_m(\xi)$ denote the roots of $P_0(\xi, z) = 0$ for each real vector $\xi = (\xi_1, \ldots, \xi_{n-1})$. In the case $n > 2$ we see from the condition (2.4) that the number $r$ of the roots with the positive imaginary part is independent of $\xi \neq 0$, and in this case we shall say that $P(\xi, \eta)(P(D))$ is of determined type $r$. In the case of $n = 2$ we suppose this root-condition on $P_0(\xi, \eta)$.

3. **Coerciveness inequality (I).** The constant coefficient case.

We consider a quasi-elliptic operator $P(D)$ of weight $q = (m/m_1, \ldots, m/m_n)$ and of determined type $r (1 \leq r \leq m_n)$:

\begin{equation}
P(D) = P(D_x, D_y) = \sum_{\langle \alpha, q \rangle \leq m} a_\alpha D_1^\alpha D_2^\alpha.
\end{equation}

Here we may assume the coefficient of $D_1^m$ is equal to 1. The corresponding polynomial of $P(D)$ is

\begin{equation}
P(\xi, \eta) = \sum_{\langle \alpha, q \rangle \leq m} a_\alpha \xi_1^{\alpha_1} \cdots \xi_{n-1}^{\alpha_{n-1}} \eta^n.
\end{equation}
By rearrangement if necessary we may assume that

\[(3.2) \quad \text{Im } \tau_k(\xi) > 0, \quad 1 \leq k \leq r,\]

\[(3.3) \quad \text{Im } \tau_k(\xi) < 0, \quad r < k \leq m_n,\]

Set

\[P_+ = \prod_{k=1}^{r} (\eta - \tau_k(\xi)), \quad P_- = P_0/P_+.

Similarly we consider the boundary operators of the form

\[(3.4) \quad Q_j(D) = \sum_{\langle \beta, \delta \rangle = p_j} b_\delta D^\beta, \quad j = 1, \ldots, r, \quad 0 \leq p_j \leq (m_n - 1) \frac{m}{m_n} = m - q_n.

The corresponding polynomial of \(Q_j(D)\) is

\[Q_j(\xi, \eta) = \sum_{\langle \beta, \delta \rangle = p_j} b_\delta \xi^{\delta_1} \cdots \xi^{\delta_{n-1}} \eta^\delta\]

and

\[Q_j(\xi, \eta) = \sum_{\langle \beta, \delta \rangle = p_j} b_\delta \xi^{\delta_1} \cdots \xi^{\delta_{n-1}} \eta^\delta.

**Definition 3.1 (Complementing condition).** We shall say that the \(Q_j(D)\) \((j = 1, \ldots, r)\) cover \(P(D)\) when \(Q_j(\xi, \eta)\) \((j = 1, \ldots, r)\) are linearly independent modulo \(P_+ (\xi, \eta)\) as the polynomials in \(\eta\) for every nonzero \(\xi \in \mathbb{R}^{n-1}\).

Let \(C_\xi(\mathbb{R}^n)\) denote the set of complex-valued functions which are infinitely differentiable in \(\mathbb{R}^n\), and vanish for \((x, y)\) with \(|x|^2 + y^2\) sufficiently large. We denote by \(\delta(\xi, y)\) the Fourier transform of \(v(x, y) \in C_\xi(\mathbb{R}^n)\) with respect to the variables \(x_1, \ldots, x_{n-1}:

\[\delta(\xi, y) = (2\pi)^{-(n-1)/2} \int_{\mathbb{R}^{n-1}} e^{-i\langle \xi, x \rangle} v(x, y) \, dx.

As usual we set

\[\|v\| = \left( \int_{(\mathbb{R}^n)^a} |v(x, y)|^2 \, dx \, dy \right)^{1/2} = \left( \int_0^\infty \int_{|\xi| < \infty} |\delta(\xi, y)|^2 \, d\xi \, dy \right)^{1/2}.

Corresponding to the operator (3.1) we employ the notation

\[(3.5) \quad \langle \xi \rangle = (|\xi_1|^{m_1 + \cdots + |\xi_{n-1}|^{m_{n-1}})^{1/m}.

For a real number \(p\), we shall make use of the scalar product

\[\langle v_1, v_2 \rangle_p = \int_{\mathbb{R}^{n-1}} (1 + \langle \xi \rangle^2)^p \delta_1(\xi, 0) \overline{\delta_2(\xi, 0)} \, d\xi, \quad v_1, v_2 \in C_\xi(\mathbb{R}^n)^a).

The corresponding norm is given by

\[(3.6) \quad |v|_p = (\langle v, v \rangle_p)^{1/2}.

Then we have the following two theorems.
Theorem 3.1. Let $P(D)$ and $Q_j(D)$, $j=1, \ldots, r$, be given as above and $Q_j(D)$ $(j=1, \ldots, r)$ cover $P(D)$. Then there is a constant $C$ depending only on $P^0$ and $Q_j$ $(j=1, \ldots, r)$ such that

$$
(3.7) \sum_{\langle \alpha, q \rangle = m} \| D^\alpha v \| \leq C \left( \| P^0(D)v \| + \sum_{j=1}^r |Q_j(D)v|_{m-p_j-(m/2m_n)} \right)
$$

for all $v \in C^\infty_0((R^+_n)^r)$.

Theorem 3.2. Let $P(D)$ and $Q_j(D)$, $j=1, \ldots, r$, be the same as in Theorem 3.1. Then there is a constant $C$ depending only on $P$ and $Q_j$ $(j=1, \ldots, r)$ such that

$$
(3.8) \sum_{\langle \alpha, q \rangle \leq m} \| D^\alpha v \| \leq C \left( \| P(D)v \| + \sum_{j=1}^r |Q_j(D)v|_{m-p_j-(m/2m_n)} + \| v \| \right)
$$

for all $v \in C^\infty_0((R^+_n)^r)$.

Theorem 3.1 can be proved by the modification of Schechter's method [12], we shall give the complete proof in the following section. The proof of Theorem 3.2 will be given in §5.

4. Proof of Theorem 3.1. Let $R(\xi, \eta)$ be any monomial such that

$$
(4.1) R(\xi, \eta) = \xi^a_1 \cdots \xi^a_{m_n-1} \eta^{a_n}, \quad \langle \alpha, q \rangle = m.
$$

Then there is a constant $K_1$ such that

$$
(4.2) |R(\xi, \eta)| \leq K_1 |P^0(\xi, \eta)|, \quad (\xi, \eta) \in R^n.
$$

In fact, $R(\xi, \eta)/P^0(\xi, \eta)$ is continuous on the surface $|\xi|^2 + |\eta|^2 = 1$ in $R^n$, and to replace $\xi$, by $\xi t^{1/m_n}, j=1, \ldots, n-1$ and $\eta$ by $\eta t^{1/m_n}$ in (4.2) is nothing but to multiply both sides of (4.2) by $t$ if $t>0$. Hence (4.2) is valid for every $(\xi, \eta) \in R^n$.

Now it will be convenient to make the following definition.

Definition 4.1. We shall say that a function $p(\xi, \eta)$ is homogeneous of degree $k$ (with respect to weight $q$), if for any $t>0$ it holds

$$
(4.3) p(t^{m_n/m_1}\xi_1, t^{m_n/m_2}\xi_2, \ldots, t^{m_n/m_n}\eta) = t^k p(\xi, \eta), \quad (\xi, \eta) \in R^n.
$$

We recall that $Q_j^0(\xi, \eta) = \sum_{\langle \alpha, q \rangle = p_j} b_\alpha \xi^{a_1} \cdots \xi^{a_{m_n-1}} \eta^{a_n}, 0 \leq p_j \leq m - (m/m_n)$ that is, $Q_j^0$ is homogeneous of degree $p_j$ with respect to weight $q$. We observe that by multiplying each $Q_j^0(\xi, \eta)$ by an appropriate power of $\langle \xi \rangle = (|\xi_1|^{m_1} + \cdots + |\xi_{n-1}|^{m_{n-1}})^{1/m}$, we may assume that each $Q_j^0(\xi, \eta)$ is homogeneous of degree $m - (m/m_n)$ with respect to weight $q$. Then $Q_j^0(\xi, \eta), 1 \leq j \leq r$, may no longer be polynomials in the $\xi$, but this does not affect the following argument.

For simplicity we assume that the roots $\tau_\alpha(\xi)$ of $P^0(\xi, z)=0$ are simple, because

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(1) In this paper we use the same symbol $C$ to express different constants.
it is easy to prove Theorem 3.1 for the general case. Resolving into partial fractions we have

\[
\frac{R}{p^0} = \sum_{k=1}^{m_n} \frac{e_k}{\eta - \tau_k}, \quad \frac{Q_0}{p^0} = \sum_{k=1}^{m_n} \frac{q_k}{\eta - \tau_k}
\]

where

\[
e_k(\xi) = \frac{R(\xi, \tau_k)}{\partial p^0(\xi, \tau_k)} \quad q_k(\xi) = \frac{Q_0(\xi, \tau_k)}{\partial p^0(\xi, \tau_k)}, \quad 1 \leq j \leq r, 1 \leq k \leq m_n.
\]

We can easily verify

\[
(4.4) \quad \tau_k(\xi) = \tau_k(t^{m/m_n} \xi_1, \ldots, t^{m/m_n-1} \xi_{n-1}), \quad t > 0, 1 \leq k \leq m_n,
\]

\[
(4.5) \quad e_k(t^{m/m_n} \xi_1, \ldots, t^{m/m_n-1} \xi_{n-1}), \quad t > 0, 1 \leq k \leq m_n.
\]

Similarly we have

\[
(4.6) \quad q_k(\xi) = q_k(t^{m/m_n} \xi_1, \ldots, t^{m/m_n-1} \xi_{n-1}), \quad t > 0, 1 \leq j \leq r, 1 \leq k \leq m_n.
\]

In particular, it follows that there are constants $K_2$ and $K_3$ such that

\[
(4.7) \quad |e_k(\xi)| \leq K_2 \langle \xi \rangle^{m/m_n} = K_2 \left( \sum_{j=1}^{n-1} |j|^m \right)^{1/m_n}, \quad 1 \leq k \leq m_n,
\]

\[
(4.8) \quad K_3^{-1} \langle \xi \rangle^{m/m_n} \leq |\Im \tau_k(\xi)| \leq K_3 \langle \xi \rangle^{m/m_n}, \quad 1 \leq k \leq m_n.
\]

Let $\delta(\xi, y)$ be the Fourier transform of $v(x, y) \in C_0^\infty((R^n)^n)$ with respect to the variables $x_1, \ldots, x_{n-1}$ and define it to be zero for $y < 0$. We consider $\delta(\xi, y)$ as a function of $y$ with a vector parameter $\xi$. Set

\[
\delta(\xi, \eta) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{-i\eta \cdot \varphi(\xi, y)} dy.
\]

Then we see that [recall that $D_\eta = i^{-1}(\partial/\partial y)$]

\[
[\eta, D_\eta] = \eta \delta + \frac{i}{(2\pi)^{1/2}} \delta(\xi, 0).
\]

Next define

\[
f(\xi, y) = p^0(\xi, D_\eta)\delta(\xi, y) \quad \text{for } y \geq 0
\]

\[
= 0 \quad \text{for } y < 0,
\]

and set

\[
p^0_k(\xi, \eta) = \frac{p^0(\xi, \eta)}{\eta - \tau_k(\xi)}, \quad 1 \leq k \leq m_n.
\]

Then by (4.9)

\[
(4.10) \quad \tilde{f} = (\eta - \tau_k)p^0_k(\xi, D_\eta)\delta + W_k, \quad 1 \leq k \leq m_n.
\]
where
\[ W_k = \frac{i}{(2\pi)^{1/2}} P_k^0(\xi, D_y)\psi(\xi, 0), \quad 1 \leq k \leq m. \]

Since
\[ R = \sum_{k=1}^{m} e_k P_k^0, \]
we have by (4.9)
\[
[R(\xi, D_y)\psi]^* = \sum_{k=1}^{m} e_k [P_k^0(\xi, D_y)\psi]^* = \sum_{k=1}^{m} e_k \frac{f - W_k}{\eta - \tau_k} = \frac{R}{P^0} \sum_{k=1}^{m} \frac{e_k W_k}{\eta - \tau_k}
\]
and hence
\[
|[R(\xi, D_y)\psi]^*| \leq K_1 |f| + K_2 |\xi|^{m/m} \sum_{k=1}^{m} \left| \frac{W_k}{\eta - \tau_k} \right|
\]

Thus by Parseval's formula and (4.8) we see
\[
\int_{-\infty}^{\infty} |R(\xi, D_y)\psi|^2 \, dy \leq C \left( \int_{-\infty}^{\infty} |f|^2 \, d\eta + |\xi|^{2m/m} \sum_{k=1}^{m} |W_k|^2 \int_{-\infty}^{\infty} \frac{d\eta}{|\eta - \tau_k|^2} \right)
\]
(4.11)
\[
\leq C' \left( \int_{-\infty}^{\infty} |f|^2 \, d\eta + |\xi|^{2m/m} \sum_{k=1}^{m} |W_k|^2 \right).
\]

We note that Im \( \tau_k \)<0 for \( r<k \leq m \). Paley-Wiener's theorem implies
\[
\int_{-\infty}^{\infty} \frac{f}{\eta - \tau_k} \, d\eta = -2\pi i W_k, \quad r<k \leq m.
\]

Hence by (4.8) and by Schwarz's inequality
\[
|W_k|^2 \leq \text{const.} |\xi|^{-m/m} \int_{-\infty}^{\infty} |f|^2 \, d\eta, \quad r<k \leq m.
\]
(4.12)

Therefore (4.11) becomes
\[
\int_{0}^{\infty} |R(\xi, D_y)\psi|^2 \, dy \leq C \left( \int_{-\infty}^{\infty} |f|^2 \, d\eta + |\xi|^{m/m} \sum_{k=1}^{r} |W_k|^2 \right).
\]
(4.13)

Next we observe that there is no complex vector \( \omega = (\omega_1, \ldots, \omega_r) \neq 0 \) such that
\[ \sum_{k=1}^{r} q_{jk} \omega_k = 0, \quad 1 \leq k \leq r. \] For, otherwise there would be a complex vector \( \lambda = (\lambda_1, \ldots, \lambda_r) \neq 0 \) such that
\[
\sum_{j=1}^{r} \lambda_j q_{jk} = 0, \quad 1 \leq k \leq r,
\]
(4.14)
and hence

$$P^{n-1} \sum_{j=1}^{r} \lambda_j Q^0_j = \sum_{j=1}^{r} \sum_{k=1}^{m_n} \lambda_j q_{jk} - \sum_{k=r+1}^{m_n} \sum_{j=1}^{r} \lambda_j q_{jk}$$

which shows that \( \sum_{j=1}^{r} \lambda_j Q^0_j \) is a multiple of \( P_+ \). This contradicts the complementing condition (Definition 3.1). Thus the expression

$$\sum_{j=1}^{r} \left| \sum_{k=1}^{r} q_{jk} \omega_k \right|^2$$

is positive on the compact set \( \sum_{k=1}^{r} |\omega_k|^2 = 1, |\xi| = 1 \). Hence by the homogeneity properties of the \( q_{jk} \), there exists some constant \( K_4 \)

$$\sum_{k=1}^{r} |\omega_k|^2 \leq K_4 \sum_{j=1}^{r} \left| \sum_{k=1}^{r} q_{jk} \omega_k \right|^2$$

for all \( \omega \) and \( \xi \).

Now recall that

$$Q_0^j = \sum_{k=1}^{r} q_{jk} P_k^0 + \sum_{k=r+1}^{m_n} q_{jk} P_k^0,$$

$$W_k = \frac{i}{(2\pi)^{1/2}} P_k^0(\xi, D_y) \delta(\xi, 0).$$

By the triangle inequality

$$\left| \sum_{k=1}^{r} q_{jk} P_k^0(\xi, 0) \right| \leq |Q_0^j \delta(\xi, 0)| + \left| \sum_{k=r+1}^{m_n} q_{jk} W_k \right| (2\pi)^{1/2}. \tag{4.15}$$

Hence combining (4.12), (4.13) and (4.15) we get

$$\int_{-\infty}^{\infty} |R(\xi, D_y) \delta(\xi, y)|^2 \, dy \leq C \left( \int_{-\infty}^{\infty} |\xi|^2 \, d\eta + (1 + \xi)^{m/m_n} \sum_{j=1}^{r} |Q_0^j \delta(\xi, 0)|^2 \right). \tag{4.16}$$

If we now integrate with respect to \( \xi \) we obtain (3.7). This completes the proof.

5. Proof of Theorem 3.2. First we need the following lemma.

**Lemma 5.1.** For any \( \epsilon > 0 \) there exists a constant \( C = C(\epsilon) > 0 \) such that

$$\sum_{|\beta|, \alpha \leq m_n} \| D^\beta v \| \leq \epsilon \sum_{|\alpha|, \gamma \leq m_n} \| D^\alpha v \| + C \| v \|, \quad v \in C_0^\infty ((R^n)^m). \tag{5.1}$$

**Proof.** We extend \( v = v(x, y) \in C_0^\infty ((R^n)^m) \) to the whole space \( R^n \) setting

$$v_1(x, y) = v(x, y), \quad y \geq 0$$

$$= \sum_{k=1}^{m_n} \lambda_k v(x, -ky), \quad y < 0,$$

where the \( \lambda_k \) are constants chosen so that all the derivatives \( D^j v \) for \( 0 \leq j \leq m_n - 1 \)
are continuous at \( y = 0 \). Here \( \lambda_k \) depends only on \( m_n \). We observe that for \( \alpha \) satisfying \( \langle \alpha, q \rangle \leq m \)

\[
[D^x v_1] = \xi_1^{m_1} \cdots \xi_{n-1}^{m_{n-1}} \eta^m v_1(\xi, \eta),
\]

and that, for any \( \epsilon > 0 \) and \( \beta (\langle \beta, q \rangle < m) \),

\[
|\xi_1^{m_1} \cdots \xi_{n-1}^{m_{n-1}} \eta^m| \leq \epsilon \sum_{\langle \alpha, q \rangle = m} |\xi_1^{m_1} \cdots \xi_{n-1}^{m_{n-1}} \eta^m| + C
\]

with a constant \( C \) depending only on \( \epsilon \). By using Parseval's identity the lemma follows.

By Lemma 5.1 we have

\[
\sum_{\langle \alpha, q \rangle = m} \| D^x v \| = \sum_{\langle \alpha, q \rangle < m} \| D^x v \| + \sum_{\langle \alpha, q \rangle = m} \| D^x v \|
\]

\[
\leq (1 + \epsilon) \sum_{\langle \alpha, q \rangle = m} \| D^x v \| + C(\epsilon) \| v \|, \quad v \in C^0_\alpha((R^n)^\alpha).
\]

So, from Theorem 3.1 we get

\[
\sum_{\langle \alpha, q \rangle = m} \| D^x v \| \leq C \left( \| P^0(D)v \| + \sum_{j=1}^r |Q_j^0(D)v|_{m - p_j - (m/2m_n) + \| v \|} \right), \quad v \in C^0_\alpha((R^n)^\alpha).
\]

Similarly we can see that for any \( \epsilon > 0 \) there is a constant \( C > 0 \) such that

\[
\| P^0(D)v \| \leq \epsilon \sum_{\langle \alpha, q \rangle = m} \| D^x v \| + C(\epsilon) \| v \| + \| P(D)v \|, \quad v \in C^0_\alpha((R^n)^\alpha).
\]

Taking \( \epsilon \) sufficiently small (for instance \( \epsilon = \frac{1}{2} \)) we get the inequality

\[
\sum_{\langle \alpha, q \rangle = m} \| D^x v \| \leq C \left( \| P(D)v \| + \sum_{j=1}^r |Q_j^0(D)v|_{m - p_j - (m/2m_n) + \| v \|} \right).
\]

It remains to replace \( Q_j^0(D)v \) by \( Q_j(D)v \) in (5.3). To do so, again we need to extend \( v(x, y) \) to the whole space \( R^n \) as in the proof of Lemma 5.1 and we denote the extension by \( v_1(x, y) \).

For any \( v \in C^0_\alpha((R^n)^\alpha) \), we have by Schwarz's inequality

\[
|v_1(\xi, 0)| \leq \frac{1}{2\pi} \left( \int_{-\infty}^{\infty} \left| \frac{d}{d\xi} \right|^{2} d\eta \right)^{1/2}
\]

\[
\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\eta}{1 + \langle \xi \rangle^{2m/2m_n} + \eta^2}.
\]

The last integral is equal to \( \pi(1 + \langle \xi \rangle^{2m/2m_n})^{-1/2} \). Hence we get

\[
(1 + \langle \xi \rangle)^{m - p_j - m/2m_n} |\xi^m D^x \eta^p v_1(\xi, 0)| \leq C(1 + \langle \xi \rangle)^{m - p_j} \left( \int |\xi^m \eta^p v_1|^2 d\eta \right)^{1/2}
\]

\[
+ (1 + \langle \xi \rangle)^{m - p_j - m/2m_n} \left( \int |\xi^m \eta^p v_1|^2 d\eta \right)^{1/2}.
\]
Here we put $\alpha = (\alpha', \alpha_n)$, $\alpha' = (\alpha_1, \ldots, \alpha_{n-1})$ and we assume $\langle \alpha, q \rangle = p_l$ ($p_l \leq m - m/m_n$). For the first term on the right hand side, the method used already gives

$$(1 + \langle \xi \rangle)^{m - p_l} |\xi'^{\alpha} \eta^{\alpha_n}| \leq C(1 + |\xi_1|^{m_1} + \cdots + |\eta|^{m_n}), \quad (\xi, \eta) \in \mathbb{R}^n,$$

for some constant $C$. Similarly for the second term, we have

$$(1 + \langle \xi \rangle)^{m - p_l - m/m_n} |\xi'^{\alpha} \eta^{\alpha_n - 1}| \leq C(1 + |\xi_1|^{m_1} + \cdots + |\eta|^{m_n}), \quad (\xi, \eta) \in \mathbb{R}^n.$$

Thus by extending $v$ to $v_1$ and by Parseval’s formula, we get

$$(5.5) |\mathcal{Q}_l(D)v|_{m - p_l - m/2m_n} \leq C \sum_{\langle \alpha, q \rangle \leq m} \|D^\alpha v\|, \quad v \in C_0^\infty((\mathbb{R}^n)_0)$$

for any $\alpha$ such that $\langle \alpha, q \rangle = p_l$.

By triangle inequality we have

$$|Q_l(D)v|_{m - p_l - m/2m_n} \leq |Q_l(D)v|_{m - p_l - m/2m_n} + C \sum_{\langle \alpha, q \rangle < p_l} \|D^\alpha v\|_{m - p_l - m/2m_n}.$$

Hence a slight modification of the proof of Lemma 5.1 gives for any $\epsilon > 0$

$$(5.6) |Q_l(D)v|_{m - p_l - m/2m_n} \leq |Q_l(D)v|_{m - p_l - m/2m_n} + \epsilon \sum_{\langle \alpha, q \rangle = p_l} \|D^\alpha v\|_{m - p_l - m/2m_n} + C(\epsilon)\|v\|_{m - p_l - m/2m_n}, \quad v \in C_0^\infty((\mathbb{R}^n)_0).$$

By the argument used in the proof of the inequality (5.5) we get for the third term on the right hand side of (5.6)

$$(5.7) |v|_{m - p_l - m/2m_n} \leq \epsilon \sum_{\langle \alpha, q \rangle = m} \|D^\alpha v\| + C(\epsilon)\|v\|, \quad v \in C_0^\infty((\mathbb{R}^n)_0).$$

Here we may assume $p_l > 0$, otherwise we do not need such an inequality because $p_l = 0$ implies $Q_l = \text{const.}$ and $Q_l^l = Q_l$.

Combining the inequalities (5.5), (5.6), and (5.7) we have

$$(5.8) |Q_l(D)v|_{m - p_l - m/2m_n} \leq |Q_l(D)v|_{m - p_l - m/2m_n} + 2\epsilon \sum_{\langle \alpha, q \rangle \leq m} \|D^\alpha v\| + C\|v\|.$$

Finally, taking $\epsilon$ sufficiently small we arrive at the conclusion:

$$(5.9) \sum_{\langle \alpha, q \rangle \leq m} \|D^\alpha v\| \leq C\left(\|P(D)v\| + \sum_{l = 1} T |Q_l(D)v|_{m - p_l - m/2m_n} + \|v\|\right), \quad v \in C_0^\infty((\mathbb{R}^n)_0).$$

This completes the proof of Theorem 3.2.

6. Coerciveness inequality (II). The case of variable coefficients. The conclusions of Theorem 3.1 and Theorem 3.2 can be extended to operators with variable coefficients.

Let $\Omega$ be a domain in $\mathbb{R}^n$. It is supposed that the boundary of $\Omega$ contains an open set $\omega (\neq \emptyset)$ in the plane $y = 0$. For convenience, assume the origin $(0, \ldots, 0)$ is contained in the (interior of) plane boundary $\omega$. 

We consider a linear differential operator

\[ P(x, y, D) = \sum_{\langle a, q \rangle \leq m} a_a(x, y) D^a, \]

where \( a_a(x, y) \) are complex valued functions defined on \( \Omega \cup \omega \) and infinitely differentiable. We assume that the operator (6.1) is quasi-elliptic of weight \( q = (m/m_1, \ldots, m/m_n) \) in \( \Omega \cup \omega \), more precisely, there is a constant \( K > 0 \) such that

\[ |\xi_1|^{m_1} + \cdots + |\xi_{n-1}|^{m_{n-1}} + |\eta|^{m_n} \leq K \sum_{\langle a, q \rangle = m} a_a(x, y) \xi^a \eta^a \]

for any \((\xi, \eta) \in \mathbb{R}^n\) and for any \((x, y) \in \Omega \cup \omega\). We may assume the coefficient of \( D_x^{m_n} \) is identically equal to 1.

Set \( P(D) = \sum_{\langle a, q \rangle \leq m} a_a(0, 0) D^a \) and assume that \( P(D) \) is of determined type \( r \), \( 1 \leq r \leq m \) (see §1). We consider \( r \) boundary operators \( Q_j(x, D) \) defined on \( \omega \):

\[ Q_j(x, D) = \sum_{\langle a, q \rangle \leq p_j} b_a(x) D_x^{a} D_y^{q}, \quad j = 1, \ldots, r, \]

where \( b_a(x) \) are complex valued functions defined on \( \omega \) and infinitely differentiable. Set

\[ Q_j(D) = \sum_{\langle a, q \rangle \leq p_j} b_a(0) D_x^{a} D_y^{q}, \quad j = 1, \ldots, r. \]

We denote by \( \Omega_\delta (\delta > 0) \) the hemisphere

\[ \Omega_\delta = \{(x, y); y \geq 0, |x|^2 + y^2 < \delta^2\}, \]

and denote by \( \omega_\delta \) the plane boundary of \( \Omega_\delta \):

\[ \omega_\delta = \{(x, 0); x < \delta\}. \]

**Theorem 6.1.** Let \( P(x, y, D) \) and \( Q_j(x, D), j = 1, \ldots, r \) be defined as above and assume that \( Q_j(D) (j = 1, \ldots, r) \) cover \( P(D) \). Then for sufficiently small \( \delta > 0 \) there exists a constant \( C > 0 \) such that

\[ \sum_{\langle a, q \rangle \leq m} \| D^a v \| \leq C \left( \| P(x, y, D)v \| + \sum_{j=1}^r |Q_j(x, D)v|_{m-p_j-m/2m_n} + \| v \| \right), \]

\[ v \in C_0^\infty (\Omega_\delta). \]

7. **Proof of Theorem 6.1.** In view of Theorem 3.2 it holds for some constant \( C \)

\[ \sum_{\langle a, q \rangle \leq m} \| D^a v \| \leq C \left( \| P(D)v \| + \sum_{j=1}^r |Q_j(D)v|_{m-p_j-m/2m_n} + \| v \| \right), \]

\[ v \in C_0^\infty ((R^n)^r). \]

Let us write

\[ P(D)v = P(x, y, D)v + [P(D) - P(x, y, D)]v. \]
Then for any \( \varepsilon > 0 \) we can take sufficiently small \( \delta > 0 \) such that
\[
(7.2) \quad \| P(D)v \| \leq \| P(x, y, D)v \| + \varepsilon \sum_{\langle a, \varphi \rangle \leq m} \| D^\varphi v \|, \quad v \in C^\infty_0(\Omega_6).
\]

Put \( \varepsilon = 1/2 \). Then we have
\[
(7.3) \quad \sum_{\langle a, \varphi \rangle \leq m} \| D^\varphi v \| \leq 2C \left( \| P(x, y, D)v \| + \sum_{j=1}^l \| Q_j(D)v \|_{m-p_j-m/2m} + \| v \| \right),
\]
for \( v \in C^\infty_0(\Omega_6) \).

It remains to replace \( Q_j(D) \) by \( Q_j(x, D) \) in (7.3). To do so, we shall prove some lemmas. We denote by \( \gamma_s \in \mathcal{D}'(\mathbb{R}^n-1) \) the distribution such that
\[
(7.4) \quad \hat{\gamma}_s(\xi) = (1 + \xi_1^2 + \cdots + \xi_{n-1}^2)^{s/2}
\]
for real \( s \).

**Lemma 7.1 (cf. Mizohata [9]).** Let \( s \) be real and positive. Then
\[
(7.5) \quad \| (x^s \gamma_s) \ast \varphi \| \leq \epsilon(\alpha, \delta) \| \gamma_s \ast \varphi \|, \quad \varphi \in C^\infty_0(\mathbb{R}^n-1), \quad \alpha \neq 0.
\]

Here, \( \epsilon(\alpha, \delta) \) is a constant such that \( \epsilon(\alpha, \delta) \) tends to zero when the diameter \( \delta \) of the support of \( \varphi \) tends to zero.

**Proof.** It is obvious that
\[
(7.6) \quad P_a(\xi) \gamma_s(\xi) = (-1)^{\lceil \frac{n}{2} \rceil} D^\xi \hat{\gamma}_s(\xi) \equiv P_a(\xi) \gamma_s(\xi),
\]
where \( P_a(\xi) \) tends to zero when \( |\xi| \to \infty \). Now
\[
(7.7) \quad (x^s \gamma_s) \ast \varphi(x) = (2\pi)^{-(n-1)/2} \int_{|\xi| \leq R} \exp \left( i\langle x, \xi \rangle \right) P_a(\xi) \gamma_s(\xi) \varphi(\xi) \, d\xi + \int_{|\xi| \geq R} \cdots.
\]
Take \( R \) sufficiently large. Then \( |P_a(\xi)| \leq \epsilon/2, \ |\xi| > R \). The \( L^2 \)-norm of the second term is dominated by \( \epsilon/2 \| \gamma_s \ast \varphi \| \), where \( \epsilon \) will be given later. Let \( R \) be fixed as above. The \( L^2 \)-norm of the first term is estimated by
\[
(7.8) \quad \max_{|\xi| \leq R} |P_a(\xi) \gamma_s(\xi)| \left( \int_{|\xi| \geq R} |\varphi(\xi)|^2 \, d\xi \right)^{1/2}.
\]
On the other hand, it holds that
\[
|\varphi(\xi)| \leq \int |\varphi(x)| \, dx \leq (\text{Vol. [supp. } \varphi])^{1/2}. \| \varphi \|.
\]
Hence (7.6) is estimated by
\[
\max_{|\xi| \leq R} |P_a(\xi) \gamma_s(\xi)| (\text{Vol. [supp. } \varphi])^{1/2}. \| \varphi \| \cdot (\text{Vol. } B(R))^{1/2},
\]
where \( B(R) \) is a ball whose radius is \( R \). Clearly \( \| \varphi \| \leq \| \gamma_s \ast \varphi \| \) for \( s > 0 \). So the lemma is proved.
Remark. We note that \(x^\alpha y_s(x) \in L^2(\mathbb{R}^{n-1})\) if we take \(\alpha\) such that \(|\alpha|\) is sufficiently large.

**Lemma 7.2 (cf. [9]).** Let \(s\) be real, positive and \(b(x) \in C^\infty(\mathbb{R}^{n-1})\). Then

\[
\|b(x)(y_s \ast \varphi) - y_s \ast (b(x)\varphi)\| \leq \epsilon(\delta, s)\|y_s \ast \varphi\|, \quad \varphi \in C_0^\infty(\mathbb{R}^{n-1}).
\]

Here \(\epsilon(\delta, s)\) is a constant, which tends to zero when the diameter of the support of \(\varphi\) tends to zero.

**Proof.** The commutator, which we must estimate, is

\[
b(x)y_s \ast \varphi - y_s \ast b(x)\varphi = \int [b(x) - b(x')]y_s(x-x')\varphi(x') \, dx'.
\]

We use the Taylor's formula for \(b(x')\) around \(x\):

\[
b(x') - b(x) = \sum_{1 \leq |\alpha| \leq l-1} \frac{(x'-x)^\alpha}{\alpha!} D^\alpha b(x) + \sum_{|\alpha| = l} b_a(x, x')(x'-x)^\alpha, \quad b_a(x, x') \in C^\infty.
\]

Then (7.8) is written as

\[
\sum_{1 \leq |\alpha| \leq l-1} \frac{(-1)^\alpha}{\alpha!} D^\alpha b(x) \ast (x^\alpha y_s) \ast \varphi
\]

\[
+ \sum_{|\alpha| = l} \int b_a(x, x')(x-x')^\alpha y_s(x-x')\varphi(x') \, dx'.
\]

By Lemma 7.1 it is clear that every expression \((D^\alpha b(x)/\alpha!)(x^\alpha y_s) \ast \varphi\) in the first term of (7.9) has the property (7.7). We take \(l\) sufficiently large so that \(x^\alpha y_s(x) \in L^2\), \(|\alpha| = l\) (cf. Remark of Lemma 7.1). Then every term in the second summation is estimated as follows:

\[
sup |b_a(x, x')| \int |(x-x')^\alpha y_s(x-x')| |\varphi(x')| \, dx' = sup |b_a(x, x')|(|x^\alpha y_s| \ast |\varphi|). (\star)
\]

By Hausdorff-Young's inequality

\[
\|x^\alpha y_s \ast |\varphi|\|_{L^2} \leq \|x^\alpha y_s\|_{L^2} \cdot \|\varphi(x)\|_{L^1}.
\]

Moreover

\[
\|\varphi(x)\|_{L^1} \leq (\text{Vol. [supp. } \varphi])^{1/2} \|\varphi(x)\|_{L^2}.
\]

Combining these estimates we obtain the inequality (7.7).

**Corollary 1.** Let \(\varphi \in C_0^\infty(\mathbb{R}^{n-1})\), supp. \(\varphi \subset \{x; |x| \leq \delta\}\equiv B(\delta)\) and \(b(x) \in C^\infty(\mathbb{R}^{n-1})\) with \(b\) and each \(D^\alpha b\) bounded on \(\mathbb{R}^{n-1}\). Then

\[
\|y_s \ast b(x)\varphi\| \leq C|b|_k\|y_s \ast \varphi\|, \quad k = k(s), -\infty < s < \infty,
\]

\(^{(*)}\) It follows easily from our hypotheses that \(b_a(x, x')\) is bounded on \(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}\).
where

$$|b|_k = \sum_{|a|=k} \sup_{x \in \mathbb{R}^{n-1}} |D^a b(x)|.$$  

In particular, if $b(0)=0$, then

$$(7.10) \quad \|\gamma_s * b(x)\varphi\| \leq \epsilon(\delta, s) \|\gamma_s * \varphi\|,$$

where $\epsilon(\delta, s)$ is a constant such that $\epsilon(\delta, s) \to 0$ when the diameter $\delta$ of the support of $\varphi$ tends to zero.

The proof is easy, so we omit it here.

Now let us continue the proof of Theorem 6.1. We note that the norms $|v|_s$ and $\|\gamma_s * v(x, 0)\|_{L^2(\mathbb{R}^{n-1})}$ are equivalent to each other. Let us write

$$Q(D)v(x, 0) = Q(x, D)v + (Q(D) - Q(x, D))v, \quad v \in C_0^\infty(\Omega_0).$$

If we assume that $\delta < 1$, we may (by altering the coefficients of the $b_j(x, D)$ outside the unit ball) assume that coefficients of $b_j(x, D)$ and each of its derivatives is bounded on $\mathbb{R}^{n-1}$. Then the corollary stated above yields

$$\|Q(D)v\|_{m-p_j-m/2m_a} \leq \||Q_j(x, D)v\|_{m-p_j-m/2m_a}$$

$$+ \epsilon(\delta, p_j) \sum_{|a| \leq p_j} \|D^a v\|_{m-p_j-m/2m_a}, \quad v \in C_0^\infty(\Omega_0).$$

As we have seen in the proof of Theorem 3.2 there exists a constant $C$ not depending on $\delta$ such that

$$\sum_{|a| \leq p_j} \|D^a v\|_{m-p_j-m/2m_a} \leq C \sum_{|a| \leq m} \|D^a v\|, \quad v \in C_0^\infty(\Omega_0).$$

Hence if we take $\delta$ sufficiently small, we get the estimate:

$$\sum_{|a| \leq p_j} \|D^a v\| \leq C \left( \|P(x, y, D)v\| + \sum_{j=1}^k \|Q_j(x, D)v\|_{m-p_j-m/2m_a} + \|v\| \right), \quad v \in C_0^\infty(\Omega_0),$$

which completes the proof.

**Remark.** We note that the necessity of our complementing condition (Definition 3.1) for the validity of the estimate (3.8) can be proved by the analogous way in [1, §10].

8. Regularity at the boundary. We introduce function spaces $\Phi^m$ and $H^s$.

**Definition 8.1.** By $\Phi^m(R^n_+) = \Phi^m$ we mean the set of all $u \in L^2(R^n_+)$ such that $D^\alpha u \in L^2(R^n_+)$ for any $\alpha$ satisfying $\langle \alpha, q \rangle \leq m$. The space $\Phi^m$ is a Hilbert space with norm

$$\|u\|^2_m = \sum_{|\alpha, q| \leq m} \|D^{\alpha, q} u\|^2.$$

We denote by $H^s(R^{n-1}) = H^s$ (for a real $s \geq 0$) the set of all $u \in L^2(R^{n-1})$ such that

$$\left(1 + \langle \xi \rangle^s\right)^{\alpha} u(\xi) \in L^2(R^{n-1}).$$
The space $H^s$ is a Hilbert space with norm

$$\|u\|_s = \|(1 + \langle \xi \rangle^2) \hat{u} (\xi)\|_{L^2(R^{n-1})} \simeq \|\gamma_s * u\|_{L^2(R^{n-1})},$$

where we also denote by $\hat{u} (\xi)$ the Fourier transform of $u$ with respect to variables $x = (x_2, \ldots, x_{n-1})$.

**Definition 8.2.** We shall say $u \in \Phi^m_{\text{loc}}(\Omega_0)$ if $\varphi u \in \Phi^m$ for all $\varphi \in C_0^\infty(\Omega_0)$. Similarly we shall say $u \in H^m_{\text{loc}}(\omega_0)$ if $\varphi u(x) \in H^s$ for all $\varphi \in C_0^\infty(\omega_0)$.

Concerning with the properties of the spaces $\Phi^m$ and $H^s$ we have the following lemmas.

**Lemma 8.1.** Let $s$ and $t$ be two integers such that $0 < s < t$. Then for any $\varepsilon > 0$ there exists a constant $C = C(\varepsilon)$ such that

$$\|u\|_s \lesssim \varepsilon \|u\|_t + C\|u\|_s, \quad u \in \Phi^t.$$

**Proof.** Clearly $C_0^\infty((R^*_+)^n)$ is dense in $\Phi^t$ for every $t$. Hence Lemma 8.1 can be reduced to Lemma 5.1.

**Lemma 8.2.** For every $u \in \Phi^m$ there exists the trace

$$D^su(x, 0) \in H^{m-p-m/2m}, \quad 0 \leq p \leq m-m'/m_n, \quad \langle \alpha, q \rangle \leq p,$$

and there is a constant $C = C(m)$ such that

$$\sum_{\langle \alpha, q \rangle \leq p} \|D^s u\|_{m-p-m/2m} \lesssim C\|u\|_m, \quad u \in \Phi^m.$$

The proof of Lemma 8.2 can be given by the argument stated at the end of §5. So we omit the detail.

Now take $\rho = \rho(x) \in C_0^\infty(R^{n-1})$ satisfying the following:

(i) $\int_{R^{n-1}} \rho(x) \, dx = 1$ and

(ii) $\rho(|\xi|) = O(|\xi|^k)$, $\xi \to 0$ for some sufficiently large integer $k$.

Set $\rho_\varepsilon(x) = e^{-(n-1)\rho(x_\varepsilon)} (\varepsilon > 0)$. If $v \in \Phi^m$, the regularization of $v$

$$v_\varepsilon(x, y) = e^{-(n-1)\int v(x', y) \rho(x-x'/\varepsilon) \, dx'}, \quad y \geq 0,$$

is infinitely differentiable with respect to $x$ and obviously $v_\varepsilon \in \Phi^m$, and $v_\varepsilon \to v$ in $\Phi^m(R^*_n)$ as $\varepsilon \to 0$.

As usual we denote by $\mathcal{S}((R^*_n)^n)$ the set of all the functions, each of which is a restriction in $(R^*_n)^n$ of a function in $\mathcal{S}(R^n)$.

**Lemma 8.3.** If $u \in L^2(R^*_n)$ and $a \in \mathcal{S}((R^*_n)^n)$ and if $\rho \in C_0^\infty(R^{n-1})$ satisfies (i) and (ii), then

$$\|\gamma_1 * (au_\varepsilon - (au)_\varepsilon\|_{L^2(R^*_n)} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

The lemma is derived by a simple modification of the proof of Theorem 2.4.3 in [5], so we omit the proof. (For the definition of $\gamma_1$, see (7.4).)
Theorem 8.1. Consider the boundary problem

\begin{align}
    P(x, y, D)u &= f & \text{in } & \Omega, \\
    Q_j(x, D)u &= g_j & \text{on } & \omega, \quad j = 1, \ldots, r,
\end{align}

where $P(x, y, D)$ is a quasi-elliptic operator given in §6, and the boundary operators $Q_j(x, D)$ are also given in §6.

Suppose $u \in \Phi^n_{\text{loc}}(\Omega)$. If $f \in C^\infty(\overline{\Omega})$ and if $g_j \in C^\infty(\omega)$, $j = 1, \ldots, r$, then there exists a positive number $\delta = \delta(t)$ such that $u \in \Phi^n_{\text{loc}}(\Omega_t)$ for $t = 1, 2, \ldots$.

Proof. First take any tangential derivative $D^\xi_x$ (here $\beta = (\beta_1, \ldots, \beta_{n-1}, 0)$). We want to prove that there exists a $\delta = \delta(\beta)$ such that

\begin{equation}
    D^\xi_x(\phi u) \in \Phi^n
\end{equation}

for any $\phi \in C^\infty_0(\Omega_\delta)$. To do so, it is sufficient to prove that for any positive integer $s$ there exists a $\delta = \delta(s)$ such that

\begin{equation}
    \gamma_s * (\phi u) \in \Phi^n
\end{equation}

for any $\phi \in C^\infty_0(\Omega_\delta)$. We consider the case $s = 1$.

We set $P(D) = \sum_{a, \beta} a_{\beta}(0, 0) D^{\beta}$ and $Q_j(D) = \sum_{a, \beta} b_{\beta}(0) D^{\beta}$, $j = 1, \ldots, r$, the same as in §6. As $\gamma_1 * (\phi u) \in \Phi^m(P^*_\delta)$ for any $\phi \in C^\infty(\Omega)$ and for any $\varepsilon > 0$, the coerciveness estimate yields

\begin{equation}
    \|\gamma_1 * (\phi u)_\varepsilon\|_m \leq C \|P(D)(\gamma_1 * (\phi u)_\varepsilon)\| \\
    + \sum_{j=1}^r \|Q_j(D)\gamma_1 * (\phi u)_\varepsilon\|_{m - p_j - m/2m_n} + \|\gamma_1 * (\phi u)_\varepsilon\|_m,
\end{equation}

where the constant $C$ is independent of $\phi \in C^\infty(\Omega)$ and of $\varepsilon (> 0)$.

Write

\begin{align}
    \gamma_1 * P(D)(\phi u)_\varepsilon &= \gamma_1 * P(x, y, D)(\phi u)_\varepsilon \\
    &\quad + \gamma_1 * ((P(D) - P(x, y, D))(\phi u)_\varepsilon).
\end{align}

Then, for the second term in the right hand side, we can easily see by using Corollary 1 in §7, that for any $\nu > 0$ there exist $\delta = \delta(\nu)$ and $\varepsilon_0 = \varepsilon_0(\nu)$ such that

\begin{equation}
    \|\gamma_1 * (P(D) - P(x, y, D))(\phi u)_\varepsilon\|_{L^2(\mathbb{R}^n_\delta)} \leq \nu \|\gamma_1 * (\phi u)_\varepsilon\|_m
\end{equation}

for any $\phi \in C^\infty_0(\Omega_\delta)$ and for any $\varepsilon (0 < \varepsilon \leq \varepsilon_0)$. Similarly we can get

\begin{equation}
    |\gamma_1 * (Q_j(D) - Q_j(x, D)(\phi u)_\varepsilon)|_{m - p_j - m/2m_n} \leq \nu \|\gamma_1 * (\phi u)_\varepsilon\|_m,
\end{equation}

for $j = 1, \ldots, r$, for any $\phi \in C^\infty_0(\Omega_\delta)$ and for any $\varepsilon (0 < \varepsilon \leq \varepsilon_0)$. Here also $\delta$ and $\varepsilon_0$ depend only on $\nu$. 

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Thus we get the following estimate:

\[
\| \gamma_1 \ast (\phi u) \|_m \leq C (\| \gamma_1 \ast P(x, y, D)(\phi u) \| \\
+ \sum_{j=1}^{r} | \gamma_1 \ast Q_j(x, D)(\phi u) |_{m - p_j - m/2m_n} \\
+ \| \gamma_1 \ast (\phi u) \|, \quad \phi \in C_0^\infty(\Omega_0), \quad 0 < \epsilon \leq \epsilon_0,
\]

(8.11)

where the constant \( C \) is independent of \( \phi \) and \( \epsilon \).

Next we prove that the norms

\[
\| \gamma_1 \ast P(x, y, D)(\phi u) \|
\]

and

\[
| \gamma_1 \ast Q_j(x, D)(\phi u) |_{m - p_j - m/2m_n}, \quad j = 1, \ldots, r,
\]

are uniformly bounded with respect to \( \epsilon (0 < \epsilon \leq \epsilon_0) \). By Lemma 8.3 it is sufficient to prove the uniform boundedness of the norms

\[
\| \gamma_1 \ast (P(x, y, D)(\phi u)) \|
\]

and

\[
| \gamma_1 \ast (Q_j(x, D)(\phi u)) |_{m - p_j - m/2m_n}, \quad j = 1, \ldots, r.
\]

We note that \( \gamma_1 \ast (P(x, y, D)(\phi u)) = (\gamma_1 \ast P(x, y, D)(\phi u))_x \). If we write

\[
P(x, y, D)(\phi u) = \phi P(x, y, D)u + \sum_{\beta \neq 0} \frac{D_\beta \phi}{\beta!} P_\beta(x, y, D)u,
\]

then the fact \( \phi P(x, y, D)u = \phi f \in C_0^\infty(\Omega_0) \) implies \( \gamma_1 \ast \phi f \in L^2(\mathbb{R}^n) \). Now, we take any term \( (D^\beta \phi/\beta!)a_\beta(x, y)D^{\alpha-\beta}u \) in the summation of (8.12). Take \( \psi \in C_0^\infty(\Omega_0) \) such that \( \psi = 1 \) in supp. \( \phi \). Then

\[
(D^\beta \phi/\beta!)a_\beta(x, y)D^{\alpha-\beta}u = (D^\beta \phi/\beta!)a_\beta(x, y)D^{\alpha-\beta}(\psi u).
\]

By virtue of Lemma 7.2 it is sufficient to prove \( \gamma_1 \ast D^{\alpha-\beta}(\psi u) \in L^2(\mathbb{R}^n) \). We can easily observe that if \( \langle \alpha, q \rangle \leq m, \beta \leq \alpha, \beta \neq 0 \), then there exists a constant \( C \) such that

\[
(1 + |\xi|^m + \cdots + |\xi_{n-1}|^{m_{n-1}/m})|\xi^{\alpha-\beta}q^{\alpha_\beta-n_\beta}|
\leq C(1 + |\xi|^m + \cdots + |\xi_{n-1}|^{m_{n-1}/m} + |\eta|^m)
\]

for any \( (\xi, \eta) \in \mathbb{R}^n \). Let \( (\psi u)_1 \) be the extension of \( \psi u \) in \( \Phi^m(\mathbb{R}^n) \) by the method used in the proof of Lemma 5.1. Then \( \gamma_1 \ast D^{\alpha-\beta}(\psi u)_1 \in L^2(\mathbb{R}^n) \). Therefore we see \( \gamma_1 \ast D^{\alpha-\beta}(\psi u) \in L^2(\mathbb{R}^n) \). Hence we have

\[
\gamma_1 \ast P(x, y, D)(\phi u) \in L^2(\mathbb{R}^n).
\]

Now it is obvious that

\[
(\gamma_1 \ast P(x, y, D)(\phi u))_x = \gamma_1 \ast (P(x, y, D)(\phi u))_x
\]

is uniformly bounded for \( \epsilon (0 < \epsilon \leq \epsilon_0) \).
For the norms $|\gamma_1 \ast Q_j(x, D)(\varphi u)|_{m - p_j - m/2m_n}$, we again write

$$Q_j(x, D)(\varphi u) = \varphi Q_j(x, D)u + \sum_{\beta \neq 0} \frac{D^\beta \varphi}{\beta!} D^\gamma Q_j(x, D)u.$$  

Similarly to the above, it is sufficient to prove that

$$|\gamma_1 \ast D^{\alpha - \beta}(\varphi u)|_{m - p_j - m/2m_n} < \infty$$

for any $\varphi \in C_0^\infty(\Omega_\delta)$, where $\langle \alpha, q \rangle \leq p_j, \beta \leq \alpha, \beta \neq 0$. Clearly, it holds that

$$|\gamma_1 \ast D^{\alpha - \beta}(\varphi u)|_{m - p_j - m/2m_n} \leq |D^{\alpha - \beta}(\varphi u)|_{m - (p_j - 1) - m/2m_n},$$

and $\langle \alpha - \beta, q \rangle \leq p_j - 1$. So Lemma 8.2 gives us

$$|\gamma_1 \ast D^{\alpha - \beta}(\varphi u)|_{m - p_j - m/2m_n} < \infty.$$  

Thus, by using Lemma 8.3, we can conclude that the norms

$$\|\gamma_1 \ast (P(x, y, D)(\varphi u)_j)\|$$

are uniformly bounded with respect to $\epsilon (0 < \epsilon \leq \epsilon_0)$. By the estimate (8.11), we can see that the norm $\|\gamma_1 \ast (\varphi u)\|_m$ is uniformly bounded with respect to $\epsilon (0 < \epsilon \leq \epsilon_0)$. Hence the theorem of Banach-Saks implies $\gamma_1 \ast \varphi u \in \Phi^m$.

We can repeat this procedure and get

$$D^\beta u \in \Phi^m_\infty(\Omega_\delta), \quad |\beta| \leq N, \quad \beta_n = 0, \quad \delta = \delta(N)$$

for any integer $N$. Now the equation $P(x, y, D)u = f$ can be written in the form

$$(8.13) \quad D^\alpha u = - \sum_{\langle a, q \rangle \geq m_1 a_n < m_n} a_a(x, y) D^a_x D^a_y u + f.$$  

Differentiating (8.13) with respect to $x$-variables we see

$$D^\beta_x D^\alpha u \in L^2_\infty(\Omega_\delta).$$

Moreover we have

$$D^\beta_x D^\alpha u + 1 \in L^2_\infty(\Omega_\delta).$$

We can repeat this procedure and arrive at the conclusion of Theorem 8.1.

9. Hypo-analyticity at the boundary. In the following we shall investigate the more precise estimates of the derivatives of the solutions of a quasi-elliptic boundary problem. However, we are limited to the case of simple boundary operators.

Consider a quasi-elliptic operator $P(D)$ of weight $q = (m_1, \ldots, m_{m_n - 1}, m/m_n)$ and of determined type $r (1 \leq r \leq m_n)$ given by (3.1):

$$P(D) = D^\alpha + \sum_{\langle a, q \rangle \geq m_1 a_n < m_n} a_a D^a_x D^a_y,$$

where $a_a$ are complex constants.
As the boundary operators we take only normal derivatives such that 
\[ Q_j(D) = D_{ij}^j, \quad j = 1, \ldots, r, \]
where \( k_i \neq k_j \) if \( i \neq j \) and \( 0 \leq k_j \leq m_n - 1 \). Clearly these \( r \) boundary operators cover \( P(D) \) (see Definition 3.1).

**Definition 9.1.** Let \( \Omega \) be a domain in \( \mathbb{R}^n \) and set \( d = (d_1, \ldots, d_n) \), \( d_i \geq 1, i = 1, \ldots, n \). We call \( u \) a function of the class \( G(d, \Omega) \) if \( u \) is infinitely differentiable in \( \Omega \) and if for each compact set \( K \) in \( \Omega \) there exist two constants \( C_0, C_1 \) such that

\[
\| \partial^\alpha u, K \|_\infty \leq C_0 C_1^{\sum_{i=1}^n a_i^j \alpha_i^j}
\]

or equivalently (if some \( \alpha_i = 0 \), then we define \( \alpha_i^j = 1 \) and \( |\alpha|^{\alpha_i} = 1 \) if \( |\alpha| = 0 \))

\[
\| \partial^\alpha u, K \|_\infty \leq C_0 C_1^{\sum_{i=1}^n a_i^j \alpha_i^j d_i}
\]

for any \( \alpha \), where \( \| w, K \|_\infty \) means the maximum of \( |w| \) in \( K \).

As in §6, let \( \Omega \) be a domain in \( \mathbb{R}^n \) and let the boundary of \( \Omega \) contain an open set \( \omega (\neq \emptyset) \) in the plane \( \gamma = 0 \).

Now we can state our results.

**Theorem 9.1 (cf. [3]).** Let \( P(D) \) be the quasi-elliptic operator defined as above and take the boundary operators \( D_{ij}^j \) \((j = 1, \ldots, r)\) given above. Consider the boundary problem

\[
P(D)u(x, y) = f(x, y) \quad \text{in} \quad \Omega,
\]

\[
D_{ij}^j u(x, 0) = 0, \quad j = 1, \ldots, r \quad \text{on} \quad \omega
\]

with \( f \in G(\lambda q; \Omega \cup \omega), \lambda q = (\lambda q_1, \ldots, \lambda q_n), \lambda \geq 1 \). Then any function \( u \in \Phi_{\text{loc}}(\Omega \cup \omega) \) satisfying (9.3) and (9.4) is a function in \( G(\lambda q; \Omega \cup \omega) \).

**Theorem 9.2.** Let \( P(D) \) be the same as in Theorem 9.1. Assume that a function \( u \in C^\infty(\Omega \cup \omega) \) satisfies the following conditions:

(i) For each compact set \( K \) in \( \Omega \cup \omega \) there exist two constants \( C_0, C_1 \) such that

\[
\| P^k(D)u, K \|_\infty \leq C_0 C_1^{\sum_{i=1}^n a_i^j \alpha_i^j (mk)} \quad (\lambda \geq 1).
\]

(ii) On the plane boundary \( \omega \), it holds that

\[
D_{ij}^j P^k(D)u(x, 0) = 0, \quad k = 0, 1, 2, \ldots,
\]

where \( k_j, j = 1, \ldots, r \) are the same as in Theorem 9.1. Then \( u \in G(\lambda q; \Omega \cup \omega) \).

**Remark.** As a special case of Theorem 8.1 we see that any solution \( u \in \Phi_{\text{loc}}(\Omega \cup \omega) \) of the problem (9.3), (9.4) is infinitely differentiable on \( \Omega \cup \omega \).

**10. Proof of Theorem 9.1.** To prove Theorems 9.1 and 9.2 we make use of the methods in [3], [2] and [8].

First we shall derive some preliminary lemmas by making use of the coerciveness inequalities proved in the foregoing paragraphs.
We denote by $W$ the set of all functions $v \in C^\infty_0(\Omega \cup \omega)$ which satisfy boundary conditions (9.4).

**Lemma 10.1.** Let $P(D)$ be that in Theorem 9.1. Then for any $\varepsilon > 0$ and for any $v \in W$ the following inequality holds:

$$
\sum_{\alpha, \beta \leq m} \|D^\alpha v\| \leq C(\varepsilon^n \|P(D)v\| + (1 + \varepsilon^n)\|v\|),
$$

where the constant $C$ is independent of $\varepsilon$ and $v \in W$.

**Proof.** First we see that, if $\langle \beta, q \rangle \leq m$, then

$$
|\xi^\beta \eta^q| \leq C \left( \sum_{j=1}^{n-1} |\xi_j|^{m_j} + |\eta|^{m_n} \right)^{\langle \beta, q \rangle / m},
$$

where the constant $C$ is independent of such $\beta$. In fact, replacing $\xi_j$ by $\xi_j t^{m_j}$, $j=1, \ldots, n-1$ and $\eta$ by $\eta t^{m_n}$ in (10.2) means multiplying both sides by $t$ if $t > 0$. Hence (10.2) is valid. Next noting that $\langle \beta, q \rangle ^{\langle \xi, \eta \rangle} \leq m$ we can easily verify that, if $\langle \beta, q \rangle \leq m$ then

$$
|\xi^\beta \eta^q| \leq C \left( \sum_{j=1}^{n-1} |\xi_j|^{m_j} + |\eta|^{m_n} \right) + 1
$$

for another constant $C$ independent of such $\beta$ and of $\langle \xi, \eta \rangle \in \mathbb{R}^n$. By the same way in the proof of Lemma 5.1, we get for any $\alpha, \langle \alpha, q \rangle \leq m$ and for any $v \in W$

$$
\|D^\alpha v\| \leq C \left( \sum_{j=1}^{n-1} \|D_j^\alpha v\| + \|v\| \right).
$$

Finally we apply the coerciveness estimate (3.8) and we get Lemma 10.1.

**Lemma 10.2 (cf. [3], [10]).** For every compact set $K \subset (R^n)^a$ and for every $h, 0 < h \leq 1$, there are a function $\psi = \psi_{K, h}$ and constants $C_a$ independent of $h$ such that

$$
\psi \in C^\infty_0(K_h), \psi = 1 \text{ on } K \text{ and }
$$

$$
\|D^\alpha \psi\| \leq C_a h^{-\langle \alpha, q \rangle} \text{ for every } \alpha,
$$

where $K_h = \{x \in (R^n)^a; \text{dis}(x, K) \leq h\}$. It may be assumed that $D^\alpha \psi(x, 0) = 0, i=1, \ldots, m_n$.

Now we introduce some notation. For convenience we assume that the plane boundary $\omega$ contains the origin $(0, \ldots, 0)$. We denote by $V$ the hemisphere $\{(x, y); |x|^2 + y^2 < R^2, y > 0\}$ included in $\Omega$ and put $V_{-r} = \{(x, y); |x|^2 + y^2 < (R-r)^2, y > 0\}, 0 < r < R \leq 1$. We set for arbitrary $l \geq 0$,

$$
\|D^\alpha u; l + \langle \beta, q \rangle, V\| = \sup_{0 < r < R} \sup_{r^2 < R} \sup_{r^1 < r \leq R} r^{l+\langle \beta, q \rangle} \|D^\alpha u, V_{-r}\|,
$$

and

$$
\|u; q, \mu, l, V\| = \sup_{\beta \in \mathbb{R}^n, \beta_i > 0} \left( \prod_{i=1}^{n-1} \frac{\beta_i}{\beta_i + 1} \right)^{\beta_i q_i} \|D^\alpha u; l + \langle \beta, q \rangle, V\|,
$$

$$
\mu = (\mu_1, \ldots, \mu_{n-1}), \mu_i > 0, 1 \leq i \leq n-1.
$$
The following lemma is essential in our proofs of Theorem 9.1 and Theorem 9.2.

**Lemma 10.3 (cf. [2]).** Let $P(D)$ be the same as in Theorem 9.1, that is, $P(D)$ is quasi-elliptic of weight $q$ and of determined type $r$. Then

$$
\sum_{\langle a, q \rangle \leq m} \| D^a u, V_{-r} \| e^{\langle a, q \rangle} \leq C \left\{ e^n \| P(D) u, V_{-(r-\delta)} \| + e^n \delta^{-m} \right. \\
\sum_{\langle a, q \rangle \leq m} \delta^{\langle a, q \rangle} \| D^a u, V_{-(r-\delta)} \| + \| u, V_{-(r-\delta)} \| \right\}
$$

(10.8)

for any $\epsilon > 0$ and for any $u \in C^\infty(V)$ satisfying the boundary condition (9.4). The constant $C$ is independent of $r, \delta$ ($0 < \delta < r$, $0 < r < +\infty$) and $u$.

**Proof.** Take $\psi = \psi_{V-r, \delta}$ defined in Lemma 10.2. Then $\psi u$ also satisfies the boundary condition (9.4). Hence $\psi u$ satisfies the inequality (10.1). So

$$
\sum_{\langle a, q \rangle \leq m} \| D^a u, V_{-r} \| e^{\langle a, q \rangle} \leq C \{ e^n \| P(D) (\psi u) \| + (1 + e^n) \| \psi u \| \}.
$$

By the Leibniz formula we have

$$
P(D)(\psi u) = \psi P(D) u + \sum_{\langle a, q \rangle \leq m} \sum_{\beta_i \in \mathbb{Z}^n, \beta_i \neq 0} a_{a, \beta} D^{a-\beta} u D^\beta \psi,
$$

where $C_{a, \beta}$ are appropriate constants. Hence we have

$$
\| P(D)(\psi u) \| \leq C \| P(D) u, V_{-(r-\delta)} \| + C \sum_{\langle a, q \rangle \leq m} \sum_{\beta_i \in \mathbb{Z}^n, \beta_i \neq 0} \delta^{-\langle a, q \rangle} \| D^{a-\beta} u, V_{-(r-\delta)} \|,
$$

$$
\leq C \| P(D) u, V_{-(r-\delta)} \| + C \sum_{\langle a, q \rangle \leq m} \delta^{-m+\langle a, q \rangle} \| D^a u, V_{-(r-\delta)} \|.
$$

Thus the lemma is proved.

Now in (10.8) we put $\epsilon = \chi \cdot t \cdot r, \delta = rt$ with sufficiently small $\chi, t$ ($t > 0$) determined later. Then (10.8) turns into

$$
\sum_{\langle a, q \rangle \leq m} \| D^a u, V_{-r} \| \chi^{\langle a, q \rangle} (tr)^{\langle a, q \rangle} \leq C \left\{ \chi^n (tr)^m \| P(D) u, V_{-(r-1-t)} \| \\
+ \chi^n \sum_{\langle a, q \rangle \leq m} (rt)^{\langle a, q \rangle} \| D^a u, V_{-(r-1-t)} \| \\
+ \| u, V_{-(r-1-t)} \| \right\}.
$$

(10.9)

Multiplying both sides by $(tr)^l (l \geq 0)$ we have

$$
\sum_{\langle a, q \rangle \leq m} \| D^a u, V_{-r} \| (tr)^l + \chi^{\langle a, q \rangle} (tr)^{\langle a, q \rangle} + 1
$$

$$
\leq C \left\{ \| P(D) u, V_{-(r-1-t)} \| (r(1-t))^{l+\langle a, q \rangle} \chi^n \left( \frac{t}{1-t} \right)^{l+m} \\
+ \chi^n \sum_{\langle a, q \rangle \leq m} \| D^a u, V_{-(r-1-t)} \| (r(1-t))^{l+\langle a, q \rangle} \left( \frac{t}{1-t} \right)^{l+m} \\
+ \| u, V_{-(r-1-t)} \| (r(1-t))^{l+\langle a, q \rangle} \left( \frac{t}{1-t} \right)^{l+m} \right\}.
$$
Hence we get by (10.6)
\[
\sum_{\langle \alpha, q \rangle \leq m} \| D^s u; \ l + \langle \alpha, q \rangle, V \|^{|(\alpha, q)|} t \chi^m + \chi^m \leq C \left\{ \| P(D)u; \ l + m, V \| \left( \frac{t}{1-\tau} \right)^{1+m} + \| u; \ l, V \| \left( \frac{t}{1-\tau} \right)^{1} \right\}.
\]

Now assume \( 0 < t \leq 1/(l+m) \). Then there is a constant \( c > 0 \) such that
\[
(t/(1-t))^{1+m} \leq t^{1+m} c^e
\]
for any \( \alpha \) satisfying \( \langle \alpha, q \rangle \leq m \). Taking \( \chi \) sufficiently small here we get, for another constant \( C \),
\[
\sum_{\langle \alpha, q \rangle < m} \| D^s u; \ l + \langle \alpha, q \rangle, V \|^{|(\alpha, q)|} t \chi^m + \chi^m \leq C \left\{ \| P(D)u; \ l + m, V \| t^{m} + \| u; \ l, V \| \right\}
\]
with any \( t \) such that \( 0 < t \leq 1/(l+m) \).

We note that in Lemma 10.3 the terms \( t^{(\alpha, q)} \) and \( s^{(\alpha, q)} \) can be replaced by \( t^{(\alpha, \lambda q)} \) and \( s^{(\alpha, \lambda q)} \) with any \( \lambda \geq 1 \) respectively. Thus we can obtain the following lemma.

**Lemma 10.4.** There exists a constant \( C \) such that
\[
(10.10) \sum_{\langle \alpha, q \rangle \leq m} \| D^s u; \ l + \langle \alpha, q \rangle, V \|^{|(\alpha, q)|} \leq C \left\{ \| P(D)u; \ l + m, V \| t^{m} + \| u; \ l, V \| \right\}
\]
for all \( u \in C(\Omega \cup \omega) \) satisfying the condition (9.4), provided that \( 0 < t \leq 1/(l+m), \lambda \geq 1 \).

By making use of these lemmas we can prove Theorems 9.1 and 9.2. For simplicity we consider the case \( \lambda = 1 \). It will be convenient to use the notation:

\[
(10.11) (D^s P(D)u)_i = t^{(\beta, q)} + m D^s P(D)u, \quad (D^s u)_i = t^{(\beta, q)} D^s u;
\]

\[
(10.12) B_0(D^s u) = \| (D^s u)_i; \ l + \langle \beta, q \rangle, V \|^{(\beta, q)};
\]

\[
B_{i+1}(u) = \max_{\langle \beta, q \rangle \leq m; \beta_i = 0} B_i(D^s u), \quad i \geq 0;
\]

\[
B_0(D^s Pu) = \| (D^s Pu)_i; \ l + \langle \beta, q \rangle + m, V \|^{(\beta, q)};
\]

\[
B_{i+1}(Pu) = \max_{\langle \beta, q \rangle \leq m; \beta_i = 0} B_i(D^s Pu), \quad i \geq 0;
\]

\[
\| u; \ q, \mu; \ l, V \| = \sup_{\beta \leq 0, \beta_n = 0} \left( \prod_{i=1}^{n-1} \left( \frac{\mu_i}{\beta_i} \right)^{\mu_i} \right) \| D^s u; \ l + \langle \beta, q \rangle, V \|.
\]

\[
(10.7) \| Pu; \ q, \mu; \ l, V \| = \sup_{\beta \leq 0, \beta_n = 0} \left( \prod_{i=1}^{n-1} \left( \frac{\mu_i}{\beta_i} \right)^{\mu_i} \right) \| D^{sp} u; \ l + m + \langle \beta, q \rangle, V \|.
\]

\(^(*)\) The \( B_i \) are functions of \( t \) also.
Lemma 10.5. There is a constant $C > 1$ such that

\begin{equation}
C^{-1}B_{j}(u) \leq \max \left\{ \max_{1 \leq k \leq j} C^{k-1}B_{j-k}(Pu), B_{0}(u) \right\}
\end{equation}

for $j = 1, 2, \ldots$ and for all $u \in C^{\infty}(V)$ satisfying (9.4), provided that $0 < t \leq 1/(l+jm)$.

Proof. We see that (10.10) means

\begin{equation}
B_{1}(u) \leq \max \{CB_{0}(Pu), CB_{0}(u)\}
\end{equation}

with some positive constant $C$. The inequality (10.15) shows that (10.14) is true when $j = 1$ and $0 < t \leq 1/(l+m)$. If we replace $u$ by $D^{j}u$ in (10.8), multiply both sides by $e^{\langle \beta, \alpha \rangle}$, and then proceed as in the proof of Lemma 10.4, we find that

\begin{equation}
B_{2}(u) \leq \max \{CB_{1}(Pu), CB_{1}(u)\},
\end{equation}

provided that $0 < t \leq 1/(l+2m)$. Again by (10.15) we obtain

\begin{equation}
B_{2}(u) \leq \max \{CB_{1}(Pu), C^{2}B_{0}(Pu), C^{2}B_{0}(u)\},
\end{equation}

provided that $0 < t \leq 1/(l+2m)$. Proceeding in this way, we can prove (10.14) for all $j$.

Lemma 10.6. Let $B_{0}$ be defined by (10.12), (10.13) with $t_{i} = 1/(l+jm)$ for $l$ and $j$ fixed. Then there are constants $c < 1$ and $C_{1}$, independent of $j$, such that

\begin{equation}
C^{-1} \|u; q, c_{\mu}; l, V\| \leq C^{-1}B_{j}(Pu, t_{j}) + B_{0}(u)
\end{equation}

and

\begin{equation}
C^{-1}B_{j}(Pu, t_{j}) \leq \|Pu; q, \mu; l, V\|,
\end{equation}

where $\mu = (C^{1/q_{1}}, \ldots, C^{1/q_{n}-1})$.

Proof. Put $N = \|Pu; q, \mu; l, V\|$, where $\mu = (C^{1/q_{1}}, \ldots, C^{1/q_{n}-1})$, and suppose that $t = 1/(l+jm)$. Then

\[
\max C^{-1}B_{j}(Pu) \leq \max_{\langle \beta, \alpha \rangle \in m; \beta_{n} = 0} \left| C^{-1}\frac{1}{l+jm} \right| \left( \frac{1}{l+jm} \right)^{m+\langle \beta, \alpha \rangle} \|D^{\alpha}Pu; l+m+\langle \beta, \alpha \rangle, V\|
\]

\[
\leq \max_{\langle \beta, \alpha \rangle \in m; \beta_{n} = 0} \left| C^{-1}\frac{1}{l+jm} \right| \left( \frac{1}{l+jm} \right)^{m+\sum_{i=1}^{n-1} \frac{\beta_{i}}{\mu_{i}}} N
\]

\[
\leq \max_{\langle \beta, \alpha \rangle \in m; \beta_{n} = 0} \left( \frac{\beta_{1}}{l+jm} \right)^{\frac{\beta_{q_{1}}}{\beta_{n}}} \left( \frac{1}{l+jm} \right)^{m} N \leq N.
\]

This proves (10.18).
Next, with the same $t$, $\mu$ as above we have, for $c$ determined later,

$$
C^{-1}B_t(u) \leq \max_{(j-1)m \leq \langle \beta, q \rangle \leq jm} C^{-1} \left( \frac{1}{1+jm} \right)^{\langle \beta, q \rangle} \| D^\beta u; l + \langle \beta, q \rangle, V \|
$$

$$
\leq \max_{(j-1)m \leq \langle \beta, q \rangle \leq jm} C^{-1} \left( \frac{1}{1+jm} \right)^{\langle \beta, q \rangle} \prod_{i=1}^{n-1} \left( \frac{\beta_i}{c_{\mu_i}} \right)^{\delta_i q_i} \left( \frac{c_{\mu_i}}{\beta_i} \right)^{\delta_i q_i} \| D^\beta u; l + \langle \beta, q \rangle, V \|
$$

$$
\leq C' \max_{(j-1)m \leq \langle \beta, q \rangle \leq jm} \prod_{i=1}^{n-1} \left[ \frac{\beta_i}{c(l+jm)} \right]^{\delta_i q_i} \| D^\beta u; l + \langle \beta, q \rangle, V \|
$$

where $C'$ and $C_0$ are constants independent of $j$. Put

$$
K = \prod_{i=1}^{n-1} \left[ \frac{\beta_i}{c(l+jm)} \right]^{\delta_i q_i}, \quad (j-1)m \leq \langle \beta, q \rangle \leq jm.
$$

Then

$$
(C'K)^{-1} = (C')^{-1} \prod_{i=1}^{n-1} \left[ \frac{c(l+jm)}{\beta_i} \right]^{\delta_i q_i} \leq C_1 c^{-\langle \beta, q \rangle}, \quad (j-1)m \leq \langle \beta, q \rangle \leq jm,
$$

is finite if $c$ is sufficiently small. This proves (10.17).

Finally we obtain the following estimate:

$$
\| u; q, c\mu; l, V \| \leq C\{ \| Pu; q, \mu; l, V \| + \| u; l, V \| \},
$$

for all $u \in C^\infty(V)$ satisfying (9.4). In the same way we can get the estimate of the type

$$
\| D^\alpha u; q, c\mu; l + \langle \alpha, q \rangle, V \| \leq C\{ \| Pu; q, \mu; l, V \| + \| u; l, V \| \}
$$

for any $\alpha$ such that $\langle \alpha, q \rangle \leq m$.

Now we can complete the proof of Theorem 9.1. Let $f(x, y)$ be in $G(q, \Omega \cup \omega)$. Then for any hemisphere $K=\{(x, y); |x|^2 + y^2 \leq r, y \geq 0\} \subset V$, there are constants $C_0, C_1$ such that

$$
\| D^\beta f, K \| \leq C_0 C_1^\langle \beta \rangle |\beta|^{\langle \beta, q \rangle}
$$

for all $\beta$.

By the inequality (10.20) we have for new constants $C_0, C_1$

$$
\| D^\beta D^\beta u, K \| \leq C_0 C_1^\langle \beta \rangle |\beta|^{\langle \beta, q \rangle}
$$

for any $\beta$ ($\beta_n = 0$) and $\alpha$ ($\langle \alpha, q \rangle \leq m$). We may assume that the corresponding constants in (10.21) and (10.22) are the same.

The equation $P(D)u = f$ can be written in the form

$$
D^n u = f - \sum_{\langle \alpha, q \rangle \geq m} a_\alpha D^\alpha u.
$$

We note that $\langle \alpha, q \rangle \leq m$ and $a_n = m_n$ imply $a_i = 0$, ($i = 1, \ldots, n-1$), and that for
any positive integer \( k (\leq m_n) \), \( \langle \alpha, q \rangle \leq m \) and \( \alpha_n = m_n - k \) imply \( \sum_{n=1}^{m-1} \alpha q_n \leq k q_n \). Differentiating (10.23) with respect to \( x\)-variables and applying (10.21) and (10.22), we have

\[
\| D_x^k D_y^m u, K \| \leq C_0 C_1^{\beta + (\beta, q) + B C_0 C_1^{\beta + (\beta, q)},}
\]

where we put \( B = 1 + \sum_{n} |a_n| \). Again differentiating (10.23) we have

\[
D_x^k D_y^m u = D_x^k D_y^m f - \sum_{\langle \alpha, q \rangle \leq m, \alpha_n = m_n - 1} a_n D_x^k D_y^m u + \sum_{\langle \alpha, q \rangle \leq m, \alpha_n = m_n - 2} a_n D_x^k (D_y^m u)
\]

where we put \( a_n = 0 \) when \( \alpha_n < 0 \). Applying (10.24) and Lemma 10.1 we have

\[
\| D_x^k D_y^m u, K \| \leq C_0 C_1^{\beta + (\beta, q) + q_n}
\]

where we put \( a_n = 0 \) when \( \alpha_n < 0 \). Applying (10.24) and Lemma 10.1 we have

\[
\| D_x^k D_y^m u, K \| \leq (B+1)^{\beta + (\beta, q) + q_n + m} + B \sum_{n=0}^{m} C_0 C_1^{\beta + (\beta, q) + q_n + m} + BC_0 C_1^{\beta + (\beta, q) + q_n + m}.
\]

Repeating the procedure we can obtain, by a simple induction argument on \( k \),

\[
\| D_x^k D_y^m u, K \| \leq (B+1)^{k+1} C_0 C_1^{\beta + (\beta, q) + q_n + m} + B C_0 C_1^{\beta + (\beta, q) + q_n + m},
\]

where we put \( a_n = 0 \) when \( \alpha_n < 0 \). Applying (10.24) and Lemma 10.1 we have

\[
\| D_x^k D_y^m u, K \| \leq C_0 C_1^{\beta + (\beta, q) + q_n}
\]

for any \( \beta (\beta_n = 0) \) and for any \( k \geq 0 \).

Applying Sobolev's lemma to the inequality (10.27) we obtain Theorem 9.1.

11. Proof of Theorem 9.2. From the estimate (10.10) we easily obtain

\[
\max_{\langle \alpha, q \rangle \leq m, \langle \beta, q \rangle \leq km_1} \| D^{\beta + \alpha} u, l + \langle \beta + \alpha, \lambda q \rangle, V \| \leq C \max_{\langle \alpha, q \rangle \leq m, \langle \beta, q \rangle \leq km_1} \| D^{\beta + \alpha} u, l + \langle \beta + \alpha, \lambda q \rangle, V \|
\]

for all \( u \in C^\infty(\Omega \cup \omega) \) satisfying (9.6), provided that \( 0 < t \leq 1/(l + km) \), \( k = 0, 1, 2, \ldots \).

In a quite similar manner to that used in the proof of Lemma 10.6 we obtain the estimate of the form

\[
\max_{\langle \alpha, q \rangle \leq m} \| D^{\alpha} u, \lambda q, \mu; l + \langle \alpha, \lambda q \rangle, V \| \leq C \sup_k \left( \frac{\mu}{mk} \right)^{mk} \| P^k u, l + \lambda km, V \|
\]

where

\[
\| D^{\alpha} u, \lambda q, \mu; l + \langle \alpha, \lambda q \rangle, V \| = \sup_{\beta \geq 0, \beta_n = 0} \prod_{i=1}^{n-1} \left( \frac{\beta_i}{\beta_i} \right)^{\beta_i q_i} \| D^{\alpha} u, l + \beta + \langle \alpha, \lambda q \rangle, V \|
\]

The same argument as in the end of the proof of Theorem 9.1 completes the proof of Theorem 9.2.
Remark. The conclusion of Theorems 9.1 and 9.2 can be extended to operators with variable coefficients. The proof can be obtained by a quite similar argument to the proof of Theorem 9.1 and Theorem 9.2 (cf. [2], [8]).

Added in Proof. The conclusion of Theorems 9.1 and 9.2 can be extended to the general quasi-elliptic boundary problems defined in §3. Details will be given in a future publication.

REFERENCES