ON PRIMITIVE ELEMENTS IN DIFFERENTIALLY
ALGEBRAIC EXTENSION FIELDS

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It is well known that if $F$ is a field of characteristic zero and $K=F(\alpha_1, \ldots, \alpha_n)$ is a finite algebraic extension of $F$, then $K$ contains a primitive element, i.e. an element $\alpha$ such that $F(\alpha_1, \ldots, \alpha_n)=F(\alpha)$. Moreover, by means of Galois theory, it is possible to characterize those elements of the extension field which are primitive.

In the case of finite differentially algebraic extensions the theorem without further restrictions is false. Let $Q$ be the field of rational numbers and $\delta$ the usual derivation, i.e., $\delta q=0$ for every $q \in Q$. Let $c_1, \ldots, c_n$ be algebraically independent complex numbers over $Q$. If $(Q\langle c_1, \ldots, c_n \rangle, \delta)$ is the differentially algebraic extension of $Q$ where $\delta c=0$ for every $c \in Q\langle c_1, \ldots, c_n \rangle$, then the underlying set of $Q\langle c_1, \ldots, c_n \rangle$ is identical with that of $Q\langle c_1, \ldots, c_n \rangle$, whence it is clear that there is no element $c \in Q\langle c_1, \ldots, c_n \rangle$ such that $Q\langle c_1, \ldots, c_n \rangle=Q\langle c \rangle$. Kolchin [2] (also [5, p. 52]) has shown the existence of primitive elements in the case where the differential field $F\langle \alpha_1, \ldots, \alpha_n \rangle$ has one derivation operator and the field $F$ has an element $f$ such that $\partial f \neq 0(f)$. The differential fields $(F\langle x_0, \ldots, x_p \rangle, D)$ considered in this paper are differentially algebraic over $F$, but $F$ does not contain nonconstant elements. We prove the existence of primitive elements in the case where the derivation operator satisfies the conditions

$$Df = 0 \quad \text{for every } f \in F, \quad Dx_0 = 1, \quad x_0 \cdot x_1 \cdot \cdots \cdot x_{k-1}Dx_k = 1 \quad \text{for } 0 < k \leq p.$$  

An example of such a differential field is $(C\langle e^z, z, \log z, \log \log z \rangle, \delta)$ where $C$ is the field of complex numbers, $\delta = e^{-z}D$ and $D$ is the usual derivation of functions of a complex variable, i.e., $\delta e^z = e^{-z}De^z = 1$, $\delta \log z = e^{-z}D \log z = (e^z \cdot z)^{-1}$, $\delta \log \log z = e^{-2z}D \log \log z = (e^{2z} \cdot z \cdot \log z)^{-1}$.

In the sequel, for differentially algebraic extension fields which satisfy conditions (1) not only do we establish the existence of primitive elements, but we give explicit formulas for such elements. In §§7 and 8 we apply these formulas to the asymptotic theory of ordinary differential equations. More precisely, in [6] W. Strodt introduced the concept of the "principal monomials" and "principal solutions" for a certain class of differential equations whose coefficients belong to a logarithmic domain. In [8] Strodt characterized the principal monomials by the concept of

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(*) In the partial case (more than one derivation operator) $F$ must contain a set of elements whose Jacobian does not vanish (Kolchin [2, §4]).
stability. S. Bank [1] investigated all the logarithmic monomials at which an $n$th order differential polynomial of a certain class is unstable; such logarithmic monomials were called "critical monomials" of the differential polynomial. The algorithm which produced the principal monomials in [6] and the critical monomials in [1] consists essentially of the repeated applications of the transformation $x = e^u, y = ve^{mu}$. The effectiveness of this transformation depends upon two crucial lemmas ([6, Lemma 61] and [1, Lemma 13]), to the effect that whenever a transformation $x = e^u, y = ve^{mu}$ is applied to a homogeneous, isobaric differential polynomial of positive weight $W$, with constant coefficients, the transformed differential polynomial always effectively involves at least one term whose weight is less than $W$, unless the differential polynomial is of the form $cY^d - wY^w$ and $m = 0$. In this note we generalize these lemmas and prove them with the aid of a result on the transcendence degree of differential field extensions (Theorem 7.1 below).

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1. Preliminaries. This section contains some elementary results on differential field extensions. All differential fields considered here are of characteristic zero. The notations are the same as in [3], [4], and [5]. The differential field defined over a field $G$ by a derivation $\delta$ will be denoted by $(G, \delta)$.

**Lemma 1.1.** Let $(H, \delta)$ be a subdifferential field of $(G, \delta)$. Suppose the subfield $C$ of elements of $G$ annihilated by $\delta$ is contained in $H$. If $\alpha \in G$ is such that $\delta \alpha \in H$, then either $\alpha$ is transcendental over $H$ or $\alpha \in H$.

**Proof.** Follows along the same lines as the proof of Lemma 3.9, Kaplansky [4].

**Lemma 1.2.** Let $(H, \delta)$ be a differential field and $(H(\alpha), \delta)$ a differential extension field of $(H, \delta)$ such that $\alpha$ is transcendental over $H$ and $\delta \alpha = 1$ (3). Then there is no element in $H(\alpha)$ whose derivative is $\alpha^{-1}$.

**Proof.** Since $\delta \alpha = 1 \in H$, the underlying sets of $H(\alpha)$ and $H(\alpha)$ are identical. Thus every $g \in H(\alpha)$, $g \neq 0$, can be written in the form $\alpha^n(P/Q)$ where $n$ is an integer, and $P$ and $Q$ are polynomials in $\alpha$ with coefficients in $H$ such that $P(0) \neq 0, Q(0) \neq 0$. The integer $n$ is uniquely determined by $g$; by direct calculation $\delta(\alpha^n(P/Q)) \neq \alpha^{-1}$.

We will need the following well-known lemma:

**Lemma 1.3.** Let $(C(\alpha), \delta)$ be a differential field where $C$ is the subfield of elements of $C(\alpha)$ annihilated by $\delta$. If $C(\alpha)$ has transcendence degree $p + 1$ over $C$, then $\alpha, \delta \alpha, \ldots, \delta^p \alpha$ are algebraically independent over $C$. Moreover

$$C(\alpha) = C(\alpha, \delta \alpha, \ldots, \delta^{p+1} \alpha).$$

(3) A proof of the case $m \neq 0$ in the setting of graduated logarithmic fields, independently of ours, is given by Strodt [8].

(3) Such a differential field extension can always be constructed. See Corollary 1 of Theorem 39, page 124, Vol. 1 of [9].
2. The logarithmic differential fields. In this section $K$ is a differential field with derivation $D$. $K$ contains a distinguished sequence $x_0, x_1, \ldots$ of elements called a logarithmic sequence such that $Dx_0=1$ and $x_0 \cdot x_1 \cdot \cdots \cdot x_{p-1} D x_p = 1$ for $p = 1, 2, \ldots$.(1) $C = \{c \in K, Dc=0\}$. For any $p \geq 0$ the subdifferential field

$$(C(x_0, x_1, \ldots, x_p), D)$$

will be called a logarithmic differential field.

**Lemma 2.1.** Let $F_p = C(x_0, x_1, \ldots, x_p)$ for each nonnegative integer $p$, and $F_{-1} = C$. Then for $p = 0, 1, 2, \ldots$,

**Proof.** By induction on $p$. Since $Dx_0=1 \neq 0$, for $p = 0$ we have $x_p = x_0 \notin C = F_{p-1}$.

Suppose for the nonnegative integer $q$, $x_q \notin F_{q-1}$. Since $(F_{q-1}, D)$ is a differential field and $Dx_q = (x_0 \cdot x_1 \cdot \cdots \cdot x_{q-1})^{-1} \in F_{q-1}$, by Lemma 1.1, $x_q$ is transcendental over $F_{q-1}$. Let $\delta_q = x_0 \cdot x_1 \cdot \cdots \cdot x_{q-1} D$, then $(F_{q-1}, \delta_q)$ is a differential field.

Since $x_q$ is transcendental over $F_{q-1}$ and $\delta_q x_q = 1$, by Lemma 1.2, there is no element $y$ in $(F_{q-1} \langle x_q \rangle, \delta_q)$ such that $\delta_q y = x_q^{-1}$. Since $\delta_q x_{q+1} = x_{q+1}^{-1}$, $x_{q+1} \notin F_{q-1} \langle x_q \rangle = F_q$. This completes the induction.

**Lemma 2.2.** $x_0, x_1, \ldots$ are algebraically independent over $C$.

**Proof.** Follows from Lemmas 1.1 and 2.1.

3. Partial order in $(C(x_0, x_1, \ldots, x_p), D)$. Let $K, C$, and the sequence $x_0, x_1, \ldots$ be as in §2. We will introduce a partial order in the subfield $F_p = C(x_0, x_1, \ldots, x_p)$ of $K$ as follows. Let

$$V_p = \{x_0^{i_0}x_1^{i_1} \cdots x_p^{i_p} : (i_0, i_1, \ldots, i_p) \in Z^p \}$$

where $Z$ is the ring of integers. $V_p$ is a subgroup of the multiplicative group of $F_p$. Let $\mathcal{N}_p = \{ev : c \in C - \{0\}, v \in V_p\}$. Let $M = ax_0^{i_0}x_1^{i_1} \cdots x_p^{i_p}$ and $N = bx_0^{j_0}x_1^{j_1} \cdots x_p^{j_p}$ be elements of $\mathcal{N}_p$. We will write $M < N$ if $m_0 < n_0$, or for some natural number $q$, $0 < q \leq p$, $m_k = n_k$ for $k = 0, 1, \ldots, q-1$ and $m_q < n_q$. If $(m_0, m_1, \ldots, m_p) = (n_0, n_1, \ldots, n_p)$ we write $M \approx N$. If $f \in C[x_0, x_1, \ldots, x_p]$, $f \neq 0$, it can be written in the form

$$f = \sum_{t=1}^{\infty} c_t M_t$$

where $c_t \in C - \{0\}$ and $M_t \in V_p$ such that $M_i \neq M_j$, if $i \neq j$. For some positive integer $s$, $1 \leq s \leq n$, $c_s M_s < c_s M_s$ for $i \neq s$. We will call $c_s M_s$ the dominating monomial of $f$. If $g \in F_p - \{0\}$ then $g$ can be written $g = \sum c_t M_t / \sum b_t N_t$ where $c_t, b_t \in C - \{0\}$ and $M_t, N_t \in V_p$. If $c_s M_s$ is the dominating monomial of $\sum c_t M_t$ and $b_t N_t$ is the dominating monomial of $\sum b_t N_t$, we say $g^* = (c_s M_s)(b_t N_t)^{-1}$ is the dominating monomial of $g$. If $g, h \in F_p - \{0\}$ we write $g < h$ if $g^* < h^*$, and $g \sim h$ if $g^* = h^*$. We extend this definition of order to $F_p$ by setting $0 < g$ for every $g \in F_p - \{0\}$.(4) It is clear now that if $f, g, h, k \in F_p$,

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(1) The notion of logarithmic sequences was introduced by Strodt (cf. [6] and [8]).

(2) $F_p$ with the partial order $<$ is a field with asymptotic order (cf. Strodt [7]).
(i) if \( f < g \), then \( f \neq g \);
(ii) \( 0 < 1 \);
(iii) if \( f < g \) and \( h \neq 0 \), then \( fh < gh \);
(iv) if \( f < g \) and \( h < k \), then \( fh < gk \);
(v) if \( f < g \) and \( h < g \), then \( f + h < g \);
(vi) if \( f < g \) and \( h < g \), then \( f < g + h \);
(vii) if \( f \sim g \) then \( f - g < g \) and conversely;
(viii) if \( f \sim g \) and \( g \sim h \), then \( f \sim h \);
(ix) if \( f \sim g \) and \( h \sim k \), then \( fh \sim gk \);
(x) if \( f \sim g \) then \( f \neq 0 \) and \( g \neq 0 \).

**Definition 3.1.** Let

\[
 f = \frac{\sum b_k x_0^k \cdot x_1^k \cdot \ldots \cdot x_p^k}{\sum c_j x_0^j \cdot x_1^j \cdot \ldots \cdot x_p^j} \in F_p.
\]

Define \( Ef \in F_{p+1} \) to be

\[
 Ef = \frac{\sum b_k x_1^k \cdot x_2^k \cdot \ldots \cdot x_{p+1}^k}{\sum c_j x_1^j \cdot x_2^j \cdot \ldots \cdot x_{p+1}^j}.
\]

**Lemma 3.1.** Let \( f, g \in F_p \), then

(i) \( f < g \) implies \( Ef < Eg \).
(ii) \( f \sim g \) implies \( Ef \sim Eg \).

**Proof.** Follows from the definitions.

4. The functions \( S_{i,j}(m) \).

**Definition 4.1.** Let \( m \) be a variable; define

\[
 S_{k,k}(m) = m(m-1)(m-2)\cdots(m-k+1) \quad \text{if } k > 0,
\]

and

\[
 S_{k,k-j}(m) = (1/j!) S_{k,j}^{(j)}(m) \quad \text{if } j > 0,
\]

where \( S_{k,j}^{(j)}(m) \) is the \( j \)th derivative of \( S_{k,k}(m) \). We will make the convention that \( S_{0,0}(m) = 1 \).

It is clear that \( S_{i,1}(m) \) is the elementary symmetric function of degree \( j \) in \( m, m-1, \ldots, m-i+1 \). Thus

\[
 (1) \quad \sum_{j=0}^{k} x^j S_{k,k-j}(m) = (x+m)(x+m-1)\cdots(x+m-k+1)
\]

(cf. [6, §58]). We remark that \( S_{k,0}(m) = 1 \) and \( S_{k,1}(m) = 0 \) if \( j < 0 \).

**Lemma 4.1.** If \( k \geq 1 \), \( m = (-1)^{k+1} \cdot k \cdot S_{k,k}(m) - (-1)^k \cdot m \cdot \sum_{j=1}^{k-1} (-1)^j S_{k,k-j}(m) \).
Proof. By (1) above \( \sum_{j=0}^{k-1} (-1)^j S_{k,j}(m) = (m-1)(m-2) \cdots (m-k) \). Therefore

\[
m \sum_{j=1}^{k-1} (-1)^j S_{k,j}(m) = m[(m-1)(m-2) \cdots (m-k) - S_{k,k}(m) - (-1)^k S_{k,0}(m)]
\]

\[
= m[(m-1) \cdots (m-k) - m(m-1) \cdots (m-k+1) + (-1)^{k+1}]
\]

\[
= m[(m-1) \cdots (m-k+1)(m-k-m) + (-1)^{k+1}]
\]

\[
= -kS_{k,k}(m) + (-1)^{k+1}m.
\]

5. The elements \( (x_0, x_1, \ldots, x_k)^m \) of \( (C(x_0, x_1, \ldots, x_p), D) \). Here and in the next section \( m \) is a nonzero integer.

**Definition 5.1.** Let \( E_0 = E \) where \( E \) is as in Definition 3.1, and for the positive integer \( p \), \( E_p = [(m-p+1) + x_0 D]E_{p-1} \).

**Lemma 5.1.** Let \( V = V(x_0, x_1, \ldots, x_k) \in C(x_0, x_1, \ldots, x_p), k < p \). Then \( x_0 DEV = EDV \).

**Proof.**

\[
x_0 DEV(x_0, x_1, \ldots, x_k) = x_0 DV(x_1, x_2, \ldots, x_k + x)
\]

\[
= x_0 \left[ \frac{1}{x_0} \frac{\partial EV}{\partial x_1} + \frac{1}{x_0 x_1} \frac{\partial EV}{\partial x_2} + \cdots + \frac{1}{x_0 x_1 \cdots x_k} \frac{\partial EV}{\partial x_k+1} \right]
\]

\[
= \frac{\partial EV}{\partial x_1} + \frac{1}{x_1} \frac{\partial EV}{\partial x_2} + \cdots + \frac{1}{x_1 x_2 \cdots \cdots x_k} \frac{\partial EV}{\partial x_k+1}
\]

\[
= EDV.
\]

Let \( T_k = (x_0 \cdot x_1 \cdot \cdots \cdot x_k)^m \) where \( m \) is a nonzero integer, then:

**Corollary.**

\( E_p T_k = (m-p+1 + x_0 D)(m-p+2+x_0 D) \cdots (m-1+x_0 D)(m+x_0 D)ET_k \)

\[
= \sum_{i=0}^{p} S_{p,i}(m)ED^{p-i}T_k.
\]

**Proof.** By straightforward calculation and Lemma 5.1.

**Lemma 5.2(6).** \( D^p T_0 = S_{p,k}(m)x_0^{m-p} \) and for positive integers \( p \) and \( k \), \( D^p T_k = x_0^{m-p} E_p T_{k-1} \).

**Proof.** By induction on \( p \). \( D^0 T_k = T_k = x_0^2 E_0 T_{k-1} \) by the definitions of \( T_k \) and \( E_0 \).

Suppose \( D^{p-1} T_k = x_0^{m-p+1} E_{p-1} T_{k-1} \). Then

\( D^p T_k = x_0^{m-p}[(m-p+1) + x_0 D]E_{p-1} T_{k-1} = x_0^{m-p} E_p T_{k-1} \)

by the definition of \( E_p T_{k-1} \).

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\( (*) \) This is a special case of [8, Lemma 66(c)].
Lemma 5.3. For any pair of integers \((p, q)\) such that \(0 \leq p < q\), \(D^q T_k < D^p T_k\) if \(D^p T_k \neq 0\).

Proof. If \(D^p T_k = 0\), then \(D^q T_k < D^p T_k\). Suppose \(D^q T_k \neq 0\). By Lemma 5.2,
\[
D^q T_k = x_0^{q - p} E_p T_{k - 1} \quad \text{and} \quad D^p T_k = x_0^{q - p} E_q T_{k - 1}.
\]
Since \(E_p T_{k - 1}\) and \(E_q T_{k - 1}\) are elements of \(C(x_1, x_2, \ldots, x_k)\) and \(m - q < m - p\), it follows from the definition of \(<\) that \(D^q T_k < D^p T_k\).

Notation. For \(f \in C(x_0, x_1, \ldots, x_k)\), by \(\partial f\) we will mean the formal partial derivative of \(f\) with respect to \(x_j\).

Lemma 5.4. For any pair of integers \((p, q)\) such that \(0 \leq p < q\), \(\partial_n D^q T_k < \partial_n D^p T_k\) if \(\partial_n D^q T_k \neq 0\) and \(0 \leq n \leq k\).

Proof. Similar to the proof of Lemma 5.3.

6. The differential field \((C(T_p), D)\). In this section we will show that the transcendence degree of \((C(T_p), D)\) is \(p + 1\) over \(C\).

Notation. In the sequel the minor of \(\partial_p D_{T_p - 1}\) in the Jacobian determinant
\[
\frac{\partial(T_{p - 1}, DT_{p - 1}, \ldots, D^p T_{p - 1})}{\partial(x_0, x_1, \ldots, x_p)}
\]
will be denoted by \(A_i\).

Lemma 6.1. Suppose the Jacobian determinant
\[
J_{p - 1} = \frac{\partial(T_{p - 1}, DT_{p - 1}, \ldots, D^p T_{p - 1})}{\partial(x_0, x_1, \ldots, x_p)} \neq 0,
\]
then \(A_i < J_{p - 1}\) for \(i = 0, 1, 2, \ldots, p - 1\).

Proof. Write \(D^p T_{p - 1} = x_0^{p - 1} E_i T_{p - 2}\), by Lemma 5.2, and write \(J_{p - 1}\) and \(A_i\) in the determinant form. By direct calculation
\[
(3) \quad J_{p - 1} \sim dx_0^{d(m - (p - 1)/2) - 1} \cdot x_1^{k_1} \cdot x_2^{k_2} \cdot \ldots x_{p - 1}^{k_{p - 1}} \quad \text{with} \quad d \in C - \{0\},
\]
and either \(A_i = 0\) or
\[
(4) \quad A_i \sim cx_0^{d(m - (p - 1)/2) - 1 - (p - 1)/2} \cdot x_1^{j_1} x_2^{j_2} \cdot \ldots x_{p - 1}^{j_{p - 1}}
\]
where \(k_1, k_2, \ldots, k_{p - 1}, j_1, j_2, \ldots, j_{p - 1}\) are integers. \(A_i < J_{p - 1}\) for \(i = 0, 1, \ldots, p - 1\) by comparing the right sides of (3) and (4).

Corollary. If \(J_{p - 1} \neq 0\), \( \sum_{i=0}^m E_i A_i \sim m E_0 J_{p - 1}\).

Proof. Follows from Lemma 6.1 and the identity \(A_p = J_{p - 1}\).

Lemma 6.2. If \(J_{p - 1} \neq 0\), then
\[
A = \sum_{i=0}^{p-1} (-1)^i D^i T_{p - 1} A_i < T_{p - 1} \cdot J_{p - 1}.
\]
Proof. By the properties (iii) and (iv) of §3, and Lemmas 5.3 and 6.1,
\[ (-1)^{i}D^{i}T_{p-1}A_{i} < T_{p-1}J_{p-1} \text{ for } i = 0, \ldots, p-1 \]
and
\[ (-1)^{p}D^{p}T_{p-1}A_{p} = (-1)^{p}D^{p}T_{p-1}J_{p-1} < T_{p-1}J_{p-1} \]
by Lemma 5.3 and the property (iv) of §3. Therefore by (v) of §3, \( A < T_{p-1}J_{p-1} \).

Corollary. \( E_{0}A < E_{0}T_{p-1}E_{0}J_{p-1} \).

Proof. By Lemma 3.1.

Lemma 6.3. Let \( B_{p} = \sum_{i=1}^{p} mE_{0}A_{i} \) and \( B = \sum_{i=1}^{p} (-1)^{i} \cdot S_{i,i}(m)B_{i} \) where \( B_{i} \) is the minor of \( \partial(E_{0}T_{p-1}, E_{1}T_{p-1}, \ldots, E_{p}T_{p-1}) \)
\[ \partial(x_{0}, x_{1}, \ldots, x_{p}) \]
then \( B_{p} = B \).

Proof. \( B_{p} \) and \( B \) may be viewed as the expansions of two \((p+1) \times (p+1)\) determinants by the minors of the first columns. These first columns are respectively \((0, -m, +m, \ldots, (-1)^{m}m)\) and \((0, -S_{1,1}(m), -2S_{2,2}(m), \ldots, -pS_{p,p}(m))\). We introduce the \((p+1) \times (p+1)\) determinants \( C_{0}, C_{1}, \ldots, C_{p} \) recursively as follows. Define \( C_{0} = B \) and define \( C_{k+1} \) as the determinant obtained from \( C_{k} \) by adding \(-S_{k+1,k+2-j}(m)\) times the \( j^{th} \) row of \( C_{k}, j = 1, 2, \ldots, k+1 \), to the \((k+2)\)th row of \( C_{k} \). Obviously \( B = C_{0} = C_{1} = \ldots = C_{p} \). Evidently \( C_{0} \) has the form of \( B_{p} \) in the first row, and the form of \( B \) in the remaining rows. Suppose now that for some \( k \) in \( \{0, 1, \ldots, p-1\} \), \( C_{k} \) has the form of \( B_{p} \) in the first \( k+1 \) rows and the form of \( B \) in the remaining rows. By the above construction \( C_{k+1} \) has the form of \( B_{p} \) in the first \( k+2 \) rows. In fact, the \((k+2)\)th row of \( C_{k+1} \) is
\[ \left\{ -(k+1)S_{k+1,k+1}(m)-m \sum_{i=1}^{k} (-1)^{i}S_{k+1,k+1-i}(m), \partial_{E_{0}D^{k+1}T_{p-1}}, \ldots, \partial_{pE_{0}D^{k+1}T_{p-1}} \right\}. \]
Hence the first entry in this row is
\[ (-1)^{k+1} \left\{ -(k+1)S_{k+1,k+1}(m)-m \sum_{i=1}^{k} (-1)^{i}S_{k+1,k+1-i}(m) \right\} = (-1)^{k+1}m \text{ by Lemma 4.1.} \]

The remaining entries of this row have the asserted form by the Corollary of Lemma 5.1.

Corollary. Suppose \( J_{p-1} \neq 0 \), then \( B \sim mE_{0}J_{p-1} \).

Proof. Follows from the Corollary of Lemma 6.1 and Lemma 6.3.
Lemma 6.4. If $J_{p-1} \neq 0$, then

$$J_p = \frac{\partial (T_p, DT_p, \ldots, D^p T_p)}{\partial (x_0, x_1, \ldots, x_p)} \sim mx_0^{p+1}(m-p/2-1)\cdot E_0 T_{p-1} \cdot E_0 J_{p-1}.$$  

Proof. Write the Jacobian $J_p$ in the determinant form, then

$$J_p = x_0^{(p+1)(m-p/2)-1} \cdot \{mB + C\}$$

where $B$ and $C$ are $(p+1) \times (p+1)$ determinants whose expansions in the minors of the first columns are respectively $\sum_{i=0}^{p} (-1)^i E_i T_{p-1} B_i$ and $\sum_{i=0}^{p} (-1)^{i+1} \cdot i \cdot E_i T_{p-1} B_i$. We assert that

$$B = E_0 \sum_{i=0}^{p} (-1)^i D^i T_{p-1} A_i.$$  

This can be verified by applying to $B$ the same row operations as were applied to $B$ in Lemma 6.3. Each column of $B$, except the first, is thereby transformed into the corresponding column of a determinant $E_0 A$ whose expansion in the minors of the first column is the right side of (5). The verification that the first column of $B$ is transformed into the first column of $E_0 A$ is done inductively as in Lemma 6.3 and turns upon the identity

$$E_{k+1} T_{p-1} = \sum_{j=1}^{k+1} S_{k+1,k+2-j}(m) E_0 D^{j-1} T_{p-1} = E_0 D^{k+1} T_{p-1}$$

which follows from the Corollary of Lemma 5.1. As for $C$, $C$ can be written

$$C = (E_0 T_{p-1})B + \sum_{i=0}^{p} (-1)^{i+1} \cdot i \cdot \sum_{j=0}^{i} S_{i,j}(m) (E_0 D^{i-j+1} T_{p-1})B_i.$$  

It can be shown by the same reduction as employed in Lemma 6.3 that the second term in the right side of (6) is $< E_0 T_{p-1} E_0 J_{p-1}$. It now follows from the Corollary of Lemma 6.3 that $C \sim mE_0 T_{p-1} E_0 J_{p-1}$. Thus, by the Corollary of Lemma 6.2, $mB + C \sim mE_0 T_{p-1} E_0 J_{p-1}$. This establishes the lemma.

Corollary. If $J_{p-1} \neq 0$, then $J_p \neq 0$.

Theorem 6.1. The differential field $(C \langle T_p \rangle, D)$ is of transcendence degree $p+1$ over $C$.

Proof. Since $(C \langle T_p \rangle, D)$ as a field is contained in the field $C(x_0, x_1, \ldots, x_p)$, the transcendence degree of $C \langle T_p \rangle$ over $C$ is at most $p+1$. It is enough, therefore, to show that $T_p, DT_p, \ldots, D^p T_p$ are algebraically independent over $C$. By induction on $p$ and the Corollary of Lemma 6.4, it is enough to show that the Jacobian $J_0 \neq 0$. This by direct calculation is $mx_0^{p+1} \neq 0$.

Theorem 6.2. (i) $(C(x_0, x_1, \ldots, x_p), D) = (C \langle x_0, x_1, \ldots, x_p \rangle, D)$, (ii) $(C(x_0, x_1, \ldots, x_p), D) = (C \langle x_p \rangle, D)$. 

Proof of (i). By Theorem 6.1, \((C(x_0, x_1, \ldots, x_p), D)\) is of transcendence degree \(p+1\) over \(C\). Thus \((C(x_0, x_1, \ldots, x_p), D)\) is an algebraic extension of
\[(C(x_0, x_1, \ldots, x_p), D).\]

By Lemma 1.1, it is enough to show \(Dx_0, Dx_1, \ldots, Dx_p\) are in \((C(x_0, x_1, \ldots, x_p), D)\).

\(Dx_0 = 1\); therefore \(x_0 \in (C(x_0, x_1, \ldots, x_p), D)\). Suppose
\[x_0, x_1, \ldots, x_{k-1} \in (C(x_0, x_1, \ldots, x_p), D).\]

Since \(Dx_k = (x_0 - x_1 \cdot \cdots \cdot x_{k-1})^{-1}\), then \(x_k \in (C(x_0, x_1, \ldots, x_p), D)\).

Proof of (ii). Since \(Dx_p = (x_0 - x_1 \cdot \cdots \cdot x_{p-1})^{-1}\), \(x_0, x_1, \ldots, x_{p-1} \in (C(x_p), D)\).

Thus \(x_0, x_1, \ldots, x_p \in (C(x_p), D)\). Therefore
\[(C(x_0, x_1, \ldots, x_p), D) \subseteq (C(x_p), D) \subseteq (C(x_0, x_1, \ldots, x_p), D).\]

7. Imbedding of \((C(x_0, x_1, \ldots, x_p), D)\) in a graduated logarithmic field\(^7\).

Let \(K, C, D\) and the logarithmic sequence \(x_0, x_1, \ldots\) be as in §2. Recall that \(K\) is a differential field with derivative \(D\), \(C\) is the subfield of constants: \(C = \{c : c \in K\} \text{ such that } Dc = 0\), and \(x_0, x_1, \ldots\) is a logarithmic sequence in \(K\). Suppose further that \(K\) contains a multiplicative subgroup \(U'\) such that for every \(f \in U'\) and every integer \(r \geq 1\) there is a unique \(g \in U'\) such that \(g^r = f\); we will denote \(g\) by \(f^{1/r}\). Furthermore, suppose \(U'\) contains the set \(\{x_0, x_1, \ldots\}\). Let \(U_p\) be the subgroup of \(U'\) generated by the elements of the form \(x^m \in U'\), where \(m\) is rational and \(0 \leq i \leq p\). Let \(G_p\) be the differential subfield of \(K\) generated by \(U_p\) over \(C\). We observe that \((C(x_0, x_1, \ldots, x_p), D)\) is a differential subfield of \((G_p, D)\). Moreover, \((G_p, D)\) is an algebraic extension of \((C(x_0, x_1, \ldots, x_p), D)\). Since the transcendence degree of \((C(x_0, x_1, \ldots, x_p), D)\) over \(C\) is \(p+1\) by Lemma 2.2, the transcendence degree of \((G_p, D)\) is \(p+1\) over \(C\). Let \((G, U, D) = \lim_{p \to \infty} (G_p, U_p, D)\). It is clear that \((G, D)\) is a differential field.

We will introduce for the differential field \((G_p, D)\) a partial order \(<\), whose restriction to the differential subfield \((C(x_0, x_1, \ldots, x_p), D)\) coincides with the partial order \(<\) defined on \((C(x_0, x_1, \ldots, x_p), D)\) in §3, and which is such that the quadruple \((G_p, <, U_p, C)\) is a graduated field as defined in [7]. Define the order relation \(<\) in \(U_p\) as follows. Let \(\{M, N\} \subseteq U_p\), then \(M = x_0^{n_0} \cdot x_1^{n_1} \cdot \cdots \cdot x_p^{n_p}\) and \(N = x_0^{m_0} \cdot x_1^{m_1} \cdot \cdots \cdot x_p^{m_p}\) where the exponents are rational numbers. Set \(M < N\) if \(m_0 < n_0\), or for some natural number \(q\), \(0 < q \leq p\), \(m_k = n_k\) for \(k = 0, 1, \ldots, q-1\) and \(m_q < n_q\). Let \(M = C^* \cdot U_p\), where \(C^* = C - \{0\}\). Define the order relation in \(M\) as follows. If \(\{g^*, h^*\} \subseteq M\), then \(g^* = cM\) and \(h^* = dN\) for some \(M\) and \(N\) in \(U_p\) such that \(c, d \in C^*\). Set \(g^* < h^*\) if \(M < N\) and \(g^* \approx h^*\) if \(M = N\). It is clear that the order

\(^7\) The graduated differential field \((X, D)\) (see Lemma 7.3) is a graduated logarithmic field if \(U\) contains a logarithmic sequence (see also [8, p. 14]).
relation $<$ in $\mathcal{M}$ is compatible with the order relation $<$ in $U_p$. We now define an order relation $<$ in $G_p - \{0\}$. Suppose first that $f \in G_p - \{0\}$ and $f = \sum_{i=1}^{n} c_i N_i$ with $c_i \in \mathbb{C}^*$ and $N_i \in U_p$ (with $i \neq j \Rightarrow N_i \neq N_j$), then for some $r$, $c_i N_i \leq c_r N_r$ for $i = 1, 2, \ldots, n$, with strict inequality if $i \neq r$. The representation $f = \sum_{i=1}^{n} c_i N_i$ is unique. In fact for an arbitrary nonzero $p+1$-tuple $(r_0, r_1, \ldots, r_p)$ of rational numbers, $x_0^{r_0}, x_1^{r_1}, \ldots, x_p^{r_p}$ are algebraically independent over $\mathbb{C}$. In this case we write $f \sim c_r N_r$, and we say $c_r N_r$ is the dominating monomial of $f$. If now $f$ is any element of $G_p - \{0\}$, then $f = \sum c_i M_i / \sum d_i N_i$. Let $g^*$ and $h^*$ be the dominating monomials of the numerator and the denominator respectively. We write $g^* \cdot h^{*-1} \sim f$, and call $g^* \cdot h^{*-1}$ the dominating monomial of $f$. Let $(c, u)$ be the unique element of $\mathbb{C} \times U_p$ such that $g^* \cdot h^{*-1} = cu$. Then $u$ is called the gauge of $f$ and is denoted by $f[(^8)]$. If $f_1$ and $f_2$ belong to $G_p - \{0\}$, we say $f_1 \sim f_2$ if and only if $f_1[<]f_2$. If $f_1^*$ and $f_2^*$ are the dominating monomials of $f_1$ and $f_2$ respectively and $f_1^* = f_2^*$, then we write $f_1 \sim f_2$. It is clear that this order relation in $G_p - \{0\}$ is compatible with the order relation in $\mathcal{M}$. We extend this definition of order by setting $0 < f$ for every $f \in G_p - \{0\}$, and $0|0=0$. Thus $f_1 \sim f_2$ if and only if $f_1^* = f_2^*$, and hence if and only if $f_1 - f_2 < 0$. It is now clear that the partial order $<$ defined here restricted to the subfield $\mathbb{C}(x_0, x_1, \ldots, x_p)$ of $G_p$ coincides with the partial order $<$ defined in §3. We observe that this partial order can be extended to the differential field $(G, D)$.

**Lemma 7.1.** If $M \in U_p - \{1\}$ and $e \in U_p$, such that $e < 1$, then $M De < DM$.

**Proof.** If $M = x_0^{m_0} \cdots x_p^{m_p}$ with $m \neq 0$, then by routine calculations $m(x_0 \cdots x_n)^{-1} \sim M^{-1} DM$. On the other hand $e = x_0^{b_0} \cdots x_p^{b_p}$ with $b < 0$. Then $De \sim b(x_0 \cdots x_n)^{-1} \cdot e < M^{-1} DM$ by lexicographic comparison of exponents. Thus $M De < DM$.

**Lemma 7.2.** If $\{M, N\} \subseteq U_p, N \neq 1$, such that $M < N$, then $DM < DN$.

**Proof.** $M < N$ implies $M = Ne$ for some $e \in U_p$ with $e < 1$. Hence $DM = e DN + N De$. Now $e DN < DN$ since $e < 1$ and $N De < DN$ by Lemma 7.1. Thus $DM < DN$.

**Lemma 7.3.** Let $G_p, G, D, C, U_p, U, <$ and the logarithmic sequence $\{x_0, x_1, \ldots\}$ be as defined above, then

(i) the ordered pairs $(G_p, <), (G, <)$ are fields with asymptotic order (for the definition of asymptotic order (see [7, p. 231])).

(ii) the ordered quadruples $X_p = (G_p, <, U_p, C)$ and $X = (G, <, U, C)$ are graduated fields (see [7, p. 231]).

(iii) $(X_p, D)$ is a graduated differential field. This means $DC = \{0\}$ and $D$ is stable at $U_p - \{1\}$.(^b). (See also [8, Definition 20].)

(iv) $(X, D)$ is a graduated logarithmic field.

(^b) See Definition 17 of [7].

(^b) $D$ is stable at $U_p - \{1\}$ if $M < N$ implies $DM < DN$ whenever $M \in G_p$ and $N \in U_p - \{1\}$ (see also [8, Definition 18]).
Proof. (i) and (ii) follow from the order relation defined in $G_p$. To show (iii) it is enough to show $D$ is stable on $U_p - \{1\}^p$. This follows from Lemma 7.2. (iv) is obvious.

**Lemma 7.4.** Let $y \in (C(x_0, x_1, \ldots, x_p), D)$. Let $m$ be a nonzero rational number. Then $(C\langle y \rangle, D)$ and $(C\langle y^m \rangle, D)$ have the same transcendence degree over $C$.

**Proof.** Suppose $m = a/b$ with $a$ and $b$ integers, $b > 0$. Then $(C\langle y \rangle, D)$ is an algebraic extension of $(C\langle y^a \rangle, D)$. Hence the transcendence degree of $(C\langle y^a \rangle, D)$ is same as the transcendence degree of $(C\langle y \rangle, D)$. Similarly $(C\langle y^{ab} \rangle, D)$ is an algebraic extension of $(C\langle y^a \rangle, D)$. Thus $(C\langle y^m \rangle, D), (C\langle y^a \rangle, D)$ and $(C\langle y \rangle, D)$ have the same transcendence degree over $C$.

**Theorem 7.1.** Let $m$ be a nonzero rational number, then

(i) the transcendence degree of $(C\langle (x_0, x_1, \ldots, x_p)^m \rangle, D)$ is $p\! + \! 1$ over $C$.

(ii) the transcendence degree of $(C\langle x_0^m \rangle, D)$ is $p\! + \! 1$ over $C$.

(iii) let $V_p = (x_0^m, x_1^m, \ldots, x_p^m)$, then $V_p, DV_p, \ldots, D^pV_p$ are algebraically independent over $C$.

(iv) $x_0^m, Dx_0^m, \ldots, D^p x_0^m$ are algebraically independent over $C$.

(v) $V_p$ and $x_0^m$ satisfy no algebraic differential equation of order less than $p\! + \! 1$.

**Proof.** (i) and (ii) follow from Theorems 6.1 and 6.2 and Lemma 7.4 above. (iii) and (iv) follow from (i) and (ii). (v) follows from (iii) and (iv).

8. Applications.

**Definition 8.1.** Let $P$ be the algebraic differential operator defined by

(I) $P(y) = \sum a_i y_0^{i_0} (Dy)^{i_1} \cdots (D^n y)^{i_p}, \quad a_i \in C$.

We say $P$ is homogeneous of degree $d$ if $i_0 + i_1 + \cdots + i_p = d$, and is isobaric of weight $W$ if $i_1 + 2i_2 + \cdots + pi_p = W$ for every monomial effectively present in the right side of formula (I).

**Lemma 8.1.** Let $P$ be the algebraic differential operator given by formula (I) above. Let $d$ and $W$ be positive integers. Let $P$ be homogeneous of degree $d$ and isobaric of weight $W$. Let $m$ be a rational number and $q = dm - W$. Then under the substitution $y = x_0^m z$, the expression $x_0^m z P(y)$ is transformed into

$$Q(z) = \sum b_j z^{i_0} (x_0 Dz)^{i_1} \cdots ((x_0 D)^{i_p} z)^{i_p}$$

where $b_j \in C$.

**Proof.** By formula 66(c) of [8], or induction on $k$

(II) $D^k [x_0^m z] = x_0^{m-k} \sum_{i=0}^k S_{k;i}(m)(x_0 D)^{k-i} z$
for any positive integer \( k \). Substitution of \( y = x_0^p z \) in \( P(y) \) with the aid of formula (II) establishes the Lemma.

**Definition 8.2.** The operator \( Q \) of Lemma 8.1 will be called the \( m \)-image of the operator \( P \) given by the formula (I).

**Theorem 8.1 (Strodt).** Under the hypotheses of Lemma 8.1, if \( m \) is nonzero (rational), then \( Q \) effectively involves terms of weight less than \( W \).

**Note.** This theorem was proven by W. Strodt in the general context of a graduated logarithmic field (cf. [8, §69]). The proof in [8] depends upon the analytic proof of a special case of this theorem [6, §61]. We eliminate here this dependence upon the analytic proof.

**Proof of Theorem 8.1.** \( P(Y) = \sum a_i Y^i (D Y)^i \cdots (D^p Y)^i \) and

\[
Q(Y) = \sum a_i Y^i (x_0 D Y + S_{11}(m) Y)^i \cdots \left( [x_0 D]^p Y + \sum_{i=1}^{p} s_{pi}(m) [x_0 D]^{p-i} Y \right)^i
\]

\[
= P(Y, x_0 D Y, \ldots, [x_0 D]^p Y) + H(Y, x_0 D Y, \ldots, [x_0 D]^p Y)
\]

where

\[
P(Y, x_0 D Y, \ldots, [x_0 D]^p Y) = \sum a_i Y^i (x_0 D Y)^i \cdots ([x_0 D]^p Y)^i.
\]

It is clear that all the terms of \( H(Y, x_0 D Y, \ldots, [x_0 D]^p Y) \) have coefficients in \( C \) and are of weight less than \( W \). Suppose \( H(Y, x_0 D Y, \ldots, [x_0 D]^p Y) = 0 \), then

\[
(III) \quad Q(Y) = P(Y, x_0 D Y, \ldots, [x_0 D]^p Y).
\]

If \( y \in C(x_0, x_1, \ldots, x_k) \), \( k < p \), then

\[
(IV) \quad P(\varepsilon y, x_0 D \varepsilon y, \ldots, [x_0 D]^p \varepsilon y) = P(\varepsilon y, E \varepsilon y, \ldots, E D^p \varepsilon y)
\]

\[
= E P(y) \quad \text{by Lemma 5.1},
\]

where \( \varepsilon y(x_0, x_1, \ldots, x_k) = y(x_1, x_2, \ldots, x_k + 1) \), (Definition 3.1). Since \( W > 0 \), \( P(1) = 0 \) hence

\[
Q(1) = x_0^{-dm + W} P(x_0^p) = P(1, x_0 D 1, \ldots, [x_0 D]^p 1) \quad \text{by (III)}
\]

\[
= 0.
\]

Thus \( P(x_0^p) = 0 \). Now suppose that for \( k > 0 \), \( P((x_0 \cdot x_1 \cdots x_{k-1})^m) = 0 \). Then

\[
x_0^{-dm + W} P((x_0 \cdot x_1 \cdots x_{k})^m) = Q((x_1 \cdot x_2 \cdots \cdots x_k)^m)
\]

\[
= P((x_1 \cdot x_2 \cdots \cdots x_k)^m, x_0 D (x_1 \cdot x_2 \cdots \cdots x_k)^m, \ldots, [x_0 D]^p (x_1 \cdot x_2 \cdots \cdots x_k)^m) \quad \text{by (III)}
\]

\[
= E P((x_0 \cdot x_1 \cdots x_{k-1})^m) \quad \text{by (IV)}
\]

\[
= 0.
\]

It thus follows by induction that \( P((x_0 \cdot x_1 \cdots x_p)^m) = 0 \). This contradicts Theorem 7.1(v) and completes the proof.
Theorem 8.2 (S. Bank). Under the hypothesis of Lemma 8.1 if \( m = 0 \) and \( P(Y) \neq c Y^{d - w} \cdot (DY)^w, \ c \in C - \{0\} \), then \( Q \) effectively involves terms of weight less than \( W \).

Note. In the case where \( C \) is the field of complex numbers and \( x_0 = x, \ x_1 = \log x, \ldots, x_p = \log x_{p-1} \), this theorem has been proven by S. Bank [1, Lemma 13].

Proof of Theorem 8.2. As in the proof of the previous theorem

\[
Q(Y) = P(Y, x_0DY, \ldots, [x_0D]^pY) + H(Y, x_0DY, \ldots, [x_0D]^pY).
\]

If we suppose \( P(Y) = c Y^{d - w} \cdot (DY)^w + T(Y) \) where \( T(Y) \neq 0 \) and has its coefficients in \( C \) and is homogeneous of degree \( d \), isobaric of weight \( W \) and is of order \( \geq 2 \), then \( Q(Y) = c Y^{d - w} \cdot (x_0DY)^w + R(Y, x_0DY, \ldots, [x_0D]^pY) \) where \( R \) is the 0-image of \( T(Y) \) (see Definition 8.2). If the conclusion of the theorem is assumed false, then \( H(Y, x_0DY, \ldots, [x_0D]^pY) \equiv 0 \). Thus

\[
(V) \quad T(Y, x_0DY, \ldots, [x_0D]^pY) \equiv R(Y, x_0DY, \ldots, [x_0D]^pY).
\]

Since every term of \( T(Y) \) is of order \( \geq 2 \), \( T(x_0) = 0 \). Now suppose that for \( k > 0 \), \( T(x_{k-1}) = 0 \). Then

\[
x_0^wT(x_k) = R(x_k, x_0Dx_k, \ldots, [x_0D]^p x_k)
= T(x_k, x_0Dx_k, \ldots, [x_0D]^p x_k) \quad \text{by (V)}
= ET(x_{k-1}) \quad \text{by (IV)}
= 0.
\]

It follows by induction that \( T(x_p) = 0 \). This contradicts Theorem 7.1(v) and establishes the theorem.

References


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