THE SPECTRA OF OPERATORS HAVING RESOLVENTS OF FIRST-ORDER GROWTH

BY

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1. Introduction. Throughout this paper, $\mathcal{H}$ will denote an infinite-dimensional Hilbert space of vectors $x, y, \ldots$, with inner product $\langle x, y \rangle$, and $T$ will denote a bounded operator on $\mathcal{H}$ with spectrum $\text{sp}(T)$. As usual, let $\|T\| = \sup \|Tx\|$ where $\|x\| = 1$. If $\lambda$ does not belong to $\text{sp}(T)$, put $R_\lambda = (T - \lambda I)^{-1}$, and let $d(\lambda)$ denote the distance from $\lambda$ to $\text{sp}(T)$, thus,

\begin{equation}
    d(\lambda) = \inf |\lambda - \mu|, \quad \text{where } \mu \in \text{sp}(T).
\end{equation}

There will be studied certain properties of points $\lambda_0$ in the boundary of $\text{sp}(T)$ when the resolvent $R_\lambda$ satisfies the growth condition

\begin{equation}
    d(\lambda)\|R_\lambda\| \to 1 \quad \text{as } \lambda \to \mu
\end{equation}

for all $\mu$ in the boundary of $\text{sp}(T)$ and in some neighborhood of $\lambda_0$.

It is well known and easy to show that for any $T$ one has $d(\lambda)\|R_\lambda\| \geq 1$ for all $\lambda \notin \text{sp}(T)$ and that $d(\lambda)\|R_\lambda\| \to 1$ as $|\lambda| \to \infty$. The extreme possibility

\begin{equation}
    d(\lambda)\|R_\lambda\| = 1 \quad \text{for } \lambda \notin \text{sp}(T),
\end{equation}

which of course implies (1.2), certainly holds for normal operators as well as for others which are "nearly" normal; for instance, it is satisfied by seminormal operators $T$, so that

\begin{equation}
    T^*T - TT^* \text{ is semidefinite.}
\end{equation}

See Stampfli [11], also the references given there to Donoghue and Nieminen.

Recall that a sequence $\{x_n\}$ of vectors is said to converge weakly to a limit $x$ as $n \to \infty$ (notation $w: x_n \to x$) if $\langle x_n, y \rangle \to \langle x, y \rangle$ as $n \to \infty$ for all $y$ in $\mathcal{H}$. It will be convenient to define for any bounded operator $T$ the sets $A(T)$ and $B(T)$ by

\begin{equation}
    A(T) = \{ \lambda: T_\lambda x_n \to 0, T_\lambda^* y_n \to 0 \text{ for some pair of sequences } \{x_n\}, \{y_n\} \text{ satisfying } \|x_n\| = \|y_n\| = 1 \text{ and } w: x_n \to 0, w: y_n \to 0 \text{ as } n \to \infty \}
\end{equation}

and

\begin{equation}
    B(T) = \{ \lambda: T_\lambda x_n \to 0, T_\lambda^* x_n \to 0 \text{ for some sequence } \{x_n\} \text{ satisfying } \|x_n\| = 1 \text{ and } w: x_n \to 0 \text{ as } n \to \infty \}.
\end{equation}

(Here and in the sequel, $T_\lambda = T - \lambda I$.)

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Let \( W(T) \) be the set of points of \( \text{sp} \ (T) \) which are invariant under all completely continuous perturbations of \( T \), thus, \( W(T) = \bigcap_c \text{sp} \ (T + C) \), where \( C \) is completely continuous. (For a characterization of \( W(T) \), see Schechter [9].) It is clear that

\[
B(T) \subset A(T) \subset W(T).
\]

For any \( T \), let \( E(T) \) denote the set of limit (cluster) points of \( \text{sp} \ (T) \) together with all points of the point spectrum of \( T \) of infinite multiplicity. Then, in case \( T \) is normal, \( E(T) \) constitutes the essential spectrum of \( T \) and, according to Weyl's theorem (Weyl [12]), \( E(T) = B(T) \).

It is easy to see that if \( T \) is normal, \( E(T) = W(T) \). It was shown by Coburn [2], using properties of \( W(T) \) given in Schechter [9], that this last equation remains valid even for seminormal operators. The corresponding assertion, even for the direct sum of two seminormal operators, is false however. In fact, if \( V \) denotes the isometric operator on the sequential Hilbert space \( \mathfrak{H} \) of vectors \( x = (a_1, a_2, \ldots) \) defined by \( V: (a_1, a_2, \ldots) \rightarrow (0, a_1, a_2, \ldots) \), then \( V \) and \( V^* \) are seminormal, each has as its spectrum the unit disk \( |\lambda| \leq 1 \) on \( \mathfrak{H} \) and hence so also does \( T_0 = V \oplus V^* \). Then, \( \text{sp} \ (T_0) = \mathfrak{H} \oplus \mathbb{R} \), and, in particular, \( E(T_0) = \text{sp} \ (T_0) = \{ \lambda : |\lambda| \leq 1 \} \). Although \( T_0 \) is not seminormal, it does satisfy (1.3), since \( V \) and \( V^* \) do separately. But there exist completely continuous operators \( C \), even with \( C \) arbitrarily small, on \( \mathfrak{H} = \mathbb{R} \oplus \mathbb{R} \), such that \( \text{sp} \ (T + C) \) is the boundary, \( |\lambda| = 1 \), of \( \text{sp} \ (T_0) \); see Putnam [7] (also Halmos [5, Solutions 85, 144]). Thus \( W(T_0) \) is a subset of the set \( |\lambda| = 1 \). It follows from Theorem 2 below however that the set \( |\lambda| = 1 \) is contained in \( B(T_0) \) and, since \( B(T_0) \subset W(T_0) \), \( W(T_0) \) is precisely the set \( |\lambda| = 1 \).

There will be proved the following theorems:

**Theorem 1.** Let \( T \) be a bounded operator and let \( \lambda_0 \) be a nonisolated point of the boundary of \( \text{sp} \ (T) \). Then \( \lambda_0 \) belongs to the set \( A(T) \) of (1.5).

**Theorem 2.** Let \( T \) be a bounded operator and let \( \lambda_0 \) be a nonisolated point of the boundary of \( \text{sp} \ (T) \) for which (1.2) holds. Then \( \lambda_0 \) belongs to the set \( B(T) \) of (1.6).

It follows from Theorem 1 and (1.7) that \( W(T) \) always contains the set of nonisolated boundary points of \( \text{sp} \ (T) \). That \( W(T) \) may coincide with this latter set, even if (1.3) holds, is seen from the example mentioned above.

In §4, some other spectral implications of Theorems 1 and 2 will be derived.

**Remarks.** The author is indebted to M. Schechter for the present formulation of Theorem 1 and its proof below. The author's original version involved an added hypothesis on the growth of the resolvent as well as a considerably longer proof.

**2. Proof of Theorem 1.** Since \( \lambda_0 \) is a nonisolated boundary point of \( \text{sp} \ (T) \) if and only if \( \lambda_0 \) is a nonisolated boundary point of \( \text{sp} \ (T^*) \), it is clearly sufficient to prove the existence of the sequence \( \{x_n\} \), for \( \lambda = \lambda_0 \), in (1.5). Also, according to Wolf [13, p. 215], the existence of such a sequence is equivalent to the statement that either \( \Re(T_{\lambda_0}) \) is not closed or that the dimension \( \alpha(T_{\lambda_0}) \) of the null space of \( T_{\lambda_0} \) is
infinite. The proof will be completed then by showing that the assumptions that $\Re(T_{\lambda_0})$ be closed and that $\alpha(T_{\lambda_0})$ be finite lead to a contradiction.

To this end, note that $\Re(T_\lambda)$ is closed and $\alpha(T_\lambda)$ is finite for all $\lambda$ sufficiently close to $\lambda_0$; see Wolf [13, p. 216]. Also, the range of a bounded operator is closed if and only if the range of its adjoint is also closed; cf. Goldberg [4, p. 95]. Since $\alpha(T_{\lambda_0})$ and $\alpha(T_{\lambda_0}^*)$ are not both infinite, so that $T_{\lambda_0}$ has an index, then $\alpha(T_\lambda)$ and $\alpha(T_\lambda^*)$ are constant on a punctured neighborhood of $\lambda_0$. This follows from results of Gohberg and Kreîn [3]; see Goldberg [4, p. 114], wherein can be found a generalization. Since $\lambda_0$ is a boundary point of the resolvent set of $T$ it follows that for some $\delta > 0$, $\alpha(T_\lambda) = \alpha(T_\lambda^*) = 0$, and both $\Re(T_\lambda)$ and $\Re(T_\lambda^*)$ are closed for $0 < |\lambda - \lambda_0| < \delta$. Thus the set $\{\lambda : 0 < |\lambda - \lambda_0| < \delta\}$ belongs to the resolvent set of $T$, a contradiction.

3. Proof of Theorem 2. For use below it will be convenient to have the following

**Lemma.** Let $\lambda_n \to \lambda_0$ as $n \to \infty$ and suppose that for each $n$, there exists a sequence $(x_{nk})$, $k=1, 2, \ldots$, of unit vectors satisfying $w: x_{nk} \to 0$ and $T_{\lambda_n} x_{nk} \to 0$ as $k \to \infty$. Then there exists a sequence $(x_n)$ of the form $x_n = x_{nk_n}$ ($k_1 < k_2 < \cdots$) satisfying

$$\tag{3.1} \|x_n\| = 1, \quad w: x_n \to 0 \quad \text{and} \quad T_{\lambda_0} x_n \to 0 \quad \text{as} \quad n \to \infty.$$

**Proof.** Let $\mathfrak{M}$ denote the (separable) space spanned by $(x_{nk})$ for $n, k=1, 2, \ldots$, and let $\mathfrak{N} = \langle \phi_j \rangle$ be a complete (countable) orthonormal system spanning $\mathfrak{M}$. It is clear that one can choose $k = k_n$ satisfying $k_1 < k_2 < \cdots$ and so that $\|(x_{nk_n}, \phi_j)\| < 1/n$ for $j=1, 2, \ldots, n$ and $\|T_{\lambda_n} x_{nk_n}\| < 1/n$. If $x$ is any vector in $\mathfrak{N}$, then $x = y + z$, where $y \in \mathfrak{M}$ and $z \in \mathfrak{N}$. Then $(x, x_{nk_n}) = (y, x_{nk_n}) = 0$ as $n \to \infty$, and it is clear that the sequence $(x_n)$ defined by $x_n = x_{nk_n}$ satisfies the required conditions. This completes the proof of the Lemma.

Next, let an eigenvalue $\lambda$ of any operator $T$ be called a normal eigenvalue if $\lambda$ is an eigenvalue of $T^*$ and if the null spaces of $T_\lambda$ and $T_\lambda^*$ coincide. It was proved by Stampfli [11] that if $T$ satisfies (1.3) then any isolated point of $\text{sp}(T)$ must be a normal eigenvalue. (The special case in which $T$ satisfies (1.4) was treated in Stampfli [10].) The same argument shows that if $\lambda_0$ is an isolated point of $\text{sp}(T)$ and if $d(\lambda) \|R_\lambda\| \to 1$ as $\lambda \to \lambda_0$ then $\lambda_0$ is a normal eigenvalue.

In order to prove Theorem 2, note that if there exists a sequence $(\lambda_n)$ of distinct isolated points $\lambda_n$ in $\text{sp}(T)$ satisfying $\lambda_n \to \lambda_0$ as $n \to \infty$ then the $\lambda_n$ must be normal eigenvalues. Hence $(x_n, x_{n}) = 0$ if $T_{\lambda_n} x_n = 0$ and $T_{\lambda_n} x_m = 0$ and $n \neq m$. Thus $(x_n)$ is an infinite orthonormal sequence satisfying $T_{\lambda_0} x_n \to 0$ and $T_{\lambda_0}^* x_n \to 0$ (and, of course, $w: x_n \to 0$), as $n \to \infty$, so that the assertion of Theorem 2 follows.

If $\lambda_0$ is not the limit of a sequence of isolated points of $\text{sp}(T)$ then it is easy to see that there exists a sequence $(\lambda_n)$ of distinct points satisfying $\lambda_n \to \lambda_0$ as $n \to \infty$, such that each $\lambda_n$ is a nonisolated boundary point of $\text{sp}(T)$ and, for each $\lambda_n$, there is a disk $C_n$ containing $\lambda_n$ on its boundary and having an interior free of points of
sp (T). By Theorem 1, for each \( \lambda_n \), there exists a sequence \( \{x_{nk}\} \), \( k = 1, 2, \ldots \), of unit vectors satisfying \( w: x_{nk} \to 0 \) and \( T_{\lambda_n} x_{nk} \to 0 \) as \( k \to \infty \). It will next be shown that also \( T^*_n x_{nk} \to 0 \) as \( k \to \infty \).

For \( \lambda \notin \text{sp} (T) \), \( T_{\lambda_n} x_{nk} = T_{\lambda_n} x_{nk} + (\lambda_n - \lambda) x_{nk} \) and hence, on multiplying by \( R_{\lambda} \),

\[
(\lambda - \lambda_n) R_{\lambda} + I)x_{nk} \to 0 \quad \text{as} \quad k \to \infty,
\]

for \( \lambda, n \) fixed. But \( 0 \leq \|(\lambda - \lambda_n) R_{\lambda}^* + I)x_{nk}\|^2 = |\lambda - \lambda_n|^2\|R_{\lambda}^* x_{nk}\|^2 + (x_{nk}, (\lambda - \lambda_n) R_{\lambda} x_{nk}) + ((\lambda - \lambda_n) R_{\lambda} x_{nk}, x_{nk}) + \|x_{nk}\|^2. \) In view of (3.2) this implies that

\[
\|(\lambda - \lambda_n) R_{\lambda}^* + I)x_{nk}\|^2 \leq |\lambda - \lambda_n|^2\|R_{\lambda}^* x_{nk}\|^2 - 1 + \eta_k,
\]

where \( \eta_k \to 0 \) as \( k \to \infty \) (for \( \lambda, n \) fixed).

Now, let \( r_n \) denote the radius of \( C_n \) containing the point \( \lambda_n \). Since \( d(\lambda) = |\lambda - \lambda_n| \) for \( \lambda \) on \( r_n \), it follows from (1.2) and (3.3) that for each \( \epsilon > 0 \) there exists a \( \delta_\epsilon > 0 \) and a positive integer \( N_\epsilon \) with the property that \( \|(\lambda - \lambda_n) R_{\lambda}^* + I)x_{nk}\| < \epsilon \) provided \( \lambda \) (fixed) is on \( r_n \) and \( 0 < |\lambda - \lambda_n| < \delta_\epsilon, \ k > N_\epsilon. \) Since \( T_{\lambda}(\lambda - \lambda_n) R_{\lambda}^* + I) = T_{\lambda_n} \), it is clear that \( T_{\lambda_n} x_{nk} \to 0 \) as \( k \to \infty \).

Consequently, it has been shown that there exist \( \lambda_n \to \lambda_0 \) and sequences \( \{x_n\} \) of unit vectors satisfying \( w: x_n \to 0 \) and both limit relations \( T_{\lambda_n} x_{nk} \to 0 \) and \( T_{\lambda_n}^* x_{nk} \to 0 \) as \( k \to \infty \). An application of the Lemma then yields the existence of a sequence \( \{x_n\} \) of unit vectors satisfying \( w: x_n \to 0 \) and for which both \( T_{\lambda_0} x_n \to 0 \) and \( T_{\lambda_0}^* x_n \to 0 \) as \( n \to \infty \). Thus \( \lambda_0 \) is in the set \( B(T) \) of (1.6) and the proof of Theorem 2 is now complete.

4. Some spectral properties. Let \( T \) be seminormal or, more generally, satisfy (1.3), or even

\[
d(\lambda) = \|R_{\lambda}\| \to 1 \quad \text{as} \quad \lambda \to \lambda_0 \quad (\lambda \notin \text{sp} (T)),
\]

for all \( \lambda_0 \) in the boundary of \( \text{sp} (T) \). If \( C = T^* T - T T^* \), it is clear that

\[
C = T^*_\lambda T_{\lambda} - T_{\lambda} T^*_\lambda \quad \text{for arbitrary} \ \lambda.
\]

Hence, as a consequence of Theorem 2, if the boundary of \( \text{sp} (T) \) contains at least one nonisolated point (that is, if \( \text{sp} (T) \) is an infinite set), there exists a sequence \( \{x_n\} \) of unit vectors satisfying \( w: x_n \to 0 \) and \( C x_n \to 0 \) as \( n \to \infty \). This result holds also if \( \text{sp} (T) \) contains only a finite number of points, for, as noted earlier, each such point must be a normal eigenvalue, and one of these, therefore, must be of infinite multiplicity (the Hilbert space being infinite dimensional). Hence \( \{x_n\} \) can be chosen to be an orthonormal sequence of eigenvectors corresponding to this eigenvalue. Thus, there has been proved the following:

**Theorem 3.** If \( T \) is a bounded operator satisfying (4.1) for all \( \lambda_0 \) in the boundary of \( \text{sp} (T) \), then \( 0 \) belongs to the essential spectrum of \( T^* T - T T^* \).

**Remarks.** It is seen that if, in particular, \( T \) is seminormal then it is necessary that \( 0 \) belong to the essential spectrum of \( T^* T - T T^* \). It is easy to see also that if \( S \) is any
compact set of nonnegative (or nonpositive) real numbers containing 0, then there exists a seminormal operator $T$ for which $sp(T^*T - TT^*) = S$. In fact, if $V$ is the isometric operator considered in §1, it is seen that $D = V^*V - VV^*$ is the diagonal matrix all elements of which are 0 except for 1 in the (1, 1) position. In particular, $sp(D)$ consists of 0 and 1. If $\{r_n\}$ is any countable set of nonnegative real numbers whose closure is $S$ it is seen that for the direct sum operator $T = \bigoplus_{n=1}^{\infty} r_n^1/2 V$ on the space $\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathbb{R}$, one has $sp(T^*T - TT^*) = S$.

**Added in proof.** In fact, it follows from a result of H. Radjavi ([J. Math. Mech. 16 (1966), 19–26]) that a nonnegative operator $C$ on an infinite-dimensional separable Hilbert space has 0 in its essential spectrum if and only if it is of the form $C = T^*T - TT^*$.

Another consequence of Theorem 2 is the following:

**Theorem 4.** Let $T$ be a bounded operator satisfying (4.1) for all $\lambda_0$ in the boundary of $sp(T)$ and let $T$ have the Cartesian representation $T = H + iJ$, where $H$ and $J$ are selfadjoint. Then the projections of $sp(T)$ onto the $x$ and $y$ axes are, respectively, contained in the sets $sp(H)$ and $sp(J)$.

In order to see this, it is sufficient to consider the operator $H (= \frac{1}{2}(T + T^*))$ only. Let $\lambda_0 \in sp(T)$. If $\lambda_0$ is an isolated point of $sp(T)$, it is a normal eigenvalue (cf. §3 above) and clearly $Re(\lambda_0)$ is in $sp(H)$. If $\lambda_0$ is not an isolated point of $sp(T)$ then it is clear that there exists some nonisolated boundary point $\lambda_1$ of $sp(T)$ satisfying $Re(\lambda_1) = Re(\lambda_0)$. Hence, by Theorem 2, $Re(\lambda_0)$ is in $sp(H)$, in fact, $Re(\lambda_0)$ is in the essential spectrum of $H$.

**Remark.** In case $T$ is seminormal, that is, if (1.4) holds, the sets $sp(H)$ and $sp(J)$ are precisely the projections of $sp(T)$ onto the $x$ and $y$ axes (Putnam [6]). Whether the corresponding assertion holds if the condition of seminormality is relaxed to (4.1) for all $\lambda_0$ in the boundary of $sp(T)$, or even to (1.3), is apparently not known.

An assertion related to that of Theorem 4 can be made for any operator $T$ for which $C = T^*T - TT^*$ is completely continuous. For, by (4.2), it is clear that $A(T) = B(T)$. It follows from Theorem 1 that the projections of all nonisolated boundary points of $sp(T)$ onto the $x$ and $y$ axes belong to the spectra (even essential spectra) of $H$ and $J$ respectively.

**References**


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