A CHARACTERIZATION OF THE RIESZ SPACE OF MEASURABLE FUNCTIONS

BY

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1. Introduction. In [2], Kakutani has shown that any abstract (L)-space (Banach lattice such that \( \|u + v\| = \|u\| + \|v\| \) for \( u \geq 0 \) and \( v \geq 0 \)) can be represented as a space \( L_1(X, \mathcal{S}, \mu) \) of equivalence classes of \( \mu \)-summable functions, where \( \mu \) is a completely additive measure on a \( \sigma \)-algebra of subsets of some set \( X \). This result can also be viewed as a characterization of the class of spaces \( L_1(X, \mathcal{S}, \mu) \) by properties of the norm and order.

The purpose of this note is to show that an analogous classification exists for the class of spaces \( M(X, \mathcal{S}, \mu) \) (the set of equivalence classes of almost-everywhere finite valued \( \mu \)-measurable functions on the measure space \( (X, \mathcal{S}, \mu) \)) as long as \( \mu \) is restricted to be a \( \sigma \)-finite measure. Since such spaces can not be normed, the characterization will involve only order properties. The above-mentioned classical theorem of Kakutani will be of significant value in obtaining our result.

The principal condition needed is a reflexivity property for Riesz spaces (vector lattices) which we will study in §2. Lastly, we examine the non \( \sigma \)-finite case.

2. Riesz space preliminaries. Let \( L \) be an Archimedean Riesz space throughout this paper. A linear functional \( \varphi \) defined on \( L \) is said to be a normal integral if \( 0 \leq u \downarrow 0 \) in \( L \) implies that \( \inf |\varphi(u)| = 0 \). The space \( L_\varphi \) of normal integrals is a closed ideal in the order-bound dual of \( L \). (Further explanation of the definitions and notations used may be found in [3].)

In [3], the concept of the normal integral is generalized by considering functionals that are normal integrals for some dense ideal \( D \subset L \). This gives rise to an extended dual space which we denote by \( \Gamma(L) \). The space \( \Gamma(L) \) is a Dedekind complete and universally complete Riesz space, containing \( L_\varphi \) as an ideal [3, Theorems 2.5 and 2.6].

If \( \Gamma(L) \) is separating on \( L \), then \( L \) is an order-dense Riesz subspace of \( \Gamma^0(L) \). If \( \Gamma^0(L) = L \), we say that \( L \) is perfect in the extended sense. This is the reflexivity condition referred to in the introduction. The following result [3, Theorem 2.7] is a characterization which will be used:

(A) A Riesz space \( L \), for which \( \Gamma(L) \) is separating, is perfect in the extended sense if and only if \( 0 \leq u, u \uparrow \) and

\[
\sup \varphi(u) < \infty
\]
for every $0 \leq \varphi \in \Gamma(L)$ which belongs to some order-dense ideal in $\Gamma(L)$, implies that $\sup (u_i) = u$ exists.

Notice that the condition $\Gamma^2(L) = L$ implies that $L$ is Dedekind complete but that in the general non $\sigma$-finite case, $\mathcal{M}(X, \mathcal{S}, \mu)$ need not be Dedekind complete. This gives rise to the restriction on the measure $\mu$.

We recall some definitions and facts about order completions of a Riesz space. $L$ is said to be universally complete if $0 \leq u_i \in L$ and $\inf (u_{i_1}, u_{i_2}) = 0$ for $i_1 \neq i_2$ imply that $u = \sup (u_i)$ exists in $L$. Nakano has shown [5] that every Archimedean Riesz space can be embedded as a dense Riesz subspace, in a Dedekind complete and universally complete Riesz space $L'$ in such a way that suprema taken in $L$ are preserved when taken in $L'$. The latter space is unique (up to isomorphism) and is called the universal completion of $L$. Moreover, if $L$ is Dedekind complete, then it is an ideal in $L'$. From the uniqueness it follows that if $L$ is universally complete and Dedekind complete and $D$ is a dense ideal in $L$, then $L$ is the universal completion of $D$. Hence, if $\Gamma(L)$ is separating on $L$, implying $L$ is dense in $\Gamma^2(L)$, then $L$ is Dedekind complete and universally complete if and only if $L$ is perfect in the extended sense.

3. The characterization. Given any element $\psi$ in $\Gamma(L)$, there is a unique largest order-dense ideal to which $|\psi|$ can be extended finitely. We denote it by $D_\psi$. $\psi|D_\psi$ is a normal integral. Moreover, a positive element $u$ in $L$ is in $D_\psi$ if and only if

$$\sup \{ \psi(v) : 0 \leq v \leq u, v \in I_\psi \} < \infty,$$

where $I_\psi$ is any dense ideal on which $\psi$ is defined (that is, finitely defined) [3]. Moreover, Theorem 2.5 of the paper just cited says that if $\Gamma(L)$ is separating on $L$, there exists in $\Gamma(L)$ a strictly positive linear functional $\varphi$, that is, one such that $\varphi(u) > 0$ for all $u \in D_\varphi$, $u > 0$. We now state the central result.

**Theorem.** Let $L$ be a Riesz space which is Dedekind complete, universally complete and such that $\Gamma(L)$ is separating on $L$. Then $L$ contains a dense ideal which is an abstract $(L)$-space.

**Proof.** Let $\varphi$ be a strictly positive element in $\Gamma(L)$. Let $D$ be the (unique) largest order-dense ideal to which $|\varphi|$ can be extended finitely. For any $u$ in $L$ define $\|u\| = \varphi(|u|)$. Then $\|u\| \geq 0$; equals zero if and only if $u = 0$ since $\varphi$ is strictly positive. $\|au\| = |a| \|u\|$ for a scalar $a$ and $\|u + v\| \leq \|u\| + \|v\|$ follow from the linearity of $\varphi$. That $\|u\| \leq \|v\|$ implies $\|u\| \leq \|v\|$ results from the positivity of $\varphi$. So, with $\| \cdot \|$ as norm, $D$ is a normed vector lattice. If $u \geq 0$ and $v \geq 0$, $\|u + v\| = \varphi(|u + v|) = \varphi(u + v) = \varphi(u) + \varphi(v) = \varphi(|u|) + \varphi(|v|) = \|u\| + \|v\|$. To show that $D$ is an abstract $(L)$-space then, it remains only to prove that $D$ is a complete space under $\| \cdot \|$.

Let $\langle u_n \rangle$ be an absolutely convergent series in $D$. So, there is $M$ such that $\sum_{n=1}^{\infty} \|u_n\| \leq M < \infty$. But then $\|u_n^+\| \leq \|u_n\|$ and $\|u_n^-\| \leq \|u_n\|$ imply that $\langle u_n^+ \rangle$ and $\langle u_n^- \rangle$ are absolutely convergent series in $D$ with the same absolute bound $M$. 

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Let \( v_n = \sum_{k=1}^{n} u_k^+ \). Then, \( \langle v_n \rangle \) is an increasing sequence in \( D \) and \( \|v_n\| \leq M, n=1, 2, \ldots \). So, \( \varphi(v_n) \leq M, n=1, 2, \ldots \) and hence \( \sup_n \varphi(v_n) < \infty \). For any \( \psi \) in \( I \), then, where \( I \) is the principal ideal generated by \( \varphi \), we have
\[
\sup_n \psi(v_n) < \infty.
\]

However, since \( \varphi \) is strictly positive, \( I \) is an order-dense ideal in \( \Gamma(L) \). Hence, the conditions in the conclusion of (A) are satisfied for the sequence \( \{v_n\} \). Since \( L \) is perfect in the extended sense (being Dedekind complete and universally complete) then, \( v = \sup (v_n) \) must exist in \( L \). Now, \( \sup \varphi(v_n) \leq M \) implies that
\[
\sup \{ \varphi(w) : 0 \leq w \leq v, w \in D \} < \infty.
\]
(For if \( 0 \leq w \leq v \), \( \sup (\inf (w, v_n)) = w \), hence \( \varphi(w) \leq M \).) By remarks made above, then, \( v \) is in \( D \), and since \( \varphi \) is a normal integral on \( D \), \( \lim \varphi(v_n) = \varphi(v) \). So, \( \lim \|v - v_n\| = 0 \), hence \( \lim \|v - \sum_{k=1}^{n} u_k^+\| = 0 \). In the same way, we obtain \( v' \) in \( D \) such that \( \lim |v' - \sum_{k=1}^{n} u_k^+| = 0 \). But then, \( \lim \|v - v' - \sum_{k=1}^{n} u_k^+\| = 0 \). We have shown that every absolutely summable series in \( D \) is summable. Hence, \( D \) is complete, finishing the proof.

Using Kakutani’s representation theorem, the ideal \( D \) is isomorphic to a space \( L_1 = L_1(X, \mathcal{S}, \mu) \) (\( \mu \) may not be \( \sigma \)-finite). Let \( \mathcal{M} = \mathcal{M}(X, \mathcal{S}, \mu) \) be the associated space of equivalence classes of measurable functions. Now \( D \) (or \( L_1 \)) may be regarded as a dense ideal in both \( \mathcal{M} \) and \( L \). Hence, using [3, Theorems 2.6 and 2.3] (along with the hypotheses of the theorem) we have \( \mathcal{M} \subset \Gamma^2(\mathcal{M}) = \Gamma^2(L) = L \). Moreover, \( L \) is the universal completion of \( \mathcal{M} \). So,

COROLLARY 1. Let \( L \) be a Riesz space which is Dedekind complete, universally complete and such that \( \Gamma(L) \) is separating on \( L \). Then, \( L \) contains a dense Riesz subspace which is isomorphic to \( \mathcal{M}(X, \mathcal{S}, \mu) \).

The Riesz space \( L \) is said to be super Dedekind complete if \( L \) is Dedekind complete and if any subset \( A \subseteq L \) which is bounded above contains an, at most, countable subset having the same least upper bound as the whole set \( A \).

This property was introduced by Luxemburg and Zaanen in [4] and studied in their later papers under the same title.

It can now be shown that if this condition is added to the hypotheses of the theorem, the measure \( \mu \) obtained in the corollary above must be \( \sigma \)-finite.

Let \( \{X_n\} \) be the class of all sets of finite \( \mu \)-measure. We must have then, that \( \chi_X = \sup (\chi_{X_n}) \) in \( L \) (\( \chi_X \) being the class containing the characteristic function of \( E \)). There must then be an at most countable subset \( \{X_n\} \) such that \( \chi_X = \sup (\chi_{X_n}) \). But then \( X = \bigcup_{n=1}^{\infty} X_n \) (except possibly for a set of \( \mu \) measure zero). Hence, \( \mu \) is a \( \sigma \)-finite measure on \( S \).

With \( \mu \) \( \sigma \)-finite, we can show that \( \mathcal{M} = L \). First of all, if \( \mu \) is \( \sigma \)-finite, then \( \mathcal{M} \) is Dedekind complete. [1, IV, p. 335.] The following two facts are also readily obtained
(a) \( \mathcal{M} \) contains a complete element i.e., there is \( 0 \leq e \in \mathcal{M} \) such that \( \inf (e, x) = 0 \) implies \( x = 0 \).

(b) \( \mathcal{M} \) contains the supremum of any countable disjoint set of its elements.

Nakano has shown [5] that for a Dedekind complete Riesz space, (a) and (b) are necessary and sufficient for universal completeness. So, \( \mathcal{M} \) is universally complete and hence, \( \mathcal{M} = L \).

We have obtained one half of the characterization described in the introduction. The other half is a rather routine verification. We state the result:

**Corollary 2.** Let \( L \) be a Riesz space. In order that there exist a completely additive \( \sigma \)-finite measure space \( (X, \mathcal{S}, \mu) \) such that \( L = \mathcal{M}(X, \mathcal{S}, \mu) \), the following conditions are necessary and sufficient:

1. \( \Gamma(L) \) is separating on \( L \),
2. \( L \) is universally complete,
3. \( L \) is super Dedekind complete.

4. The non \( \sigma \)-finite case. The last corollary cannot be modified to include any measure \( \mu \), since even the condition that \( \Gamma(L) \) be separating restricts the measure \( \mu \) to have the finite subset property, that is, any set of \( \mu \)-positive measure has a subset of \( \mu \)-finite measure (a fact we used in proving the last corollary).

As can be seen from the first two corollaries, a characterization in the non \( \sigma \)-finite case will obtain if a condition on \( \mu \) can be found which guarantees that \( \mathcal{M}(X, \mathcal{S}, \mu) \) is Dedekind complete. Such a condition appears in a paper of A. C. Zaanen [6] concerning an extension of the Radon-Nikodym Theorem. Namely, the measure \( \mu \) is said to be localizable if the lattice of equivalence classes of \( \mu \)-measurable sets is complete. (The term is originally due to I. E. Segal.) We discuss a characterization of this property.

Given a set \( E \) of finite \( \mu \)-measure, let \( M_E^\oplus \) be the collection of equivalence classes \( f^* \) of \( \mu \)-measurable functions \( f \) vanishing almost everywhere off \( E \). If for each such \( E \) we select \( f_E^* \) in \( M_E^\oplus \) in such a way that for \( E \) and \( F \) of finite measure we have \( (f_{E \cap F})^* = (f_F^* f_{E \cap F})^* = (f_{F \cap E})^* \), then the collection \( \{f_E^*\} \) is called a cross-section of \( X \). Theorem 9.4 in [6] states that the measure \( \mu \) is localizable if and only if for every cross-section \( \{f_E^*\} \), there is a measurable \( f \) defined on all of \( X \) such that \( (f_{X \setminus E})^* = f_{E^c}^* \) for all \( E \) of finite measure.

The value of localizability is that it characterizes those measure spaces for which there is a Radon-Nikodym Theorem.

Assuming now that \( \mu \) is localizable, let \( \{f_n^*\} \) be a collection of nonnegative elements in \( \mathcal{M} = \mathcal{M}(X, \mathcal{S}, \mu) \), bounded above by \( g \). Then, \( (f_{axE})^* \) is in \( M_E^\oplus \) for each set \( E \) of finite measure. But, \( M_E^\oplus \) is Dedekind complete since the restriction of \( \mu \) to \( E \) is a finite measure. Moreover, \( (f_{axE})^* \leq (g_{axE})^* \). Hence, \( \sup (f_{axE}) = f_E^* \) (supremum taken in \( \mathcal{M} \) since \( M_E^\oplus \) is an ideal in \( \mathcal{M} \)) exists and is an element of \( M_E^\oplus \).

Now it is easily verified that the collection \( \{f_E^*\} \) is a cross-section, and hence there exists \( f^* \) in \( \mathcal{M} \) such that \( (f_{xE})^* = f_E^* \) for each set \( E \) of finite measure. This last
statement is true, however, only if $f^* = \sup (f_n^*)$. So, $\mathcal{M}$ is complete and we have the following:

**COROLLARY 3.** Let $L$ be a Riesz space. The following conditions are necessary and sufficient that there exist a completely additive localizable measure $\mu$ such that $L = \mathcal{M}(X, \mathcal{S}, \mu)$:

1. $\Gamma(L)$ is separating,
2. $L$ is Dedekind complete and universally complete.

**REFERENCES**

2. S. Kakutani, *Concrete representation of abstract $(L)$-space and the mean ergodic theorem*, Ann. of Math. 42 (1941), 523–537.

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