ON HAMILTONIAN LINE-GRAPHS(1)

BY

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Introduction. The line-graph $L(G)$ of a nonempty graph $G$ is the graph whose point set can be put in one-to-one correspondence with the line set of $G$ in such a way that two points of $L(G)$ are adjacent if and only if the corresponding lines of $G$ are adjacent. In this paper graphs whose line-graphs are eulerian or hamiltonian are investigated and characterizations of these graphs are given. Furthermore, necessary and sufficient conditions are presented for iterated line-graphs to be eulerian or hamiltonian. It is shown that for any connected graph $G$ which is not a path, there exists an iterated line-graph of $G$ which is hamiltonian.

Some elementary results on line-graphs. In the course of the article, it will be necessary to refer to several basic facts concerning line-graphs. In this section these results are presented. All the proofs are straightforward and are therefore omitted. In addition a few definitions are given.

If $x$ is a line of a graph $G$ joining the points $u$ and $v$, written $x=uv$, then we define the degree of $x$ by $\deg x=\deg u+\deg v-2$. We note that if $w$ is the point of $L(G)$ which corresponds to the line $x$, then the degree of $w$ in $L(G)$ equals the degree of $x$ in $G$. A point or line is called odd or even depending on whether it has odd or even degree.

If $G$ is a connected graph having at least one line, then $L(G)$ is also a connected graph. For the most part then, we restrict ourselves to connected graphs for otherwise each connected component can be treated individually.

By $L^2(G)$ we shall mean $L(L(G))$ and, in general, $L^n(G)=L(L^{n-1}(G))$ for $n \geq 1$, where $L^1(G)$ and $L^0(G)$ stand for $L(G)$ and $G$, respectively.

Two classes of graphs which have easily determined line-graphs are the cycles and simple paths. In particular, the line-graph of a cycle is a cycle of the same length, and the line-graph of a simple path of length $n$, $n \geq 1$, is a simple path of length $n-1$. It therefore follows that if $G$ is a path of length $n$, $n \geq 1$, then $L^n(G)$ is the trivial path consisting of a single point while $L^n(G)$ does not exist for $m>n$. It is not difficult to see that if $G$ is a connected graph which is not a path, then $L^n(G)$ exists for all positive integers $n$. Hence, if for some graph $G$, we wish to consider the infinite sequence $\{L^n(G)\}$ of graphs, then $G$ must not be a path.

A bridge of a connected graph $G$ is a line whose removal disconnects $G$, while a cutpoint of $G$ is a point $w$ of $G$ such that the removal of $w$ and all its incident lines

Received by the editors July 25, 1967.

(1) Definitions not presented in this article may be found in [3]. Work supported in part by a grant from the National Science Foundation (GN-2544).

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results in a disconnected graph. In relation to this, we state the following three results.

**Proposition 1.** A necessary and sufficient condition that a point \( w \) of the line-graph \( L(G) \) of a connected graph \( G \) be a cutpoint is that it corresponds to a bridge \( x=uv \) of \( G \) in which neither of the points \( u \) and \( v \) has degree one.

**Proposition 2.** A necessary and sufficient condition that a line \( x=u_1u_2 \) be a bridge of the line-graph \( L(G) \) of a connected graph \( G \) is that the lines \( y_1 \) and \( y_2 \) in \( G \) which correspond to \( u_1 \) and \( u_2 \) be bridges which meet in a point of degree two.

**Proposition 3.** A necessary and sufficient condition that the iterated line-graph \( L^n(G) \) of a connected graph \( G \) contain a bridge \( x \) is that \( G \) contain a path of \( n+1 \) bridges, each consecutive two of which have a point of degree two in common.

**Eulerian line-graphs.** A graph \( G \) is called **eulerian** if it has a closed path which contains every line of \( G \) exactly once and contains every point of \( G \). Such a path is referred to as an **eulerian path.**

Eulerian graphs have been characterized by Euler [2] as those graphs which are connected and in which every point is even. It follows trivially that if \( G \) is an eulerian graph, then \( L(G) \) too is eulerian; furthermore, if \( G \) is eulerian, then the sequence \( \{L^n(G)\} \) contains only eulerian graphs. We now state necessary and sufficient conditions for a graph \( G \) in order that there exists a nonnegative integer \( n \) such that \( L^n(G) \) is eulerian. Again the proof is routine and is omitted.

**Proposition 4.** Let \( G \) be a connected graph which is not a simple path. Then exactly one of the following must occur:

1. \( G \) is eulerian,
2. \( L(G) \) is eulerian but \( G \) is not,
3. \( L^2(G) \) is eulerian but \( L(G) \) is not,
4. there exists no \( n \geq 0 \) such that \( L^n(G) \) is eulerian,

where

1. occurs if and only if every point of \( G \) is even,
2. occurs if and only if every point of \( G \) is odd,
3. occurs if and only if every line of \( G \) is odd, and
4. occurs otherwise.

**Corollary 4a.** Let \( G \) be a connected graph which is other than a simple path. If the sequence \( \{L^n(G)\} \) of iterated line-graphs of \( G \) contains an eulerian graph, then the degrees of the lines of \( G \) are of the same parity and \( L^n(G) \) is eulerian for \( n \geq 2 \).

**Hamiltonian line-graphs.** A graph \( G \) is called **hamiltonian** if \( G \) has a cycle containing all the points of \( G \); such a cycle is also called **hamiltonian.** If \( C \) is a hamiltonian cycle of hamiltonian graph \( G \), then any line of \( G \) which does not lie on \( C \) is referred to as a **diagonal** of \( C \). Clearly, every hamiltonian graph is connected and has at least three points.
The following concept will be of considerable use to us. A graph $G$ with $q$ lines, where $q \geq 3$, is called sequential if the lines of $G$ can be ordered as $x_0, x_1, \ldots, x_{q-1}, x_q = x_0$ so that $x_i$ and $x_{i+1}$, $i = 0, 1, \ldots, q-1$, are adjacent. Two types of graphs in which we are interested are sequential, as we now see.

**Proposition 5.** Every eulerian graph is sequential.

**Proof.** If $G$ is an eulerian graph, then $G$ contains a closed path $P$ containing each line of $G$ exactly once, say $P: x_0, x_1, \ldots, x_{q-1}, x_q = x_0$, where $x_i$ and $x_{i+1}$ are adjacent for $i = 0, 1, \ldots, q-1$. This ordering of the lines of $G$ serves to show that $G$ is sequential.

**Proposition 6.** Every hamiltonian graph is sequential.

**Proof.** Let $C$ be a hamiltonian cycle of a hamiltonian graph $G$ whose points are arranged cyclically as, say, $v_0, v_1, \ldots, v_{p-1}, v_p = v_0$. To show that $G$ is sequential, we exhibit an appropriate ordering of the lines of $G$. We begin the sequence of lines by selecting all those diagonals incident with $v_0$ (there may be none). These lines may be taken in any order, and, clearly, each two are adjacent with each other. We follow these with the line $v_0v_1$. The next lines in the sequence are those diagonals incident with $v_1$ (again, there may be none). As before, these lines may be taken in any order. The next line in the sequence is $v_1v_2$, followed by all those diagonals incident with $v_2$ which are not in the part of the sequence already formed. We continue this until we finally arrive at the line $v_{p-1}v_p = v_0v_1$, which is adjacent with the first line in the sequence. From the way the sequence was produced, it is now clear that every line of $G$ appears exactly once and that any two consecutive lines in the sequence are adjacent as are the first and last lines. Thus $G$ is sequential.

The primary purpose for introducing sequential graphs lies in the following theorem.

**Theorem 1.** A necessary and sufficient condition that the line-graph $L(G)$ of a graph $G$ be hamiltonian is that $G$ is sequential.

**Proof.** The result follows by simply observing that the points of $L(G)$ can be ordered $v_0, v_1, \ldots, v_{p-1}, v_p = v_0$, where $v_i$ and $v_{i+1}$ are adjacent for $i = 0, 1, \ldots, p-1$, if and only if $L(G)$ is hamiltonian, and such an ordering is possible if and only if the lines of $G$ can be ordered $x_0, x_1, \ldots, x_{p-1}, x_p = x_0$, where $x_i$ and $x_{i+1}$ are adjacent for $i = 0, 1, \ldots, p-1$. This latter condition states that $G$ is sequential.

Propositions 5 and 6 and Theorem 1 yield the following corollaries.

**Corollary 1A.** If $G$ is an eulerian graph, then $L(G)$ is both eulerian and hamiltonian. Furthermore, $L^n(G)$ is both eulerian and hamiltonian for all $n \geq 1$.

**Corollary 1B.** If $G$ is a hamiltonian graph; then $L(G)$ is hamiltonian. Furthermore, $L^n(G)$ is hamiltonian for all $n \geq 1$. 
As with eulerian graphs, we now determine for what connected graphs $G$ which are not simple paths does the sequence $\{L^n(G)\}$ contain a hamiltonian graph. Unlike the situation for eulerian graphs, however, we find that for all connected graphs $G$ which are not simple paths, the sequence $\{L^n(G)\}$ contains a (in fact, infinitely many) hamiltonian graph. A proof of this was outlined in [1]. Before proving it in detail, we present two lemmas.

**Lemma 1.** If a graph $G$ has a cycle $C$ with the property that every line of $G$ is incident with at least one point of $C$, then $L(G)$ is hamiltonian.

**Proof.** The graph $G$ stated in the lemma is sequential so that, by Theorem 1, $L(G)$ is hamiltonian. To see that $G$ is sequential, one need only observe that an appropriate ordering of the lines of $G$ can be accomplished by using the same procedure as that in Proposition 6 except that after considering all the diagonals at a given point, we next insert in the sequence all the lines of $G$ which are incident with that point but with no other point of $C$. After this, we proceed as before. The graph $G$ is therefore easily seen to be sequential.

**Lemma 2.** Let $G$ be a graph consisting of a cycle $C$, its diagonals, and $m$ paths $P_1, P_2, \ldots, P_m$ where (i) each path has precisely one endpoint in common with $C$ and (ii) for $i \neq j$, $P_i$ and $P_j$ are disjoint except possibly having an endpoint in common if this point is also common to $C$. Then, if the maximum of the lengths of the $P_i$ is $M$, $L^n(G)$ is hamiltonian for all $n \geq M$.

**Proof.** The line-graph $L(G)$ has the same properties as $G$ except that the length of each of the $m$ paths is one less than in $G$ so that the maximum length of the paths is $M - 1$. Thus, we can apply Lemma 1 to $L^{M-1}(G)$ thereby showing that $L^n(G)$ is hamiltonian for all $n \geq M$.

**Theorem 2.** Let $G$ be a connected graph which is not a simple path. If $G$ has $p$ points, then $L^n(G)$ is hamiltonian for all $n \geq p - 3$.

**Proof.** The proof is by induction on $p$. Later developments in the proof make it convenient to investigate individually all the graphs under consideration for which $p = 3, 4,$ or $5$. The only connected graph with three points which is not a path is a triangle, but this graph is already hamiltonian so that the result follows.

For $p = 4$, there are two connected graphs which are not simple paths and which are not already hamiltonian. We denote these graphs by $G_{4,1}$ and $G_{4,2}$; they are shown in Figure 1. One readily sees that the line-graph of each of these two graphs is hamiltonian, and the result is established for $p = 4$.

There are twelve connected graphs with five points which are not paths and which do not contain hamiltonian cycles. These are also shown in Figure 1. It is a routine matter to verify that $L^2(G_{5,1})$ and $L^2(G_{5,2})$ are hamiltonian and that $L(G_{5,i})$ is hamiltonian for $i = 3, 4, \ldots, 12$. This proves the theorem for $p = 5$.

Let us assume then for all connected graphs $G'$ which are not simple paths and which have $s$ points, where $s < p$ and $p \geq 5$, that $L^n(G')$ is hamiltonian for all
$n \geq s - 3$. Let $G$ be a connected graph with $p$ points which is not a simple path. We show that $L^{p-3}(G)$ is hamiltonian which, with the aid of Corollary 1B, establishes the result.

The theorem is clearly evident if $G$ itself is a cycle, so, without loss of generality, we assume $G$ is not a cycle implying the existence of a point $v$ having degree three or more. By $H$ we shall mean the connected star subgraph whose lines are all those incident with $v$, and we let $Q$ denote the subgraph whose point set consists of all the points of $G$ different from $v$ and whose lines are all those in $G$ which are not in $H$.

The subgraphs $H$ and $Q$ have $\deg v$ points in common but are line-disjoint. We
adopt the notation \( G = H \oplus Q \) to mean that the line set of \( G \) is partitioned by \( H \) and \( Q \). Also, we denote the connected components of \( Q \) by \( G_1, G_2, \ldots, G_k \), where \( G_i \) has \( p_i \) points for \( i = 1, 2, \ldots, k \). Clearly, \( \sum p_i = p - 1 \).

If \( G_i \) is a path, then \( L^{p_i}(G_i) \) does not exist whereas if \( G_i \) is not a path, then \( L^{p_i}(G_i) \) is hamiltonian for \( n \geq p_i - 3 \), by the inductive hypothesis.

The line-graph \( H_1 = L(H) \) is a complete subgraph of \( L(G) \), which, therefore, has a cycle containing all the points of \( H_1 \). Let \( J_1 \) denote the connected subgraph of \( L(G) \) consisting of \( H_1 \) and all those lines incident with a point of \( H_1 \). Thus, \( L(G) \) can be expressed as \( J_1 \oplus L(G_1) \oplus L(G_2) \oplus \cdots \oplus L(G_k) \), where \( L(G_1) \) and \( L(G_2) \) are disjoint for \( i \neq j \).

Now let \( H_2 = L(J_1) \) and let \( J_2 \) denote the connected subgraph of \( L^2(G) \) consisting of \( H_2 \) and all lines incident with a point of \( H_2 \). By Lemma 1, \( H_2 \) has a cycle containing all the points of \( H_2 \). Thus, \( L^2(G) = J_2 \oplus L^2(G_1) \oplus L^2(G_2) \oplus \cdots \oplus L^2(G_k) \).

In general, we let \( J_m \) denote the connected subgraph of \( L^m(G) \) consisting of \( H_m \) and all those lines incident with a point of \( H_m \) and let \( H_{m+1} = L(J_m) \), where \( H_{m+1} \) has a cycle containing all the points of \( H_{m+1} \) by Lemma 1. The graph \( L^m(G) \) can therefore be expressed as \( J_m \oplus L^m(G_1) \oplus L^m(G_2) \oplus \cdots \oplus L^m(G_k) \).

We now consider two cases.

Case 1. Suppose each of the components \( G_1, G_2, \ldots, G_k \) of \( Q \) is a path. (This includes the possibility that some of these components may be the trivial path consisting of a single point.)

If \( k \geq 3 \), then \( p_i \leq p - 3 \) for all \( i \). Hence, \( L^{p-3}(G) = H_{p-3} \), which contains a hamiltonian cycle. If \( k = 2 \) and neither \( p_1 \) nor \( p_2 \) exceeds \( p - 3 \), then, as before, \( L^{p-3}(G) = H_{p-3} \). If, on the other hand, \( k = 2 \) and one component, say \( G_1 \), has \( p - 2 \) points while \( G_2 \) is a single point, then \( H \) and \( G_1 \) have at least two points in common. Thus \( G \) contains a cycle plus possibly diagonals and \( j \) pairwise disjoint paths, \( 1 \leq j \leq 3 \), each path having precisely one endpoint in common with the cycle. Since none of these paths has length exceeding \( p - 4 \), it follows, by Lemma 2, that \( L^{p-4}(G) \) (and so also \( L^{p-3}(G) \)) contains a hamiltonian cycle.

If \( k = 1 \), then \( Q \) is a path having at least three points in common with \( H \) so that \( G \) consists of a cycle (with some diagonals) and \( j \) pairwise disjoint paths, \( 0 \leq j \leq 2 \), each path having exactly one endpoint in common with the cycle. If \( j = 0 \), \( G \) is hamiltonian while if \( j > 0 \), no path extending from the aforementioned cycle can have length exceeding \( p - 4 \), and by Lemma 2, \( L^{p-4}(G) \) is hamiltonian as is \( L^{p-3}(G) \).

Case 2. Assume the first \( t \) subgraphs, \( 1 \leq t \leq k \), of \( G_1, G_2, \ldots, G_k \) are not paths. Clearly, then, each of \( G_1, G_2, \ldots, G_t \) has at least three points.

If \( t < k \), then \( G_{t+1}, G_{t+2}, \ldots, G_k \) are paths, each having at most \( p - 4 \) points so that \( L^{p-4}(G) = J_{p-4} \oplus L^{p-4}(G_1) \oplus L^{p-4}(G_2) \oplus \cdots \oplus L^{p-4}(G_k) \). Since each \( G_i \), \( 1 \leq i \leq t \), has at most \( p - 1 \) points, each subgraph \( L^{p-4}(G_i) \) of \( L^{p-4}(G) \) has a cycle containing all points of \( L^{p-4}(G_i) \) by the inductive hypothesis.

For each \( i = 1, 2, \ldots, t \), there is clearly at least one line joining a point of \( H_{p-5} \)
to a point of $L^{p-5}(G_t)$. We now show that for each $i$ such a line exists with the added property that it is adjacent with at least two lines of $L^{p-5}(G_t)$.

Suppose $t=1$ so that $G_1$ is the only component of $Q$ which is not a path. If $k>1$, then $G_1$ has at most $p-2$ points so that $L^{p-5}(G_1)$ contains a hamiltonian cycle and clearly such a line exists. If $k=1$, then $Q=G_1$ and all lines of $H$ are incident with points of $G_1$. Since each line which joins $H_m$ to $L^m(G_1)$ results in one or more lines joining $H_{m+1}$ with $L^{m+1}(G_1)$, there are at least three lines joining $H_{p-5}$ and $L^{p-5}(G_1)$. If no such line is adjacent with at least two lines of $L^{p-5}(G_1)$, then each of the three or more lines joining $H_{p-5}$ and $L^{p-5}(G_1)$ is adjacent with precisely one line of $L^{p-5}(G_1)$. Hence, $L^{p-5}(G_1)$ contains at least three lines which are incident with points of degree one, i.e., $L^{p-5}(G_1)$ contains at least three bridges. By Proposition 3, $G_1$ must contain a path of $p-4$ bridges for each bridge of $L^{p-5}(G_1)$. Since the bridges of $L^{p-5}(G_1)$ are incident with points of degree one and since $L^{p-5}(G_1)$ is not itself a path, the three or more paths of $G_1$ are line-disjoint. This implies that $G_1$ contains at least $3(p-4)+1$ points but since $p \geq 6$, $3(p-4)+1 > p-1$, which contradicts the number of points in $G_1$.

Suppose next that $t>1$, i.e., suppose $Q$ contains two or more components which are not paths. Therefore, $G_1$ and $G_2$ are not paths, and each contains at most $p-4$ points. If there is a line joining a point of $H_{p-5}$ to a point of $L^{p-5}(G_1)$, say, which is adjacent with only one line of $L^{p-5}(G_1)$, then $L^{p-5}(G_1)$ contains a bridge implying that $G_1$ contains a path of $p-4$ bridges, but this contradicts the number of points of $G_1$.

We therefore conclude that for each $i=1,2,\ldots,t$, there exists a line joining $H_{p-5}$ and $L^{p-5}(G_t)$ which is adjacent to two lines of $L^{p-5}(G_t)$. This implies that for each $i=1,2,\ldots,t$, there is a point $u_i$ in $H_{p-4}$ adjacent to both endpoints of a line in $L^{p-4}(G_1)$. It is not difficult to see that $u_i \neq u_j$ for $i \neq j$. Let $x_{i1}$ and $x_{i2}$ be lines of $L^{p-4}(G)$ which join $u_i$ to the distinct endpoints of a line $y_i$ of $L^{p-4}(G_i)$.

We now claim that $L^{p-4}(G)$ is a sequential graph so that $L^{p-5}(G)$ is hamiltonian. Recall first that $L^{p-4}(G_i)$ for $1 \leq i \leq t$ has a cycle containing all the points of $L^{p-4}(G_i)$ and so is sequential by Proposition 6. Thus for $1 \leq i \leq t$, the lines of $L^{p-4}(G_i)$ can be arranged in a sequence $s_i$ such that each pair of successive lines in $s_i$ are adjacent and the first and last lines in $s_i$ are adjacent. Let $z_i$ be the term following $y_i$ in $s_i$ (or the first term of $s_i$ if $y_i$ is the last term). Now $y_i$ is adjacent to both $x_{i1}$ and $x_{i2}$, and $z_i$, being adjacent to $y_i$, is adjacent to one of $x_{i1}$ and $x_{i2}$. Therefore, by cyclically permuting the terms of $s_i$ if necessary and reversing their order if necessary, we can convert $s_i$ into a sequence $s'_i$ whose first and last terms are adjacent to $x_{i1}$ and $x_{i2}$, respectively. Now $H_{p-4}$ has a cycle $C$ containing all the points of $H_{p-4}$ and every line of $J_{p-4}$ is incident with at least one point of $C$. Therefore, the procedure of the proof of Lemma 1 enables us to order the lines of $J_{p-4}$ in a sequence $(s, \text{ say})$ such that each pair of successive lines in $s$ are adjacent as are the first and last lines. Moreover, since $x_{i1}$ and $x_{i2}$ are lines incident with the point $u_i$ of $C$ and with no other point of $C$, it is evident that, in applying the procedure of the proof
of Lemma 1, we can arrange the lines incident with \( u_i \) so that \( x_{i2} \) will immediately follow \( x_{i1} \) in \( s \) for \( i = 1, 2, \ldots, t \). If we now insert the sequence \( s_i' \) between the terms \( x_{i1} \) and \( x_{i2} \) of \( s \) for \( i = 1, 2, \ldots, t \), the resulting sequence has the properties required for \( L^{n-4}(G) \) to be a sequential graph. This completes the proof.

The preceding theorem now permits us to make the following definition. Let \( G \) be a connected graph which is not a simple path. The \textit{hamiltonian index} of \( G \), denoted \( h(G) \), is the smallest nonnegative integer \( n \) such that \( L^n(G) \) is hamiltonian. According to Theorem 2 then, if \( G \) is a connected graph with \( p \) points which is not a simple path, then \( h(G) \) exists and \( h(G) \leq p - 3 \). This bound cannot, in general, be improved since for each \( p \geq 3 \) the graph whose point set is \( \{v_i \mid 1 \leq i \leq p\} \) and whose line set is \( \{v_2v_3\} \cup \{v_{i-1}v_{i+1} \mid 1 \leq i \leq p - 1\} \) has a hamiltonian index of \( p - 3 \).

\textbf{References}


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