AN IDENTITY FOR ELLIPTIC EQUATIONS
WITH APPLICATIONS

BY
C. A. SWANSONO

1. Introduction. An elementary identity involving a linear elliptic partial
differential operator $L$ and its associated hermitian form will be used to obtain new
comparison theorems, oscillation theorems, and lower bounds for eigenvalues.
Comparison theorems will be obtained for both subsolutions and complex-valued
solutions in unbounded domains of Euclidean space, generalizing earlier results of
Hartman and Wintner [4], Protter [8], and the author [11], [12]. Oscillation
theorems of Kreith's type [6] will be extended to (i) unbounded domains; (ii) non-
self-adjoint operators; and (iii) subsolutions.
Lower bounds for the eigenvalues of $L$ arise naturally from the basic identity in
the case of bounded domains, and are extended to unbounded domains when the
coefficients of $L$ satisfy suitable conditions. The form of the lower bounds is the
same as that obtained by Protter and Weinberger [9], [10] for bounded domains.

2. The main lemma. The linear elliptic differential operator $L$ defined by

$$Lv = \sum_{i,j=1}^{n} D_i(A_{ij}D_jv) + 2\sum_{i=1}^{n} B_iD_iv + Cv$$

will be considered on unbounded domains $R$ in $n$-dimensional Euclidean space $E^n$.
The boundary $P$ of $R$ is supposed to have a piecewise continuous unit normal vector
at each point. As usual, points in $E^n$ are denoted by $x= (x_1, x_2, \ldots, x_n)$ and differ-
entiation with respect to $x_i$ is denoted by $D_i$, $i=1, 2, \ldots, n$. The coefficients
$A_{ij}$, $B_i$, and $C$ are assumed to be real and continuous in $R \cup P$ and the matrix
$(A_{ij})$ positive definite in $R$ (ellipticity condition). The domain $\mathfrak{D}_L = \mathfrak{D}(R)$ of $L$ is
defined to be the set of all complex-valued functions $v \in C^1(R \cup P)$ such that all
derivatives of $v$ involved in $Lv$ exist and are continuous at every point in $R$.

Let $T_a$ denote the $n$-disk $\{x \in E^n : |x-x_0| < a\}$ and let $S_a$ denote the bounding
$(n-1)$-sphere, where $x_0$ is a fixed point in $E^n$. Define

$$R_a = R \cap T_a, \quad P_a = P \cap T_a, \quad C_a = R \cap S_a.$$  

Clearly there exists a positive number $a_0$ such that $R_a$ is a bounded domain with
boundary $P_a \cup C_a$ for all $a \geq a_0$.

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Let $Q[z]$ be the hermitian form in $n + 1$ variables $z_1, z_2, \ldots, z_{n+1}$ defined by

$$Q[z] = \sum_{i,j=1}^{n+i} A_{ij} z_i z_j - \sum_{i=1}^{n+i} B_i (z_i z_{n+1} + z_{n+1} z_i) + G|z_{n+1}|^2$$

where $G$ is any continuous function in $R$ satisfying the inequality

$$G \det (A_{ij}) \geq \sum_{i=1}^{n} B_i B_i^*,$$

$B_i^*$ denoting the cofactor of $-B_i$ in the matrix associated with $Q[z]$. Condition (4) is known to be necessary and sufficient for $Q[z]$ to be positive semidefinite [2], [12].

Let $M_a$ be the quadratic functional defined by

$$M_a[u] = \int_{R_a} F[u] \, dx,$$

where

$$F[u] = \sum_{i,j} A_{ij} D_i u D_j \bar{u} - 2 \Re \left( u \sum_i B_i D_i \bar{u} \right) + (G - C)|u|^2.$$

Define $M[u] = \lim_{a \to \infty} M_a[u]$ (whenever the limit exists). The domain $\mathcal{D}_M = \mathcal{D}_M(R)$ of $M$ is defined to be the set of all complex-valued functions $u \in C^1(R \cup P)$ such that $M[u]$ exists and $u$ vanishes on $P$.

Define

$$[u, v]_a = \int_{C_a} u \sum_{i,j} A_{ij} n_i n_j v \, ds,$$

where $(n_i)$ denotes the unit normal to $C_a$, and define

$$[u, v] = \lim_{a \to \infty} [u, v]_a,$$

whenever the limit on the right side exists. The notation $M[u; R]$ will be used for $M[u]$ and $[u, v; R]$ will be used for $[u, v]$ in §5 when different domains are under consideration.

An $L$-subsolution (-supersolution) is a real-valued function $v \in \mathcal{D}_L(R)$ which satisfies $Lv \leq 0$ ($Lv \geq 0$) at every point in $R$.

The following are extensions of results in [12] to subsolutions and supersolutions, and to complex-valued functions $u \in \mathcal{D}_M(R)$.

**Lemma 1.** For every $u \in C^1(R)$ and every real $v \in \mathcal{D}_L(R)$ which does not vanish in $R$, the following identity is valid at each point in $R$:

$$\sum_{i,j} A_{ij} X_i \bar{X}_j - 2 \Re \left( u \sum_i B_i \bar{X}_i \right) + G|u|^2 + \sum_i D_i(|u|^2 Y_i) = F[u] + |u|^2 v^{-1} Lv,$$

where

$$X_i = v D_i (u/v), \quad Y_i = v^{-1} \sum_{j=1}^{n} A_{ij} D_j v, \quad i = 1, 2, \ldots, n.$$

The proof is a direct calculation similar to that given in [12].
Theorem 1. If there exists \( u \in \mathcal{D}_M(R) \) not identically zero such that \( M[u] < 0 \), then there does not exist an \( L \)-subsolution (-supersolution) \( v \) satisfying \( [|u|^2/v, v] \geq 0 \) which is positive (negative) everywhere in \( R \cup P \). In particular, every real solution of \( Lv = 0 \) satisfying \( [|u|^2/v, v] \geq 0 \) must vanish at some point of \( R \cup P \). In the self-adjoint case \( B_i = 0, i = 1, 2, \ldots, n, \) and \( G = 0 \), the same conclusions are valid when the hypothesis \( M[u] < 0 \) is weakened to \( M[u] \leq 0 \).

Proof. Suppose to the contrary that there exists such a positive \( L \)-subsolution. Then integration of (9) over \( R_a \) yields

\[
\int_{R_a} F[u] \, dx \geq \int_{R_a} \sum_{i} D_i(|u|^2 Y_i) \, dx
\]

since the first three terms on the left side of (9) constitute a positive semidefinite form by the hypothesis (4). Since \( u = 0 \) on \( P_a \), by the definition of \( \mathcal{D}_M \), it follows from Green's formula that the right side of (10) is equal to

\[
\int_{R_a \cup C_a} \sum_{i} |u|^2 n_i Y_i \, ds = \int_{C_a} \frac{|u|^2}{v} \sum_{i,j} A_{ij} n_i D_j v \, ds = [|u|^2/v, v].
\]

Thus (7), (10), and the hypothesis \([|u|^2/v, v] \geq 0\) imply that

\[
M[u] = \lim_{a \to \infty} \int_{R_a} F[u] \, dx \geq 0.
\]

The contradiction proves that a positive \( L \)-subsolution satisfying \([|u|^2/v, v] \geq 0\) cannot exist. The analogous statement for a negative \( L \)-supersolution \( v \) follows from the fact that \(-v\) would then be a positive \( L \)-subsolution.

To prove the second statement of Theorem 1, suppose to the contrary that there exists a real solution \( v \neq 0 \) in \( R \cup P \). Then \( v \) would be either a positive \( L \)-subsolution or a negative \( L \)-supersolution in \( R \cup P \).

The proof in the self-adjoint case is similar to that given in [12, p. 281] and will be omitted.

We remark that the condition \([|u|^2/v, v] \geq 0\) of Theorem 1 is a mild "boundary condition at \( \infty \)" generalizing the usual condition \( v \neq 0 \) on the boundary of bounded domains.

3. Lower bounds for eigenvalues. Let \( \mathcal{H} \) be the Hilbert space \( \mathcal{L}^2(R) \), with inner product \( \langle u, v \rangle = \int_R u(x) \overline{v(x)} \, dx \) and norm \( \|u\| = \langle u, u \rangle^{1/2} \). Let \( \mathcal{D} \) be the set of all complex-valued functions \( u \in \mathcal{D}_L \cap \mathcal{H} \) such that \( u \) vanishes on \( P \). In this section the elliptic operator (1), with domain \( \mathcal{D} \), is assumed to have the self-adjoint form

\[
Lv = \sum_{i,j} D_i(A_{ij} D_j v) - Cv,
\]

under the conditions described below (1). In the case of the Schrödinger operator \(-L = -\Delta + C(x)\), it is well-known [1], [3, p. 146] that the lower part of the spectrum contains only eigenvalues of finite multiplicity if \( C(x) \) is bounded from below.
In the self-adjoint elliptic case, an assumption on the coefficients $A_{ij}$ is needed as well.

Let $A^+(x)$ denote the largest eigenvalue of $(A_{ij}(x))$ and define
\[
a(r) = \max_{1 \leq |x| \leq r} A^+(x),
\]
\[
a_0(r) = \max_{1 \leq |x| \leq r} [a(1), \max \{|x|^{-2} A^+(x)|\}],
\]
which are nondecreasing functions of $r$. The following assumptions are special cases of those given by Ikebe and Kato [5].

**Assumptions.** (i) $C(x)$ is bounded from below;
(ii) $\int_0^\infty [a(r)a_0(r)]^{-1/2} dr = \infty$.

It follows in particular from (i) and (ii) that the conditions $u \in \mathcal{D}$, $Lu \in \mathcal{D}$ imply that $|u, u| = 0$ [5].

Our purpose is to obtain a useful lower bound for the eigenvalues (if any) of $-L$.

In the case of bounded domains, Protter and Weinberger [10] recently obtained results of this type by using a general form of the maximum principle. It will be shown here in the case of unbounded domains that a lower bound is available as an easy consequence of Lemma 1.

**Theorem 2.** Let $\lambda$ be the lowest eigenvalue and $u$ be an associated normalized eigenfunction of the problem $-Lu = \lambda u$, $u \in \mathcal{D}$. If $v$ is any function in $\mathcal{D}$ such that $v(x) > 0$ for $x \in \mathbb{R} \setminus P$ and $[|u|^2/v, v] \geq 0$, then
\[
\lambda \geq \inf_{x \in \mathbb{R}} \left[-Lv(x)/v(x)\right].
\]

**Proof.** With $B_i = 0$, $i = 1, 2, \ldots, n$ and $G = 0$, integration of (9) over $R_a$ yields
\[
M_a[u] + \int_{R_a} |u|^2 v^{-1}Lv \, dx \geq \int_{R_a} \sum_i D_i(|u|^2 Y_i) \, dx
\]
where the positive-definiteness of $(A_{ij})$ has been taken into account. Since $u = 0$ on $P_a$, it follows from Green's formula that
\[
M_a[u] = -\int_{R_a} \bar{u}Lu \, dx + [u, u]_a.
\]
However, $\lim_{a \to \infty} [u, u]_a = 0$ is a general consequence of $u \in \mathcal{D}$ and $Lu \in \mathcal{D}$ under the above assumptions [5], and therefore
\[
M[u] = \lim_{a \to \infty} M_a[u] = \lambda \|u\|^2 = \lambda.
\]

As in the proof of Theorem 1, the right member of (12) has the limit $[|u|^2/v, v]$ as $a \to \infty$, which is nonnegative by hypothesis. Thus
\[
\lambda + \int_{\mathbb{R}} |u|^2 v^{-1}Lv \, dx \geq 0,
\]
which implies (11).
In the bounded case, the condition $|u|^2/v, v|^0$ is vacuous and Theorem 2 reduces to a well-known result [9]. However, the proof given here is especially easy. We remark that the extra condition $|u|^2/v, v|^0$ in the unbounded case is a condition on the asymptotic behavior of $v$ as $|x| \to \infty$; it is roughly equivalent to the usual hypotheses for bounded domains that $u=0$ on the boundary, $v > 0$ in $R \cup P$, and $v \in C^1(R \cup P)$. In the case of the Schrödinger operator $-\Delta + C(x)$, it is known [3, p. 179] that $|u(x)| < Ke^{-\mu|x|}$, where $K$ and $\mu$ are constants, for every eigenfunction $u$, and hence various exponential functions can serve as the test functions $v$. As an easy example, consider the one-dimensional harmonic oscillator problem

$$\frac{d^2 u}{dx^2} + x^2 u = \lambda u, \quad 0 \leq x < \infty,$$

$$u(0) = 0.$$

The test function $v = \exp(-x^2/2)$ yields the lower bound 1 whereas the exact lowest eigenvalue is known to be 3.

4. Comparison theorems. Consider, in addition to (1), a second elliptic operator $l$ defined by

$$(13) \quad lu = \sum_{i,j=1}^n a_{ij} D_i D_j u + 2 \sum_{i=1}^n b_i D_i u + cu$$

in which the coefficients satisfy the same conditions as the coefficients in (1). In addition to (5) consider the quadratic functional defined by

$$m_a[u; Q] = \int_{Q \cap \Omega_a} \left[ \sum_{i,j} a_{ij} D_i D_j u - 2 \sum_{i} b_i D_i u - c|u|^2 \right] dx$$

for every subdomain $Q \subset R$, and let $m[u; Q] = \lim m_a[u; Q] (a \to \infty)$. The domain $\mathcal{D}_m(Q)$ of $m$ is the analogue of $\mathcal{D}_M(Q)$ (defined in §2). The variation of $L$ relative to the domain $Q$ is defined as $V[u; Q] = m[u; Q] - M[u; Q]$, that is

$$V[u; Q] = \int_{Q} \left[ \sum_{i,j} (a_{ij} - A_{ij}) D_i D_j u - 2 \sum_{i} b_i (B_i - B_i) D_i u \right]$$

$$(14) \quad +(C - c - G)|u|^2 dx,$$

with domain $\mathcal{D}_V(Q) = \mathcal{D}_m(Q) \cap \mathcal{D}_M(Q)$.

The analogues of (7), (8) for the operator $l$ relative to the domain $Q$ are

$$(15) \quad \{u, v; Q\}_a = \int_{Q \cap \Omega_a} \sum_{i,j=1} a_{ij} n_i \Re (u D_j \overline{v}) ds;$$

$$(16) \quad \{u, v; Q\} = \lim_{a \to \infty} \{u, v; Q\}_a.$$

When $Q = R$ is the only domain under consideration, the abbreviations $V[u]$, $\{u, v\}$ will be used for $V[u; R]$, $\{u, v; R\}$, respectively.

The following comparison theorems of Sturm's type are easy extensions of those
Theorem 3. Suppose $G$ is a continuous function in $R$ satisfying the inequality (4). If there exists a nontrivial solution $u \in \mathcal{D}_v(R)$ of $lu=0$ such that $\{u, u\} \leq 0$ and $V[u] > 0$ then there does not exist an $L$-subsolution (-supersolution) which is positive (negative) everywhere in $R \cup P$ and satisfies $[|u|^2/v, v] \geq 0$. In particular, every real solution of $Lv=0$ satisfying $[|u|^2/v, v] \geq 0$ must vanish at some point of $R \cup P$. The same conclusions hold if the hypotheses $V[u] > 0$, $[|u|^2/v, v] \geq 0$ are replaced by $V[u] \geq 0$, $[|u|^2/v, v] > 0$, respectively.

Theorem 4. With $G$ as in Theorem 3, if there exists a positive $l$-supersolution $u \in \mathcal{D}_v(R)$ such that $\{u, u\} \leq 0$ and $V[u] > 0$, then the conclusions of Theorem 3 are valid.

Theorem 5 (Self-adjoint case). Suppose $b_i = B_i = 0$, $i = 1, 2, \ldots, n$ in (1) and (13) and $G = 0$. If there exists either (i) a nontrivial complex-valued solution $u \in \mathcal{D}_v(R)$ of $lu=0$, or (ii) a positive $l$-supersolution $u \in \mathcal{D}_v(R)$, such that $\{u, u\} \leq 0$ and $V[u] \geq 0$, then an $L$-subsolution (-supersolution) $v$ satisfying $[|u|^2/v, v] \geq 0$ cannot be everywhere positive (negative) in $R \cup P$. In particular, every real solution of $Lv=0$ satisfying $[|u|^2/v, v] \geq 0$ must vanish at some point of $R \cup P$.

Proof of Theorem 3. Since $u=0$ on $P_a$, it follows from Green's formula that

$$m_a[u] = - \int_{R_a} \text{Re} (u\overline{l}u) \, dx + \{u, u\}_a.$$ 

Since $lu=0$ and $l$ has real-valued coefficients, also $\overline{l}u=0$. Since $\{u, u\} \leq 0$, we obtain in the limit $a \to \infty$ that $m[u] \leq 0$. The hypothesis $V[u] > 0$ is equivalent to $M[u] < m[u]$. Hence $M[u] < 0$ and Theorem 1 shows an $L$-subsolution (-supersolution) cannot be everywhere positive (negative) in $R \cup P$ under the hypothesis $[|u|^2/v, v] \geq 0$. The second statement of Theorem 3 also follows from Theorem 1. The last statement follows upon obvious modifications of the inequalities.

If $u$ is a positive $l$-supersolution in $R$ such that $\{u, u\} \leq 0$, it follows again from (17) that $m[u] \leq 0$. The proof of Theorem 4 is then completed in the same way as that of Theorem 3. The proof of Theorem 5 follows similarly from the statement in Theorem 1 relative to the self-adjoint case.

It follows from (14) by partial integration that

$$V[u; Q] = \int_Q \left[ \sum_{i,j} (a_{ij} - A_{ij}) D_i u D_j \overline{u} + \delta |u|^2 \right] dx + \Omega(Q),$$

where

$$\delta = \sum_{i=1}^n D_i(b_i - B_i) + C - c - G,$$

and

$$\Omega(Q) = \lim_{a \to 0} \int_{Q \cap S_a} \sum_{i} (B_i - b_i) |u|^2 n_i \, ds,$$

whenever the limit exists.
$L$ is called a strict Sturmian majorant of $l$ in $Q$ when the following conditions are fulfilled: (i) $(a_{ij} - A_{ij})$ is positive semidefinite and $\delta \geq 0$ in $Q$; (ii) $\Omega(Q) \geq 0$; and (iii) either $\delta > 0$ at some point in $Q$ or $(a_{ij} - A_{ij})$ is positive definite and $c \neq 0$ at some point. A function defined in $Q$ is said to be of class $C^{2,1}(Q)$ when all of its second partial derivatives exist and are Lipschitzian in $Q$.

**Theorem 6.** Suppose that $L$ is a strict Sturmian majorant of $l$ and that all the coefficients $a_{ij}$ involved in $l$ are of class $C^{2,1}(R)$. If there exists a nontrivial solution $u \in \mathcal{D}_v(R)$ of $lu = 0$ such that $\{u, u\} \leq 0$, then no $L$-subsolution (-supersolution) $v$ satisfying $[|u|^2/v, v] \geq 0$ can be everywhere positive (negative) in $R \cup P$. In particular, every real solution of $Lv = 0$ satisfying $[|u|^2/v, v] \geq 0$ must vanish at some point of $R \cup P$.

**Theorem 7 (Self-adjoint case).** Suppose $b_i = B_i = 0$, $i = 1, 2, \ldots, n$ in (1) and (13), $G = 0$, $C \geq c$, and $(a_{ij} - A_{ij})$ is positive semidefinite in $R \cup P$. If there exists either (i) a nontrivial complex-valued solution $u \in \mathcal{D}_v(R)$ of $lu = 0$, or (ii) a positive $l$-supersolution $u \in \mathcal{D}_v(R)$, such that $\{u, u\} \leq 0$, then the conclusion of Theorem 6 is valid.

Since the pointwise conditions $G = 0$, $C \geq c$, and $(a_{ij} - A_{ij})$ positive semidefinite obviously imply that $V[u] \geq 0$, Theorem 7 is an immediate consequence of Theorem 5. The fact that the hypotheses of Theorem 6 imply $V[u] > 0$ was demonstrated in [12, p. 283], and consequently the conclusion of Theorem 6 follows from Theorems 3 and 4.

In the special case of the Schrödinger operator $-l = -\Delta + c(x)$ with $c(x)$ bounded from below in $R$, the hypothesis $\{u, u\} \leq 0$ of Theorems 5 and 7 can be replaced by $u \in \mathcal{D}$ and $lu \in \mathcal{D}$ since these conditions imply that $\{u, u\} = 0$ [3, p. 56]. In the self-adjoint elliptic case, the same statement holds under quite general conditions on the coefficients, e.g. those stated prior to Theorem 2, as shown by Ikebe and Kato [5]. Also, the conclusion of Theorem 7 is valid even if $(A_{ii})$ is only positive semidefinite provided $L$ is a strict Sturmian majorant of $l$ and all the coefficients $a_{ij}$ are of class $C^{2,1}(R)$ [12, p. 283].

5. Oscillation theorems. In [6] Kreith obtained oscillation theorems for self-adjoint elliptic equations of the form $Lw = 0$ in the case that one variable $x_a$ is separable. He considered the case of bounded domains for which part of the boundary is singular. Here we shall obtain oscillation theorems of a general nature on unbounded domains by appealing to the comparison Theorems 3–7.

Let $T_a$ denote the complement of $T_a$ in $E^n$. A function $u$ is said to be oscillatory in $R$ at $\infty$, or simply oscillatory in $R$, whenever $u$ has a zero in $R \cap T_a$ for all $a > 0$.

A domain (not necessarily bounded) $Q \subset R$ is called a nodal domain of a function $u$ if $u = 0$ on $\partial Q$ and $\{u, u; Q\} \leq 0$. If $Q$ is bounded, the latter condition is understood to be void, and the definition reduces to the standard definition of a nodal domain. If $-l$ is the Schrödinger operator with potential $c(x)$ bounded from below, sufficient
conditions for $Q$ to be a nodal domain of $u \in D_l(Q)$ are $u=0$ on $\partial Q$, $u \in \mathcal{D}$, and $lu \in \mathcal{D}$ [3, p. 56]. A function $u$ is said to have the nodal property in $R$ whenever $u$ has a nodal domain $Q \subset R \cap T_a$ for all $a > 0$.

The following results are immediate consequences of Theorems 3–7.

**Theorem 8.** Suppose $G$ is a continuous function in $R$ satisfying (4). Suppose there exists either (i) a nontrivial complex-valued solution $u$ of $lu=0$, or (ii) a positive $l$-supersolution $u$, with the nodal property in $R$ such that $V[u; Q] > 0$ for every nodal domain $Q$. Then every real solution of $Lv=0$ is oscillatory in $R$ provided $|u|^2,v; Q| \geq 0$ for every $Q$. If the nodal domains are all bounded, every solution of $Lv=0$ is oscillatory in $R$. In the self-adjoint case $b_i = B_i = 0$, $i = 1, 2, \ldots, n$, the same conclusions hold under the weaker condition $V[u; Q] \geq 0$ for every nodal domain $Q$.

**Theorem 9.** Suppose that $L$ is a strict Sturmian majorant of $I$ and that all the coefficients involved in $I$ are of class $C^{2,1}(R)$. If there exists a nontrivial complex-valued solution of $lu=0$ with the nodal property in $R$, then every real solution of $Lv = 0$ is oscillatory in $R$ provided $|u|^2v; Q| \geq 0$ for every nodal domain $Q$. If the nodal domains are all bounded, every solution of $Lv=0$ is oscillatory in $R$. In the self-adjoint case $b_i = B_i = 0$, $i = 1, 2, \ldots, n$, the same conclusions hold under the weaker hypotheses $G = 0$, $C \geq c$, and $(a_{ij} - A_{ij})$ positive semidefinite in $R \cup P$.

Kreith has shown [6] that equations of the form

$$D_n[a(x_n)D_nu] + \sum_{i,j=1}^{n-1} D_i[a_{ij}(\bar{x})D_ju] + c(x_n)u = 0, \quad \bar{x} = (x_1, x_2, \ldots, x_{n-1}),$$

have bounded nodal domains in the form of cylinders, under suitable hypotheses, when $R$ is a bounded domain with an $(n-1)$-dimensional singular boundary. We shall show that the analogous construction for unbounded domains is valid provided $R$ is limit cylindrical, i.e. contains an infinitely long cylinder. Without loss of generality we can assume that $R$ contains a cylinder of the form

$$G \times \{x_n: 0 \leq x_n < \infty\},$$

where $G$ is a bounded $(n-1)$-dimensional domain.

Let $\mu$ be the smallest eigenvalue of the boundary problem

$$-\sum_{i,j=1}^{n-1} D_i[a_{ij}(\bar{x})D_j\phi] = \mu \phi \quad \text{in} \ G,$$

$$\phi = 0 \quad \text{on} \ \partial G.$$

**Theorem 10.** If there exists a positive number $b$ such that

$$\int_b^\infty \frac{dt}{a(t)} = \infty \quad \text{and} \quad \int_b^\infty (c(t)-\mu) \, dt = \infty,$$

then equation (18) has a solution $u$ with the nodal property in $R$. If $V[u; Q] \geq 0$ for every nodal domain $Q$, every solution of $Lv=0$ is oscillatory in $R$. In particular,
Every solution of the self-adjoint equation \( Lw = 0 \) is oscillatory provided \( C \geq c \) and \( (a_i - A_i) \) is positive semidefinite in \( R \cup P \).

**Proof.** The hypotheses (20) imply that the ordinary differential equation

\[
D_n[a(x_n)D_nw] + [c(x_n) - \mu]w = 0
\]

is oscillatory at \( x_n = \infty \) on account of well-known theorems of Leighton [7] and Wintner [13]. Let \( w \) be a solution with zeros at \( x_n = \delta_1, \delta_2, \ldots, \delta_m, \ldots \), where \( \delta_m \uparrow \infty \). If \( \phi \) is an eigenfunction of (19) corresponding to the eigenvalue \( \mu \), then the function \( u \) defined by \( u(x) = w(x_n)\phi(x) \) is a solution of (18) by direct calculation, with nodal domains in the form of cylinders

\[
G_m = G \times \{x_n : \delta_m < x_n < \delta_{m+1}\}, \quad m = 1, 2, \ldots
\]

Thus \( u \) has a nodal domain \( G_m \cap T_a' \) for all \( a > 0 \). In fact, given \( a > 0 \), choose \( m \) large enough so that \( \delta_m \geq a \). Then \( x \in G_m \) implies \( |x| \geq |x_n| > a \) so \( x \in T_a' \). Hence (18) has a solution \( u \) with the nodal property. The second statement of Theorem 10 follows from Theorem 8 and the last statement follows from Theorem 9.

**References**