ON THE EXISTENCE OF MINIMAL IDEALS IN A BANACH ALGEBRA

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Introduction. In this paper we give a sufficient condition that an algebra have a minimal left (or right) ideal. Specifically, we prove that if A is a complex semisimple Banach algebra with the property that the spectrum of every element in A is at most countable, then A has a minimal left ideal. If A is an A^*-algebra, we prove that A has a minimal left ideal if the spectrum of every self-adjoint element of A is at most countable. These two basic results are given in §2, and some applications of them are given in §§3 and 4. The main result of §3 is Theorem 3.1 which is a variant of a theorem concerning B^*-algebras that was proved through the separate efforts of M. A. Naĭmark [5] and A. Rosenberg [7]. In §4 we prove that a complex semisimple Banach algebra with the property that the spectrum of every element in the algebra has no nonzero limit points is a modular annihilator algebra. This together with a previous result of the author shows that modular annihilator Banach algebras are characterized by this property.

1. **Notation.** We deal exclusively with complex algebras A. The norm of an element $u \in A$ will be denoted by the usual symbol $\|u\|$. $\sigma_A(u)$ is the spectrum of u in the algebra A. If A is commutative, u' denotes the image of $u \in A$ by the Gelfand representation of A on its maximal ideal space. We write the radical of A as rad (A) and the socle of A as S_A. If E is any nonempty subset of A, then $L[E]$ and $R[E]$ are the left and right annihilators of E respectively ($L[E]=\{a \in A \mid aE=0\}$). When A has an involution $*$, then $u \in A$ is self-adjoint when $u=u^*$. A is an A^*-algebra if there is a norm $|\cdot|$ on A (not necessarily complete) with the property that $|uu^*| = |u|^2$ for all $u \in A$. If A is complete in this norm, A is a B^*-algebra. Definitions of most of the concepts mentioned in this paper may be found in C. Rickart's book [6] (although our notations often differ). Information concerning modular annihilator algebras may be found in [1] or [8].

2. **The existence of minimal ideals.** We first treat the special case where the algebra A is commutative in Theorem 2.1. Using this special case, we give the general result in Theorem 2.2. Finally in Theorem 2.3 we consider the case where A is an A^*-algebra.

THEOREM 2.1. *Assume that A is a semisimple commutative Banach algebra with no minimal ideals. Then there is an element in A with uncountable spectrum.*

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Proof. We assume for convenience that A has an identity. Let Ω be the maximal ideal space of A. First suppose that there is a connected subset of Ω, Γ, which contains more than one point. Then there exists $v \in A$ such that v' is not constant on Γ. But then $v'(\Gamma)$ is a connected subset of the complex plane which contains more than one point. It follows that $v'(\Gamma)$, and hence $\sigma_A(v)$, is uncountable.

In the remaining case every nonempty connected subset of Ω must consist of a single point, that is Ω is totally disconnected. If Ω contains an isolated point ϕ, then by Šilov's Theorem [6, Theorem (3.6.3), p. 168] there exists $e \in A$ such that $e'(\phi)=1$, and $e'(\psi)=0$ when $\psi \in \Omega$, $\psi \neq \phi$. Then Ae is a minimal ideal of A which is contrary to hypothesis. Thus Ω contains no isolated points, and this implies that every nonempty open subset of Ω is infinite.

Now under the assumptions that Ω is totally disconnected (t.d.) and that every nonempty open and closed subset of Ω is infinite, we construct an element of A which has uncountable spectrum. Since Ω is t.d., we may choose mutually disjoint, nonempty, open and closed subsets of Ω, V_0 and V_1 such that $\Omega = V_0 \cup V_1$. The sets V_0 and V_1 are t.d. and infinite, so we may choose as before mutually disjoint, nonempty, open and closed subsets of V_k, V_{k0} and V_{k1}, such that $V_k = V_{k0} \cup V_{k1}$, $k=0, 1$. In general by this same procedure, given $V_{k_1k_2\cdots k_n}$, $k_j=0$ or 1, we may choose mutually disjoint, nonempty, open and closed subsets of this set, $V_{k_1\cdots k_n0}$ and $V_{k_1\cdots k_n1}$, such that $V_{k_1\cdots k_n} = V_{k_1\cdots k_n0} \cup V_{k_1\cdots k_n1}$. By this construction we obtain a collection of nonempty open and closed subsets of Ω, $\{V_{k_1k_2\cdots k_m}\}$ where $k_j=0$ or 1 and $1 \leqq m < \infty$. Let $e_{k_1k_2\cdots k_m}$ be the idempotent in A such that $e'_{k_1k_2\cdots k_m}$ is the characteristic function of the set $V_{k_1k_2\cdots k_m}$ (such an idempotent exists in A by Šilov's Theorem).

Let $t_1 = e_1$, and choose an integer $m_1 \geqq 1$ such that $\|10^{-m_1}t_1\| \leqq 1$. Let $t_2 = e_{01} + e_{11}$, and choose an integer $m_2 > m_1$ such that $\|10^{-m_2}t_2\| \leqq \frac{1}{2}$. In general, having chosen m_{n-1}, let

$$t_n = \left(\sum_{k_1=0}^{1} \sum_{k_2=0}^{1} \cdots \sum_{k_{n-1}=0}^{1} (e_{k_1k_2\cdots k_{n-1}1})\right),$$

and choose an integer $m_n > m_{n-1}$ such that $\|10^{-m_n}t_n\| \leqq (\frac{1}{2})^{n-1}$. Finally define $u = \sum_{k=1}^{\infty} 10^{-m_k}t_k$ (the sum converges in A by the construction).

Now given any sequence of the digits 0 and 1, $\{j_1, j_2, j_3, \ldots\}$, the set $\bigcap_{n=1}^{\infty} V_{j_1j_2\cdots j_n}$ is nonempty by the finite intersection property. Take a point ϕ in this set. Then $u'(\phi) = \sum_{k=1}^{\infty} j_k 10^{-m_k}$. But the set of all such points is uncountable, and thus $\sigma_A(u)$ is uncountable.

THEOREM 2.2. *Assume that A is a semisimple Banach algebra with the property that the spectrum of every element of A is at most countable. Then A has minimal left and right ideals, and also $L[S_A]=0$.*

Proof. We may exclude the trivial case when A is 0. A is semisimple, and for any $u \in A$ with nonzero spectrum, $\sigma_A(u)$ must have an isolated point. Thus A contains a

nonzero idempotent; see [4, Theorem 5.5.2, p. 175]. Let E be a maximal set of nonzero commuting idempotents of A. Let C be a maximal commutative subalgebra of A containing E, and denote the space of modular maximal ideals of C by Ω. Now the algebra $C/\text{rad}\,(C)$ is a semisimple commutative Banach algebra with the property that every element has at most a countable spectrum. By Theorem 2.1, $C/\text{rad}\,(C)$ has a minimal ideal, or what is equivalent, Ω has an isolated point ϕ. Again by Šilov's Theorem [6, Theorem (3.6.3), p. 168] there is an idempotent $e \in C$ such that $e'(\phi)=1$ and $e'(\psi)=0$ when $\psi \in \Omega$, $\psi \neq \phi$. Assume that $g \in E$. Then for any $\psi \in \Omega$, $(g'(\phi)e-ge)'(\psi)=0$. Thus $g'(\phi)e-ge \in \text{rad}\,(C)$. Now $g'(\phi)=0$ or 1, and in either case $g'(\phi)e-ge$ is an idempotent and is in rad (C). But the radical of any algebra contains no nonzero idempotents, and therefore $g'(\phi)e=ge=eg$.

Consider the semisimple algebra $B=eAe$. Suppose that f is a nonzero idempotent in B. Then $f=ef=fe$. If $g \in E$, then $fg=feg=g'(\phi)fe=g'(\phi)ef=gf$. Therefore by the definition of E, $f \in E$. It follows that $ef=fe=e$, and since $fe=f$, we have $f=e$. We have shown that e is the only nonzero idempotent in B. It follows by [4, Theorem 5.5.2, p. 175] that the spectrum of each element of B is connected. Now the spectrum of any element x of eAe is the same computed in B or A, and is therefore by hypothesis at most countable. Then [9, Lemma 4, p. 375] shows that B is the one-dimensional space spanned by e. Therefore e is a minimal idempotent of A. By [6, Corollary (2.1.9), p. 46] Ae and eA are minimal left and right ideals of A, respectively.

Finally consider the two-sided ideal of A, $L[S_A]=\{a \in A \mid aS_A=0\}$. If $L[S_A]\neq 0$, then as an algebra, $L[S_A]$ satisfies the hypotheses of the theorem since A does. It follows that $L[S_A]$ has a nonzero minimal left ideal K by the preceding argument. But K is also a minimal left ideal of A, and therefore $K^2=0$, contradicting the semisimplicity of A. Thus $L[S_A]$ must be zero.

THEOREM 2.3. *Assume that A is an A^*-algebra. If the spectrum of every self-adjoint element of A is at most countable, then A has minimal left and right ideals. Also $L[S_A]=0$.*

Proof. We may assume that A is not the zero algebra. Let C be a maximal commutative *-subalgebra of A, and denote the space of modular maximal ideals of C by Ω. C is an A^*-algebra, and hence semisimple (any A^*-algebra is semisimple by [6, Theorem (4.1.19), p. 188]). For any $w \in C$, $w=u+iv$ where u and v are some self-adjoint elements of C. Clearly the function $w'=u'+iv'$ takes at most a countable number of distinct values on Ω. Then by Theorem 2.1, C contains a minimal idempotent e, and by [6, Lemma (4.10.1), p. 261], we may assume that e is self-adjoint. Now given a self-adjoint element v of A and $u \in C$, there exists a scalar λ such that $eu=ue=\lambda e$, and hence, $(eve)u=\lambda eve=u(eve)$. Also note that eve is self-adjoint. Therefore $eve \in C$, and eve must be just a scalar multiple of e. But every element of A is a linear combination of self-adjoint elements of A. Therefore e is a minimal idempotent of A, and A has minimal left and right ideals.

The fact that $L[S_A]=0$ is verified in much the same way as in the proof of Theorem 2.2.

3. **An application to A^*-algebras.** In [5], M. A. Naĭmark asked essentially the following question: If A is an irreducible B^*-algebra of operators on a separable Hilbert space $\mathscr{H}$ with the property that every nontrivial irreducible representation of A on a Hilbert space is unitarily equivalent to the given representation of A on $\mathscr{H}$, must A be the algebra of all completely continuous operators on $\mathscr{H}$? Naĭmark made progress toward an affirmative answer to this question in [5], and using his results, A. Rosenberg in [7], proved that the question has an affirmative answer. Here using the results of §2, we prove a variant of the Naĭmark-Rosenberg Theorem and at the end of this section, we indicate how their result may be obtained from ours.

THEOREM 3.1. *Let A be an irreducible Banach *-subalgebra of the algebra of bounded operators on a separable inner product space $\mathscr{H}$. Assume that every nontrivial irreducible representation of A is equivalent to the given representation of A on $\mathscr{H}$. Then A is a modular annihilator algebra. When $\mathscr{H}$ is a Hilbert space, A is dense in the algebra of all completely continuous operators on $\mathscr{H}$ in the operator norm. If in addition A is a B^*-algebra, then A is the algebra of all completely continuous operators on $\mathscr{H}$.*

Proof. Assume that $v \in A$ and $v=v^*$. If $\lambda \in \sigma_A(v)$, $\lambda \neq 0$, then either $A(\lambda - v)$ or $A(\bar{\lambda}-v)$ is a proper modular left ideal of A. Choose μ to be either λ or $\bar{\lambda}$ and such that $A(\mu-v)$ is proper. Then there exists a maximal left ideal N of A such that $A(\mu-v)\subset N$. The left regular representation of A on the quotient space $A-N$ is irreducible. Elements $u \in A$ act on elements $z+N \in A-N$ in the natural way: $u(z+N)=uz+N$. By hypothesis, the representation of A on $A-N$ is equivalent to the given representation of A on $\mathscr{H}$. Therefore there exists an isomorphism W of $A-N$ onto $\mathscr{H}$ such that for every u, $z \in A$, $u(W(z+N))=W(uz+N)$. Then $(\mu-v)(W(v+N))=W(v(\mu-v)+N)=0$, since $A(\mu-v)\subset N$. Also $W(v+N)\neq 0$. This proves that μ is an eigenvalue of v. Therefore μ is in the spectrum of v in the algebra of all bounded operators on $\mathscr{H}$, and since v is self-adjoint, μ is real. Thus $\mu=\lambda=\bar{\lambda}$. We have shown that when $v \in A$ is self-adjoint, then the nonzero spectrum of v consists entirely of eigenvalues. But then since $\mathscr{H}$ is separable and eigenvectors corresponding to distinct eigenvalues of a self-adjoint operator are orthogonal, it follows that $\sigma_A(v)$ is at most countable whenever $v=v^*$. By Theorem 2.3 A has minimal left and right ideals. Now 0 is a primitive ideal of A, and in fact the only primitive ideal of A since all irreducible representations of A are equivalent. Then A is a modular annihilator algebra since S_A is contained in no primitive ideal of A; see [1, Theorem 4.3(4), p. 570].

Now assume that $\mathscr{H}$ is a Hilbert space. Let B be the closure of A in the operator norm. Then by [2, Lemma 2.6] S_B is dense in B, and by [8, Lemma 3.11, p. 41],

this implies that B is a modular annihilator B^*-algebra of operators. It remains to be shown that an irreducible modular annihilator B^*-algebra of operators on a Hilbert space $\mathscr{H}$ is the algebra of completely continuous operators on $\mathscr{H}$. But by [8, Theorem 4.1, p. 42] such an algebra must be dual and then the result follows by [6, Corollary (4.10.20), p. 269] and [6, Theorem (4.10.22), p. 270].

Now the combination of [6, Lemma (4.9.11), p. 254] and [6, Theorem (4.9.8), p. 251] implies that any irreducible representation of a B^*-algebra is algebraically equivalent to an irreducible *-representation of the algebra on a Hilbert space. Thus if all the nontrivial irreducible *-representations on a Hilbert space of a B^*-algebra A are unitarily equivalent (or in fact merely algebraically equivalent), then all nontrivial irreducible representations of A are algebraically equivalent. These remarks show that the Theorem of Naĭmark and Rosenberg follows from Theorem 3.1.

4. **Modular annihilator algebras.** It was shown in [3] that if A is a semisimple normed modular annihilator algebra, then A has the property that the spectrum of every element has no nonzero limit point; see [3], the Corollary to Theorem 3.4. In this section we prove the converse of this theorem with the assumption that A is a Banach algebra. We obtain the result for the case where A is commutative first, and then extend this special case to the general theorem.

THEOREM 4.1. *Let A be a semisimple commutative Banach algebra with the property that for every $u \in A$, $\sigma_A(u)$ has no nonzero limit points. Then A is a modular annihilator algebra.*

Proof. We may assume that A is not finite dimensional (any semisimple finite dimensional algebra is a modular annihilator algebra). Let Ω be the space of modular maximal ideals of A, and let Γ be the set of isolated points of Ω. By Theorem 2.1, Γ is nonempty. First we prove that Γ is a closed set in Ω. Assume that $\phi \in \Omega$ is a limit point of Γ. Choose $v \in A$ such that $v'(\phi)=1$. Now 1 is an isolated point of $\sigma_A(v)$, so we may choose an open subset of the complex plane U such that $U \cap \sigma_A(v)=\{1\}$. Then the set $N=\{\psi \in \Omega \mid v'(\psi)=1\}$ is an open subset of Ω which contains ϕ. But then N must contain an infinite number of points of Γ. We choose a sequence $\{\phi_k\}$ of distinct points in $\Gamma \cap N$. There exists a corresponding sequence of minimal idempotents $\{e_k\}$ in A with the property that $e_k'(\phi_k)=1$, $e_k'(\psi)=0$, $\psi \in \Omega$, $\psi \neq \phi_k$. Finally we choose a sequence of distinct complex numbers $\{\lambda_k\}$ with the property that $|\lambda_k| \leqq 1/\|e_k\|2^k$. Set $w=v+\sum_{k=1}^{\infty} \lambda_k e_k$. Then $w'(\phi_k)=1+\lambda_k$ and $\lambda_k \to 0$ as $k \to \infty$. This is a contradiction, since 1 must be an isolated point of $\sigma_A(w)$. Therefore Γ must have no limit points in Ω, and in particular Γ is a closed set.

Now let $\Delta=\Omega-\Gamma$. Δ is hull-kernel closed since Δ is the intersection of the sets $\{\psi \in \Omega \mid e'(\psi)=0\}$ for all minimal idempotents $e \in A$. By $k(\Delta)$ we mean as usual the intersection of all the ideals in Δ. Let B be the quotient algebra $A/k(\Delta)$. Then B is semisimple and the spectrum of any element in B has no nonzero limit points. By Theorem 2.1, if B is nonzero then B must have a nonzero minimal ideal. But by

[6, Theorem (3.1.17), p. 116] Δ is homeomorphic to the maximal ideal space of B, and hence Δ must contain isolated points. Since Γ is closed, Δ is open, and then any isolated point of Δ is an isolated point of Ω. This contradiction proves that in fact B is the zero algebra and Δ must be empty. We have established that $\Gamma=\Omega$.

Suppose that M is any maximal modular ideal of A. Since M is an isolated point of Ω, there exists a minimal idempotent $e \in A$ such that $e'(M)=1$, $e'(N)=0$ if $N \in \Omega$, $N \neq M$. But then whenever $u \in M$, $(ue)'(N)=0$ for every $N \in \Omega$. Since A is semisimple, we conclude that $Me=0$. Thus A is a modular annihilator algebra by definition.

THEOREM 4.2. *Let A be any complex semisimple Banach algebra. A is a modular annihilator algebra if and only if for any $u \in A$, $\sigma_A(u)$ has no nonzero limit points.*

Proof. As we mentioned previously the "only if" part of this theorem follows from [3], the Corollary to Theorem 3.4. Thus we assume that for every $u \in A$, $\sigma_A(u)$ has no nonzero limit points. Then by Theorem 2.2, $L[S_A]=0$. This implies that every nonzero right ideal of A contains a minimal right ideal of A (and hence a minimal idempotent of A); see [8, Lemma 3.1, p. 37].

Assume that e is a nonzero idempotent of A. Let $\mathscr{M}$ be a maximal orthogonal set of minimal idempotents of A in eA. If $\mathscr{M}$ is infinite, we can choose a sequence of distinct idempotents $\{e_k\} \subset \mathscr{M}$. Then choose $\{\lambda_k\}$ a sequence of distinct complex numbers with the property that $|\lambda_k| \leqq 1/\|e_k\|2^k$. Finally set $w=e+\sum_{k=1}^{\infty} \lambda_k e_k$. Each $e_k \in eA$, and thus $e_k=ee_k$. Then $we_k=(1+\lambda_k)e_k$ for each k. Therefore the sequence $\{1+\lambda_k\}$ must be in the spectrum of w, a contradiction. It follows that $\mathscr{M}$ must be finite. Then by an argument similar to the proof of Theorem 2.2 of [3], it can be shown that $e \in S_A$. We have proved that every idempotent of A is in S_A.

Now we prove directly that A is a modular annihilator algebra. Let M be a maximal modular left ideal of A. By [8, Lemma 3.3, p. 38] either $S_A \subset M$ or $R[M] \neq 0$ (where $R[M]=\{a \in A \mid Ma = 0\}$). Suppose that $S_A \subset M$. Since M is modular, there exists $u \in A$ such that $A(1-u) \subset M$. Let C be a maximal commutative subalgebra of A containing u. Let Ω be the space of modular maximal ideals of C. For every $v \in C$, $\sigma_A(v)=\sigma_C(v)$. Thus by Theorem 4.1, $C/\text{rad}\,(C)$ is a modular annihilator algebra. Denote $C/\text{rad}\,(C)$ by B, and let π be the natural homomorphism of C onto B. Since B is a semisimple modular annihilator algebra, $\pi(u)$ is quasi-regular modulo S_B by [8, Theorem 3.4 (3), p. 38]. Therefore there exist elements $x_1, x_2, \ldots, x_n$ and w in C and minimal idempotents $f_1, f_2, \ldots, f_n$ in B such that $\pi(w)(1-\pi(u))+\pi(u) =\pi(x_1)f_1+\cdots+\pi(x_n)f_n$. Now by [6, Theorem (2.3.9), p. 58], there are idempotents $e_k \in C$ such that $\pi(e_k)=f_k$, $1 \leqq k \leqq n$. By the definition of π, there exists $r \in \text{rad}\,(C)$ such that $r=w(1-u)+u-(x_1e_1+\cdots+x_ne_n)$. Then $A(1-r) \subset A(1-w)(1-u) +A(x_1e_1+\cdots+x_ne_n)$. But $A(1-w)(1-u) \subset M$, and $A(x_1e_1+\cdots+x_ne_n) \subset S_A \subset M$ (recall that we have shown that every idempotent in A is in S_A). Therefore $A(1-r) \subset M$, and this contradicts the fact that $\{0\}=\sigma_C(r)=\sigma_A(r)$. Hence $R[M] \neq 0$, and A must be a modular annihilator algebra.

COROLLARY. *Assume A is a semisimple complex modular annihilator Banach algebra. Then if B is any closed subalgebra of A, then $B/\mathrm{rad}\,(B)$ is a modular annihilator algebra.*

Proof. It is sufficient to note that $\sigma_B(u) \cup \{0\} = \sigma_A(u) \cup \{0\}$ for all $u \in B$ by [6, Theorem (1.6.12), p. 33]. Then apply Theorem 4.2.

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