SPACES FOR WHICH THE STONE-WEIERSTRASS THEOREM HOLDS

BY

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If \( X \) is a topological space, a subset \( A \) of \( C(X) \), the set of bounded continuous real functions on \( X \), is said to separate the points of \( X \) if for every pair \( x, y \) of distinct points of \( X \) there is a function \( f \) in \( A \) with \( f(x) \neq f(y) \). A space \( X \) is called completely Hausdorff if \( C(X) \) separates the points of \( X \). The Stone-Weierstrass theorem states: If \( X \) is a compact Hausdorff space, and if \( A \) is a subalgebra of \( C(X) \) which (i) separates the points of \( X \) and (ii) contains the constants, then \( A \) is uniformly dense in \( C(X) \). In the following, we shall say the Stone-Weierstrass theorem holds for a space \( X \) provided that \( X \) is completely Hausdorff, and that every subalgebra of \( C(X) \) which satisfies (i) and (ii) is uniformly dense in \( C(X) \).

An extension space of a topological space \( X \) is a pair \((Y, h)\), where \( Y \) is a topological space, \( h \) is a homeomorphism of \( X \) into \( Y \), and \( h(X) \) is dense in \( Y \); if \( h \) is the identity map, the reference to \( h \) is omitted, and \( Y \) itself is called an extension space of \( X \); \((Y, h)\) is called proper if \( h(X) \) is a proper subset of \( Y \). We shall call a completely Hausdorff space \( X \) completely Hausdorff-complete if and only if \( X \) has no proper extension space \((Y, h)\) such that \( Y \) is a completely Hausdorff space.

A filter on a space \( X \) is called completely regular provided that it has a base \( \mathscr{B} \) of open sets such that for each \( B \) in \( \mathscr{B} \), there is a set \( B' \subseteq B \) in \( \mathscr{B} \) and a function \( f \in C(X) \) which maps \( X \) into \([0, 1] \), is 0 on \( B' \), and is 1 on \( X - B \). In [3] Banaschewski proved that the Stone-Weierstrass theorem holds for a completely Hausdorff space \( X \) if and only if every completely regular filter \( \mathscr{F} \) on \( X \) has the property that \( \bigcap \{ F \mid F \in \mathscr{F} \} \neq \emptyset \). Using this result, we shall prove the following:

**Theorem 1.** The Stone-Weierstrass theorem holds for a completely Hausdorff space \( X \) if and only if \( X \) is completely Hausdorff-complete.

**Proof.** Suppose that \( X \) has a proper extension space \((Y, h)\) such that \( Y \) is a completely Hausdorff space. Fix a point \( y \) in \( Y - h(X) \), and let \( Z = \{ f \in C(h(X)) \mid f = g \mid h(X) \} \), where \( g \) is in \( C(Y) \), and \( g(y) = 0 \). For each \( f \in Z \) and number \( 0 < t \), define \( W(f, t) = \{ z \mid -t < f(z) < t \} \), and let \( \mathscr{F} \) be the filter on \( h(X) \) generated by the collection of all finite intersections of elements of \( \{ W(f, t) \mid f \in Z, 0 < t \} \). It is not difficult to see that \( \mathscr{F} \) is a completely regular filter on \( h(X) \): take \( \bigcap \{ W(f, t_i) \mid i = 1, \ldots, n \} \in \mathscr{F} \); for each integer \( i \), \( 1 \leq i \leq n \), choose a number

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(2) For definitions not given here, see [8] or [13].
Let $g \in C(X)$. Since $[g(X)]^-$ is compact, there is a number $t$ in the adherence of the filter base $g(\mathcal{F})$. Since $\mathcal{F}$ is a maximal completely regular filter, the inverse image under $g$ of each neighborhood of $t$ is an element of $\mathcal{F}$, i.e., $g(\mathcal{F})$ converges to $t$. Thus the function $g'$ defined by $g' = g$ on $X$, and $g'(\mathcal{F}) = \lim g(\mathcal{F})$ is an extension of $g$ in $C(Y)$.

It would be interesting to know when the Stone-Weierstrass theorem holds for the product of a collection of topological spaces. As far as the author knows, the following problem is unsolved: If $\{X_a \mid a \in A\}$ is a collection of topological spaces such that the Stone-Weierstrass theorem holds for each $X_a, a \in A$, does the Stone-Weierstrass theorem hold for $\prod \{X_a \mid a \in A\}$? The next theorem gives a partial answer to this question.

**Theorem 2.** If $X_1$ is a compact Hausdorff space, and if $X_2$ is a space for which the Stone-Weierstrass theorem holds, then the Stone-Weierstrass theorem holds for $X_1 \times X_2$. If $\{X_a \mid a \in A\}$ is a collection of spaces with the property that the Stone-Weierstrass theorem holds for their product $X = \prod \{X_a \mid a \in A\}$, then the Stone-Weierstrass theorem holds for each $X_a, a \in A$.

**Proof.** For the first part, assume $X_1$ and $X_2$ have the given properties. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be distinct points of $X_1 \times X_2$. Then $x_i \neq y_i$, $i = 1$, or $i = 2$, so there is a function $f$ in $C(X_i)$ such that $f(x_i) \neq f(y_i)$. Hence $f \circ pr_i$ is in $C(X_1 \times X_2)$ and satisfies $f \circ pr_i(x) \neq f \circ pr_i(y)$. Suppose that $\mathcal{G}$ is a completely regular filter on $X_1 \times X_2$. We shall show that $\mathcal{G} \neq \{C \mid C \in \mathcal{G}\}$. The filter generated by $pr_2(\mathcal{G})$ has a base consisting of open sets since $pr_2$ is an open mapping. Furthermore, it is completely regular, for take an open set $C \in pr_2(\mathcal{G})$, say $C = pr_2(B)$, where $B \in \mathcal{G}$. There exists a set $B' \subseteq B$ in $\mathcal{G}$ and a function $g$ in $C(X_1 \times X_2)$ which maps $X_1 \times X_2$ into $[0, 1]$, is 0 on $B'$, and is 1 on $(X_1 \times X_2) - B$. For each number $t \in (0, 1)$, let $U_t$ be
the open set \( pr_2(g^{-1}(0, t))) \). Then whenever \( 0 < s < t < 1 \), we have \( pr_2(B') \subseteq U_s \subseteq U_t \subseteq U \), for since \( X_t \) is compact, \( pr_2 \) is a closed mapping. Define \( h : X_2 \rightarrow [0, 1] \) by \( h(x) = 0 \) if \( x \in U_t \) for all \( t \in (0, 1) \), and \( h(x) = \sup \{ t : x \notin U_t \} \) otherwise. If \( 0 < a < 1 \), \( h^{-1}((0, a)) = \bigcup \{ U_t : |t| < a \} \) and \( h^{-1}((a, 1)) = \bigcup \{ X_t - U_t : |t| > a \} \) are open sets. Hence \( h \in C(X_2) \), \( h(pr_2(B')) = 0 \), and \( h(X_2 - C) = 1 \). By hypothesis the Stone-Weierstrass theorem holds for \( X_2 \); thus there is a point \( x \in \bigcap \{ pr_2(C) : C \in \mathcal{C} \} \).

Then \( \mathcal{D} = \{ C \cap (X_1 \times \{ x \}) : C \in \mathcal{C} \} \) is a filter base on \( X_1 \times X_2 \). \( pr_1 \{ \mathcal{D} \} \), a filter base on a compact space, has an adherent point \( y \). Therefore, for \( C \in \mathcal{C} \) and an arbitrary open set \( Y \subseteq X_1 \), \( y \in Y \) implies \( C \cap (X \times \{ x \}) = C \cap (X_1 \times \{ x \}) \cap (X_2 \times \{ x \}) \neq \emptyset \), i.e., \( (y, x) \) is an adherent point of \( \mathcal{C} \). Since the adherence of a completely regular filter is the same as the intersection of all the sets belonging to it, \( (y, x) \in \bigcap \{ C : C \in \mathcal{C} \} \).

For the second statement, we assume the Stone-Weierstrass theorem holds for \( X = \bigcap \{ X_a : a \in A \} \) and consider a factor space \( X_b \).

Let \( x_b \) and \( y_b \) be distinct points of \( X_b \). For each \( a \in A - \{ b \} \), fix \( z_a \in X_a \), and define \( x_a = y_a = z_a \). Set \( x = (x_a : a \in A) \) and \( y = (y_a : a \in A) \). \( x \neq y \), so there is a function \( f \in C(X) \) such that \( f(x) \neq f(y) \). Define \( Z_a = X_a, Z_0 = \{ z_0 \} \) if \( a \neq b \), and \( Z = \bigcap \{ Z_a : a \in A \} \). Set \( i = pr_0 | Z^{-1} \) and \( f' = f \circ i \). Then \( f' \in C(X_b) \), and \( f'(x_b) \neq f(y_b) \).

Let \( \mathcal{F} \) be a completely regular filter on \( X_b \), and consider the filter generated by \( pr_0^{-1} (\mathcal{F}) \). Take \( G = pr_0^{-1} (F) \), \( F \) a set in \( \mathcal{F} \). There is a function \( g \in C(X_b) \) and an open set \( F' \subseteq F \) in \( \mathcal{F} \) such that \( g(F') = 0 \), \( g(X_b - F) = 1 \), and \( g(X_b) \subseteq [0, 1] \). The open set \( G' = pr_0^{-1} (F') \subseteq G \) is in \( pr_0^{-1} (\mathcal{F}) \), and the function \( g \circ pr_0 \in C(X) \) satisfies \( g \circ pr_0 (G') = 0 \), \( g \circ pr_0 (X_b - G') = 1 \), and \( g \circ pr_0 (X_b) \subseteq [0, 1] \). Hence the filter generated by \( pr_0^{-1} (\mathcal{F}) \) is completely regular and has an adherent point \( x \). \( pr_0 (x) \in \bigcap \{ F : F \in \mathcal{F} \} \).

By similar reasoning one can prove:

**Theorem 3.** Let \( X \) be a space for which the Stone-Weierstrass theorem holds, and suppose that \( R \) is an equivalence relation on \( X \). Then the Stone-Weierstrass theorem holds for the quotient space \( X/R \) if and only if \( X/R \) is completely Hausdorff.

Although the Stone-Weierstrass theorem does not hold for every completely Hausdorff space, it can be shown that every completely Hausdorff space has an extension space for which it does hold.

Let \( X \) be a completely Hausdorff space, and let \( \mathcal{M} \) be the set of all maximal completely regular filters on \( X \) which have empty adherences. We shall denote by \( X' \) the topological space whose points are the elements of \( X \cup \mathcal{M} \) and whose open sets are generated by \( \{ V^* : V \text{ is open in } X \} \), where \( V^* = V \cup \{ \mathcal{F} \in \mathcal{M} : V \in \mathcal{F} \} \). We shall call \( X' \) the completely Hausdorff-completion of \( X \). In general, if \( T \) is an extension space of a topological space \( S \), the trace filters of \( T \) are the filters \( \mathcal{F}(t), t \in T - S \), where \( \mathcal{F}(t) \) is the filter on \( S \) generated by the traces \( U \cap S \) of the neighborhoods \( U \subseteq T \) of \( t \). In case \( S \) is completely Hausdorff and \( T = S' \), \( \mathcal{F}(\mathcal{F}) = \mathcal{F} \) if \( \mathcal{F} \in S' - S \), i.e., the trace filters of \( S' \) are the maximal completely regular filters \( \mathcal{F} \) on \( S \) such that \( \bigcap \{ G : G \in \mathcal{F} \} = \emptyset \).
If \( X \) and \( Z \) are topological spaces, we shall denote by \( C(X, Z) \) the set of all continuous mappings of \( X \) into \( Z \).

Suppose that \( Y \) is a completely Hausdorff space which is completely regular. Then \( Y' \) is the Stone-Čech compactification of \( Y \) (see [1]). It is well known that \( Y' \) has the following properties: if \( Z \) is a compact Hausdorff space, then each function in \( C(Y, Z) \) has a unique extension in \( C(Y', Z) \); \( Y' \) is locally connected if and only if \( Y \) is locally connected and pseudocompact [11]; \( Y' \) is connected if and only if \( Y \) is connected; \( C(Y) \) and \( C(Y') \) are isomorphic, and if \( R \) is the set of all real numbers, \( C(Y') \) and \( C(Y, R) \) are isomorphic only if \( Y \) is pseudocompact. The next theorem shows that almost all of these properties hold for \( Y' \) if \( Y \) is completely Hausdorff, but not necessarily completely regular.

**Theorem 4.** Let \( X \) be a completely Hausdorff space. The completely Hausdorff-completion \( X' \) of \( X \) has the following properties.

(i) If \( Z \) is a compact Hausdorff space, then each function in \( C(X, Z) \) has a unique extension in \( C(X', Z) \).

(ii) The Stone-Weierstrass theorem holds for \( X' \).

(iii) \( X' \) is locally connected if and only if \( X \) is locally connected and each trace filter has a base consisting of connected open sets.

(iv) \( X' \) is not locally connected unless \( X \) is locally connected and pseudocompact.

(v) \( X' \) is connected if and only if \( X \) is connected.

(vi) \( C(X') \) and \( C(X) \) are isomorphic, and if \( R \) is the set of all real numbers, \( C(X') \) and \( C(X, R) \) are isomorphic only if \( X \) is pseudocompact.

(vii) If \((Y, h)\) is an extension space of \( X \) such that \( Y \) is completely Hausdorff-complete, each element of \( C(h(X)) \) has an extension in \( C(Y) \), and each trace filter of \( Y \) is a completely regular filter on \( h(X) \), then there is a one-to-one function \( g \in C(Y, X') \) for which \( g(Y) = X' \) and \( g \circ h \) is the identity on \( X \).

**Proof.** (i) Let \( f \in C(X, Z) \). By almost the same argument as one given in the proof of Theorem 1, one can show that \( f(\mathcal{G}) \) is a convergent filter base if \( \mathcal{G} \) is a maximal completely regular filter on \( X \). Define \( f' \) by \( f'(x) = f(x) \) if \( x \in X \), and \( f'(\mathcal{H}) = \lim f(\mathcal{H}) \) if \( \mathcal{H} \in X' - X \). Take \( \mathcal{F} \in X' - X \), and choose open sets \( O \) and \( P \) such that \( f'(\mathcal{F}) \in O \subseteq \overline{O} \subseteq P \). As \( f'(\mathcal{F}) = \lim f(\mathcal{F}) \), there is a set \( V \subseteq \mathcal{F} \) open in \( X \) such that \( f(V) \subseteq O \). Necessarily the open neighborhood \( V^* \) of \( \mathcal{F} \) has the property that \( f'(V^*) \subseteq P \), for suppose there is a filter \( \mathcal{G} \in V^* \) such that \( \lim f(\mathcal{G}) \notin O \) : there exists an open set \( f'(\mathcal{G}) \) in \( W \), with \( O \cap W = \emptyset \); \( f'(\mathcal{G}) \) converges to \( f'(\mathcal{F}) \); also \( V \subseteq \mathcal{F} \) since \( \mathcal{G} \in V^* \); but then \( \emptyset = V \cap f^{-1}(W) \subseteq \mathcal{G} \), which is impossible. Therefore, \( f' \) is continuous at \( \mathcal{F} \). The proof that \( f' \) is continuous at an arbitrary point of \( X \) is similar. Thus \( f' \in C(X', Z) \). Clearly \( f'|X = f \). Since \( X \) is dense in \( X' \), and the space \( Z \) is Hausdorff, \( f' \) is unique.

(ii) Since \( X \) is completely Hausdorff, and since by (i) each function in \( C(X) \) has an extension in \( C(X') \), \( C(X') \) separates the points of \( X \). If \( x \in X \), and if \( \mathcal{F} \in X' - X \), then because \( \bigcap \{ F \mid F \in \mathcal{F} \} = \emptyset \), we can choose a function \( f \in C(X) \) such that
$f(x) = 1$, and $f(F) = 0$, some $F \in \mathcal{F}$; the extension $f'$ of $f$ in $C(X')$ has the property that $f'(x) = 1 \neq 0 = f'(\mathcal{F})$. $C(X')$ also separates the points of $X' - X$, for suppose that $\mathcal{G}, \mathcal{H} \in X' - X$, $\mathcal{G} \neq \mathcal{H}$: as $\mathcal{I}$ and $\mathcal{H}$ are distinct maximal completely regular filters on $X$, there exist sets $G \in \mathcal{I}$ and $H \in \mathcal{H}$ such that $G \cap H = \varnothing$; furthermore, $G$ and $H$ can be chosen so that there is a function $g \in C(X)$ such that $g(G) = 0$, and $g(H) = 1$; then the extension $g'$ of $g$ in $C(X')$ satisfies $g'(\mathcal{G}) = 0 \neq 1 = g'(\mathcal{H})$. Hence $X'$ is completely Hausdorff.

Let $F$ be a completely regular filter on $X'$. We wish to show that $\bigcap \{F \mid F \in \mathcal{F}\} \neq \varnothing$. If there is a point $x \in X$ such that $x \in \bigcap \{F \mid F \in \mathcal{F}\}$, we are done. Suppose that $\bigcap \{F \cap X \mid F \in \mathcal{F}\} = \varnothing$, and let $\mathcal{G}$ be the filter on $X$ generated by $\{F \cap X \mid F \in \mathcal{F}\}$. Then $\mathcal{G}$ is completely regular, so there is a maximal completely regular filter $\mathcal{H}$ on $X$ such that $\mathcal{G} \subset \mathcal{H}$, and $\bigcap \{H \mid H \in \mathcal{H}\} = \varnothing$. Then $\mathcal{H} \in X' - X$, and since $\mathcal{G} \subset \mathcal{H}$, $F \cap X \cap H \neq \varnothing$, for every $F \in \mathcal{F}$ and $H \in \mathcal{H}$. Therefore, $F \cap H \neq \varnothing$, for every $F \in \mathcal{F}$ and $H \in \mathcal{H}$. This implies $\mathcal{H}$ is in the adherence of $\mathcal{F}$, for $\{V^* \mid V \text{ is open in } X\}$ is closed under the taking of finite intersections and so actually is a base for the topology of $X'$. As $\mathcal{F}$ is completely regular, $\varnothing \neq \bigcap \{F \mid F \in \mathcal{F}\}$.

(iii) A filter $\mathcal{G}$ on a topological space $E$ is called open provided that it has a base consisting of open sets. An open filter $\mathcal{G}$ is called connected provided that whenever $O \cup P \in \mathcal{G}$, $O$ and $P$ disjoint open sets, either $O \in \mathcal{G}$ or $P \in \mathcal{G}$. In [4] it is shown that a maximal completely regular filter on a space $E$ is connected. The principal result of [2] is: Let $F$ be an extension space of $E$ each of whose trace filters is connected. Then $F$ is locally connected if and only if $E$ is locally connected and each trace filter has a base consisting of connected open sets.

(iv) By (iii) $X'$ is not locally connected unless $X$ is locally connected. Suppose that $X$ is not pseudocompact, and let $Y$ be the completely regular Hausdorff space which has the same points as $X$, and whose topology is determined by $C(X)$, or, equivalently, $C(X, R)$, where $R$ is the set of all real numbers. Then $C(X, R) = C(Y, R)$, so $Y$ is not pseudocompact.

In [10] Glicksberg proved that if $Z$ is a completely regular space, then the following are equivalent:

(a) $Z$ is pseudocompact.

(b) For every sequence $\{V_n\}$ of nonempty open sets with disjoint closures, $\{V_n\}$ has a cluster point, that is, a point $x$ such that for every $m$ and neighborhood $V$ of $x$ there exists an $n \geq m$ for which $V \cap V_n \neq \varnothing$.

In [11] it was noted that (b) is equivalent to the condition:

(c) For every sequence $\{V_n\}$ of nonempty open sets with disjoint closures, $\bigcup \{V_n\}$ is not closed if $\{V_n\}$ is infinite. Therefore, since the space $Y$ is completely regular, but not pseudocompact, it fails to satisfy (c).

Altering slightly Banaschewski's proof in [2] (that the Stone-Čech compactification of a completely regular Hausdorff space $Z$ is not locally connected unless $Z$ satisfies (c)), we shall show that $X'$ cannot be locally connected. A corollary to Banaschewski's method of proof is: If $Z$ is a completely regular space which does
not satisfy (c), then there is a sequence of nonempty open sets $O(i, k) \subseteq Z$ ($i = 1, 2, \ldots; k = 1, 2, \ldots$) with the following properties: for all $i, j, k, \ell$, $i \neq j$ implies $O(i, k) \cap O(j, \ell) = \emptyset$; for all $i, j, k, \ell$, $j \geq k$ implies $O(i, j) \subseteq O(i, k)$; the filter $\mathcal{F}$ generated by the sets $\bigcup \{O(s, i) \mid s \geq i\}$, $i = 1, 2, \ldots$ is a completely regular filter on $Z$ with empty adherence.

As $Y$ is completely regular and does not satisfy (c), we may choose a filter $\mathcal{G}$ on $Y$ and sets $O(i, k) \subseteq Y$ as above. Since $X$ and $Y$ have the same points, and since the topology of $Y$ is weaker than the topology of $X$, $\mathcal{G}$ is a completely regular filter on $X$. If $\bigcap \{C \mid C \in \mathcal{G}\} = \emptyset$, so there is a trace filter $\mathcal{F}$ of $X'$ such that $\mathcal{G} \subseteq \mathcal{F}$. By an argument identical to one given in [2], since $X$ and $Y$ have the same points, and since the topology of $Y$ is weaker than the topology of $X$, $\mathcal{G}$ is a completely regular filter on $X$. If $\mathcal{F} \subseteq \bigcup \{O(s, i) \mid s \geq i\}$, some $i$, then necessarily $\mathcal{F} \subseteq O(s, i)$, some $s \geq i$, so

$$F \cap \left[ \bigcup \{O(t, s+1) \mid t \geq s+1\} \right] = \emptyset,$$

from which it follows that $F \notin \mathcal{F}$. By (iii) $X'$ cannot be locally connected.

(v) is a consequence of (i) and the fact that $X'$ is an extension space of $X$.

The known proof (see [9]) that (vi) holds for a completely regular Hausdorff space $X$ also shows that (vi) holds for a completely Hausdorff space $X$ which is not necessarily regular.

(vii) If $y \in Y - h(X)$, we shall denote by $\mathcal{Y}(y)$ and $\mathcal{W}(y)$ the following filters:

$\mathcal{Y}(y)$ is the filter on $Y$ generated by $\{U \mid U$ is open in $Y, y \in U\}$, and for some $f \in C(Y)$, $f(y) = 0$, $f(Y - U) = 1$, and $f(Y) \subseteq \{0, 1\}$; $\mathcal{W}(y)$ is the filter on $h(X)$ generated by $\{U \cap h(X) \mid U \in \mathcal{Y}(y)\}$. If $V$ is open in $h(X)$, $\mathcal{V}$ is defined to be $V \cap \{y \in Y - h(X) \mid V \in \mathcal{Y}(y)\}$. We shall show that the following hold: (a) for each $y \in Y - h(X)$, $\mathcal{Y}(y)$ is a maximal completely regular filter on $Y$; (b) for each $y \in Y - h(X)$, $\mathcal{W}(y)$ is a maximal completely regular filter on $h(X)$; (c) the function $e$ defined by $e(y) = h^{-1}(\mathcal{W}(y))$ is a one-to-one mapping of $Y - h(X)$ onto the set of all trace filters of $X'$; (d) if $V$ is open in $h(X)$, then $\mathcal{V}$ is open in $Y$; (e) if $V$ is open in $X$, then for each $y \in [h(V)]^* - h(V)$, $e(y) \in \mathcal{V}^*$; (f) the function $g$ defined by $g(h(x)) = x$ if $x \in X$, and $g(y) = e(y)$ if $y \in Y - h(X)$ is a one-to-one continuous mapping of $Y$ onto $X'$.

(a) Let $y \in Y - h(X)$, and suppose that $\mathcal{F}$ is a completely regular filter on $Y$ such that $\mathcal{Y}(y) \subseteq \mathcal{F}$. Since $Y$ is completely Hausdorff-compact, $\bigcap \{F \mid F \in \mathcal{F}\} \neq \emptyset$. Since $Y$ is completely Hausdorff, $\{y\} = \bigcap \{V \mid V \in \mathcal{Y}(y)\}$. As $\bigcap \{F \mid F \in \mathcal{F}\} \subseteq \bigcap \{V \mid V \in \mathcal{Y}(y)\}$, $y \in \bigcap \{F \mid F \in \mathcal{F}\}$, so $\mathcal{F} \subseteq \mathcal{Y}(y)$.

(b) Let $y \in Y - h(X)$, and suppose that $\mathcal{G}$ is a completely regular filter on $h(X)$ such that $\mathcal{W}(y) \subseteq \mathcal{G}$. Take $G \in \mathcal{G}$, and choose a set $G' \subseteq G$ in $\mathcal{G}$ and a function $f \in C(h(X))$ such that $f(h(X)) \subseteq \{0, 1\}$, $f(G') = 0$, and $f(h(X) - G) = 1$. Let $g \in C(Y)$, $g|h(X) = f$. Since $\mathcal{W}(y) \subseteq \mathcal{G}$, $G' \cap U \neq \emptyset$ for all $U \in \mathcal{Y}(y)$, so if $0 < t$, $g^{-1}([0, t)) \cap U \neq \emptyset$ for all $U \in \mathcal{Y}(y)$. As $\mathcal{Y}(y)$ is a maximal completely regular filter on $Y$, $g^{-1}([0, t)) \subseteq \mathcal{Y}(y)$ for each $t > 0$. Therefore, $G \in \mathcal{W}(y)$, for $G = f^{-1}([0, 1]) = h(X) \cap g^{-1}([0, 1]) \in \mathcal{W}(y)$, so $\mathcal{G} \subseteq \mathcal{W}(y)$. 


(c) Let \( y \in Y - h(X) \). By (b) \( \mathcal{W}(y) \) is a maximal completely regular filter on \( h(X) \). Since \( Y \) is completely Hausdorff, \( \bigcap \{ W \mid W \in \mathcal{W}(y) \} = \emptyset \). As \( h^{-1} \) is a homeomorphism of \( h(X) \) onto \( X \), \( h^{-1}(\mathcal{W}(y)) \) is a maximal completely regular filter on \( X \), and \( \emptyset = \bigcap \{ W \mid W \in h^{-1}(\mathcal{W}(y)) \} \). Hence \( e(y) \) is a trace filter of \( X' \).

Take \( y, z \in Y - h(X), y \neq z \). Since \( Y \) is completely Hausdorff, we may choose a function \( f \in C(Y) \) such that \( f(y) \neq f(z) \). For each \( t > 0 \), let \( O(t) = \{ s \in Y \mid f(y) - t < f(s) < f(y) + t \} \). Then \( O(t) \in \mathcal{V}(y) \), each \( t > 0 \). Take \( u > 0 \) so that \( z \notin O(u) \), and choose a number \( 0 < v < u \) and a function \( g \in C(Y) \) such that \( g(Y) = [0, 1] \), \( g(O(v)) = 0 \), and \( g(Y - O(u)) = 1 \). Define \( j \) by \( j(s) = 1 - g(s) \) if \( s \in Y \). Then \( j \in C(Y), j(Y) \subset [0, 1], j(z) = 0 \), \( j(Y - (Y - O(v))) = 1 \), and \( z \in Y - O(v) \), so \( Y - O(v) \in \mathcal{V}(z) \).

\( O(v) \cap h(X) \in \mathcal{W}(y), h(X) - O(v) \in \mathcal{W}(z), \) and \( O(v) \cap h(X) \cap (h(X) - O(v)) = \emptyset \), so \( \mathcal{W}(y) \neq \mathcal{W}(z) \). As \( h^{-1} \) is one-to-one, \( e(y) = h^{-1}(\mathcal{W}(y)) \neq h^{-1}(\mathcal{W}(z)) = e(z) \).

Let \( \mathcal{F} \) be a trace filter of \( X' \). We wish to show that there is a point \( y \in Y - h(X) \) for which \( e(y) = \mathcal{F} \). Since \( h \) is a homeomorphism, \( h(\mathcal{F}) \) is a maximal completely regular filter on \( h(X) \), and \( \emptyset = \bigcap \{ F \mid F \in h(\mathcal{F}) \} \). Let \( Z = \{ f \in C(h(X)) \mid f(h(X)) \subset [0, 1] \} \), and for some \( G, G' \in h(\mathcal{F}), G' \subset G, f(G') = 0 \), and \( f(h(X) - G) = 1 \). For each \( f \in Z \), denote by \( f' \) the extension of \( f \) in \( C(Y) \), and let \( Z' = \{ f' \mid f \in Z \} \). For each \( f' \in Z' \) and \( t > 0 \), let \( V(f', t) = f'^{-1}([0, t]) \), and let \( \mathcal{G} \) be the filter on \( Y \) generated by the set of all finite intersections of elements of \( \{ V(f', t) \mid f \in Z' \) and \( t > 0 \)\}. By an argument similar to one given in the proof of Theorem 1, \( \mathcal{G} \) can be shown to be a completely regular filter on \( Y \). Let \( \mathcal{H} \) be a maximal completely regular filter on \( Y \) such that \( \mathcal{G} \subset \mathcal{H} \). Since \( Y \) is completely Hausdorff-complete, \( \bigcap \{ H \mid H \in \mathcal{H} \} \neq \emptyset \). If \( G \in h(\mathcal{H}), G \supseteq f^{-1}([0, \frac{1}{2}]) \), some \( f \in Z \), then \( x \in h(X) \cap \bigcap \{ H \mid H \in \mathcal{H} \} \supseteq \bigcap \{ G \mid G \in \mathcal{G} \} \cap h(X) \subset \bigcap \{ G \mid G \in h(\mathcal{F}) \} = \emptyset \). Thus there is a point \( y \in Y - h(X) \cap \bigcap \{ H \mid H \in \mathcal{H} \} \). The maximality of \( \mathcal{H} \) implies that \( \mathcal{H} = \mathcal{W}(y) \). Since \( \mathcal{F} \) is maximal, \( \mathcal{W}(y) \) is maximal, and \( \mathcal{F} = \mathcal{W}(y) \). The point \( y \) then has the property that \( e(y) = h^{-1}(\mathcal{W}(y)) = h^{-1}(\mathcal{F}) \).

(d) If \( y \in Y - h(X) \), \( \mathcal{W}(y) \) and the trace filter \( \mathcal{F}(y) \) of \( Y \) are identical: by definition \( \mathcal{W}(y) \subset \mathcal{F}(y) \); by hypothesis \( \mathcal{F}(y) \) is a completely regular filter on \( h(X) \); (b) then implies that \( \mathcal{W}(y) = \mathcal{F}(y) \).

In general, if \( S \) is a topological space, if \( T \) is an extension space of \( S \), and if \( V \) is open in \( S \), then \( V \cup \{ t \in T - S \mid V \in \mathcal{F}(t) \} \) is open in \( T \) [2].

(e) If \( V \) is open in \( X \), then for each \( y \in [h(V)]^* \), \( h(V) \in \mathcal{W}(y) \), so \( V = h^{-1}(h(V)) \in h^{-1}(\mathcal{W}(y)) = e(y) \), and hence \( e(y) \in V^* \).

(f) Since \( h^{-1} \) is a one-to-one mapping of \( h(X) \) onto \( X \), (c) implies that \( g \) is a one-to-one mapping of \( Y \) onto \( X' \). Let \( y \in Y - h(X) \), and suppose that \( W \) is open in \( X' \), with \( g(y) \in W \). Then there is a set \( V \) open in \( X \) such that \( g(y) \in V^* \subset W \). By (d) \([h(V)]^* \) is open in \( Y \), and \( y \in [h(V)]^* \), for \( g(y) \in V^* \) implies \( V \) in \( g(y) \) so that \( h(V) \in \mathcal{W}(y) \). As a consequence of (e) and the definition of \( g, g([h(V)]^*)^* \subset V^* \subset W \).
Thus $g$ is continuous at $y$. The proof that $g$ is continuous at each point of $h(X)$ is similar.

We conclude the proof of Theorem 4 with the remark that proofs of (i) and (ii) different from those given here can be obtained which depend on the properties of the Stone-Čech compactification $Y'$ of the completely regular Hausdorff space $Y$ whose points are those of $X$ and whose topology is determined by $C(X)$.

The author does not know if the converse of (iv) in Theorem 4 holds, but, as the following example shows, if $X$ is a locally connected, pseudocompact, completely Hausdorff space, and if $Y$ is a completely Hausdorff-complete extension space of $X$ such that each function in $C(X)$ has an extension in $C(Y)$, then $Y$ is not necessarily locally connected.

**Example 5.** The example given here is a slight modification of an unpublished one due to L. B. Treybig of a countably compact space $(Y, T)$, where $Y$ is “the long interval”, and $T$ is a topology which is stronger than the usual order topology put on $Y$.

**Description of the space.** Let $\Omega$ be the first ordinal with an uncountable number of predecessors, let $\Omega'$ be the set of all ordinals less than $\Omega$, and for each $x \in \Omega'$, let $I(x)$ be $\{x\} \times$ an open interval in the real line. Set $X = \Omega' \cup \{I(x) | x \in \Omega'\}$, and for $x, y \in X$, define $x < y$ if (1) $x, y \in \Omega'$, and $x < y$ in $\Omega'$, or (2) $x \in \Omega'$, $y \in I(s)$, and $x \leq s$ in $\Omega'$, or (3) $x \in I(r)$, $y \in \Omega'$, and $r < y$ in $\Omega'$, or (4) $x \in I(r)$, $y \in I(s)$, and $r < s$ in $\Omega'$, or (5) $x \in I(r)$, $y \in I(s)$, and $x < y$ in $I(s)$. Let $\Omega$ be the order topology on $X$. Let $Y = X \cup \{\Omega\}$, define $x < \Omega$ if $x \in X$, and let $\mathcal{R}$ be the order topology on $Y$. If $x \in X$, $O(x)$ will denote $\{y \in Y | x < y\}$. $\mathcal{F}$ will denote the topology on $Y$ which is generated by $\{B | B \in \mathcal{R}, \text{or for some } x \in X, B = O(x) - \Omega'\}$.

The spaces $(X, \Omega)$ and $(Y, T)$ have the following properties.

(i) $(X, \Omega)$ is a countably compact (hence pseudocompact), locally connected, completely regular Hausdorff space.

(ii) $(Y, T)$ is an extension space of $(X, \Omega)$.

(iii) The Stone-Weierstrass theorem holds for $(Y, T)$.

(iv) Every continuous real valued function on $(X, \Omega)$ has an extension in $C((Y, T))$.

(v) $(Y, T)$ is not locally connected.

**Proof.** (i) is well known [13]. (ii) holds, for $\Omega'$ is closed in $(X, \Omega)$, and $X \cap (O(x) - \Omega') \neq \emptyset$, each $x \in X$. As a consequence of the fact that each function in $C(\Omega')$ is eventually constant [9], $(Y, \mathcal{R})$ is the Stone-Čech compactification of $(X, \Omega)$, so since $\mathcal{R} \subseteq T$, and $(X, \Omega)$ is pseudocompact, (iv) follows. (v) is obvious. We prove (iii).

As $(Y, \mathcal{R})$ is compact Hausdorff, and $\mathcal{R} \subseteq T$, $(Y, T)$ is completely Hausdorff. To show that every completely regular filter on $(Y, T)$ has nonempty adherence, it suffices to show that every open filter on $(Y, T)$ has nonempty adherence.

Suppose that $\mathcal{F}$ is an open filter on $(Y, T)$ such that $\Omega \notin \bigcap\{F | F \in \mathcal{F}\}$. Then there is a set $F \in \mathcal{F} \cap T$ with the property that for some $x \in X$, $F \cap (O(x) - \Omega') = \emptyset$. Therefore, $F \cap ((O(x) - \Omega'))^c = \emptyset$. Clearly, each point of $\Omega'$ is a limit point.
of \( O(x) - \Omega' \), so \( \Omega' \subseteq [O(x) - \Omega']^- \). Hence \( F \cap O(x) \subseteq F \cap [(O(x) - \Omega')]^- = \emptyset \). As \( \emptyset \notin \mathcal{F} \), and \( F \cap G \in \mathcal{F} \), each \( G \in \mathcal{F} \), \( G \cap (Y - O(x)) \neq \emptyset \), each \( G \in \mathcal{F} \). Since \( Y - O(x) \) is compact, it follows that \( \bigcap \{ G \cap (Y - O(x)) | G \in \mathcal{F} \} \neq \emptyset \).

In [8] it is noted that a Hausdorff space is absolutely closed if and only if every open filter on it has nonempty adherence. As a consequence of the proof of (iii) and the fact that \(( Y, \mathcal{F} )\) is a one-point extension space of a countably compact space, one obtains the

**Corollary 6.** An absolutely closed, countably compact, completely Hausdorff space is not necessarily minimal Hausdorff.

As noted in [3], there does not exist a noncompact, completely regular Hausdorff space for which the Stone-Weierstrass theorem holds. It can be shown, however, that there exists a noncompact regular space (as used in this paper, the condition of regularity includes \( T_1 \) separation) for which the Stone-Weierstrass theorem holds.

An open filter is called regular if it has a base consisting of closed sets. A regular space is regular closed provided that it is closed in every regular space in which it can be embedded.

In [12] Herrlich proved that a regular space is regular closed if and only if each regular filter on it has nonempty adherence. He also showed that there is a regular space (a subspace of the minimal regular noncompact space constructed in [7]), which we shall denote by \(( S, \mathcal{W} )\), with the property that \(( S, \mathcal{W} )\) is regular closed, but not minimal regular. In particular, he showed that there exists a topology \( \mathcal{V} \subset \mathcal{W} \), \( \mathcal{V} \neq \mathcal{W} \), such that \(( S, \mathcal{V} )\) is a compact Hausdorff space.

As a consequence of the fact that every completely regular filter is a regular filter, every completely regular filter on \(( S, \mathcal{W} )\) has nonempty adherence. In addition, \(( S, \mathcal{W} )\) is completely Hausdorff, for \(( S, \mathcal{V} )\) is completely Hausdorff, and \( \mathcal{V} \subset \mathcal{W} \). \(( S, \mathcal{W} )\) is thus a noncompact regular closed space for which the Stone-Weierstrass theorem holds.

Two questions which one might consider are the following:
(i) If the Stone-Weierstrass theorem holds for a regular space \( R \), is \( R \) necessarily regular closed?
(ii) Does there exist a regular space \( R \) such that the Stone-Weierstrass theorem holds for \( R \), but \( R \) is not second category?

As Example 8 will show, not every space for which the Stone-Weierstrass theorem holds is second category. If, however, the answer to (i) is yes, then as a consequence of Theorem 7, the answer to (ii) must be no.

**Theorem 7.** Every regular closed space is second category.

**Proof.** A regular filter base is a filter base consisting of open sets which is equivalent to a filter base consisting of closed sets. In [7] it is shown that on a minimal regular space \(( \alpha \) every regular filter base which has a unique adherent point is convergent, and \( \beta \) every regular filter base has an adherent point. The proof given
in [5] that a minimal regular space is second category depends only on the fact that
\((\beta)\) holds on a minimal regular space. Clearly, \((\beta)\) holds on a topological space \(X\) if
and only if every regular filter on \(X\) has an adherent point, and, as noted above,
Herrlich has proved that every regular filter on a regular closed space has an
adherent point.

**Example 8.** Let \(X = [0, 1]\), let \(\mathcal{V}\) be the usual topology on \(X\), let \(Q\) be the set
of all rational numbers in \(X\), and define \(\mathcal{W}\) to be the weakest topology on \(X\) such
that \(\mathcal{V} \subseteq \mathcal{W}\) and \(Q \in \mathcal{W}\).

As noted in [6], \((X, \mathcal{W})\) is an absolutely closed space which is not minimal
Hausdorff. In addition, \(\mathcal{V} \subseteq \mathcal{W}\), so \((X, \mathcal{W})\) is completely Hausdorff, and the Stone-
Weierstrass theorem holds for \((X, \mathcal{W})\).

For each \(q \in Q\), the set \(F(q) = \{q\} \cup (X - Q)\) is a closed, nowhere dense set in
\((X, \mathcal{V})\), and \(X = \bigcup \{F(q) \mid q \in Q\}\), so \((X, \mathcal{W})\) is not second category.

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