THE FIRST HITTING DISTRIBUTION OF A SPHERE FOR SYMMETRIC STABLE PROCESSES

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1. Introduction. Let $X(t)$ be a symmetric stable process on $N$ dimensional Euclidean space $\mathbb{R}^N$, having exponent $\alpha$ and transition density

$$p(t, x) = (2\pi)^{-N} \int_{\mathbb{R}^N} \exp \left[-t|\theta|^\alpha \right] e^{-i(x, \theta)} d\theta.$$ 

We will always work with the version of the process $X(t)$ which is a standard Markov process. (See Chapter 1 of [1] for a complete description of a standard process.) For any $r > 0$ let $S_r = \{ y \in \mathbb{R}^N : |y| = r \}$ denote the sphere of center 0 and radius $r$. Set $T_r = \inf \{ t > 0 : |X(t)| = r \}$, and, as usual, set $T_r = \infty$ if $|X(t)| \neq r$ for all $t > 0$. The hitting measure and Green’s function of $S_r$ are respectively the quantities,

$$H_r(x, \omega) = \mathbb{P}_x(X(T_r) \in \omega, T_r < \infty)$$

and $g_r(x, y)$, where $g_r(x, y)$ is the density of the measure

$$\int_0^\infty \mathbb{P}_x(T_r > t, X(t) \in dy) dt.$$ 

The hitting probability of $S_r$ is $\Phi_r(x) = \mathbb{P}_x(T_r < \infty)$. Our purpose in this paper is to explicitly compute these as well as some related quantities.

In brief we will do the following. In §2 we introduce the radial process $Z_a(t)$ and use it to compute $\Phi_r(x)$ by the relation $\Phi_r(x) = \mathbb{P}_{|x|}(\tau_r < \infty)$, where $\tau_r = \inf \{ t > 0 : Z_a(t) = r \}$. The problem is trivial if $\alpha = 1$ (see Proposition 2.1) since $\{r\}$ is a polar set for $Z_a(t)$ in that case. Also, if the process is recurrent, then $\Phi_r(x) = 1$ so the only cases of interest are $1 < \alpha < N$. Our technique here is simply to note that

$$\Phi_r(x) = \frac{u(x, r)}{u(r, r)},$$

where $u(x, r)$ is the potential kernel of $Z_a(t)$, and to compute $u(x, y)$. But having $u(x, y)$ one may explicitly compute more elaborate probabilities, e.g.,

$$\mathbb{P}_x(\min_{1 \leq i \leq n} T_{r_i} = T_{r_n}).$$

Some of these computations will be also carried out in §2. In §3 we compute $H_r(x, \omega)$ by the method devised by M. Riesz [8] of inversion in an appropriate sphere, and in §4 we use the results of §2 and 3 to write down the Green’s function of $S_r$.

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Previously, the above quantities were computed for the solid ball by Blumenthal, Getoor, and Ray in [3] by the use of Riesz's inversion technique, and in [7] the author computed these quantities for arbitrary finite sets in the case of recurrent one-dimensional stable processes with exponent \( \alpha > 1 \).

2. The radial process. Since \( X(t) \) is isotropic, the process \( Z_\alpha(t) = |X(t)| \) is a Markov process, and since \( X(t) \) is a Feller process, it must be that \( Z_\alpha(t) \) is also a Feller process. Thus by §9 of [1], there is a realization of \( Z_\alpha(t) \) as a standard Markov process. We henceforth assume that \( Z_\alpha(t) \) is this version of the process. If \( \alpha = 2 \), then \( X(t) \) is Brownian motion, and it is well known (see [6], p. 60 or [2], §4) that the transition function of \( Z_2(t) \) is given by

\[
P_\alpha(Z_2(t) \in A) = \int_A f_2(t, x, y) \mu(dy)
\]

where \( \mu(dy) = 2^{-N/2}[\Gamma((N/2) + 1)]^{-1}y^N \, dy \), and

\[
f_2(t, x, y) = \frac{(N/2)(2t)^{-1}}{\Gamma(N/2)(xy)^{1-N/2}} \exp \left[ -\frac{(x^2 + y^2)}{4t} \right] I_{N/2-1} \left( \frac{xy}{2t} \right),
\]

where \( I_n \) is the usual modified Bessel function.

Let \( T_\beta \) be the stable subordinator of exponent \( \beta \), \( 0 < \beta < 1 \), with \( T_\beta(0) = 0 \). Then it is a familiar fact that \( Z_\alpha \) and \( Z_\alpha(T_{a/2}(t)) \) are equivalent provided that \( T_{a/2} \) and \( Z_2 \) are independent. Let \( h_{a/2}(t, u) \) denote the density function of \( T_{a/2} \). Then the transition function of \( Z_\alpha \) is given by

\[
P_\alpha(Z_\alpha(t) \in A) = \int_A f_\alpha(t, x, y) \mu(dy)
\]

where \( \mu \) was defined above and

\[
f_\alpha(t, x, y) = \int_0^\infty h_{a/2}(t, u)f_2(u, x, y) \, du.
\]

Let \( \tau_r = \inf \{ t > 0 : Z_\alpha(t) = r \} \) (\( = \infty \) if \( Z_\alpha(t) \neq r \) for all \( t > 0 \)). It is clear that \( \{r\} \) is a polar set for the radial process \( Z_\alpha(t) \) if and only if the sphere \( S_r = \{y : |y| = r\} \) is a polar set for the process \( X(t) \). In addition, if \( r \) is a regular point of \( \{r\} \) for the radial process, then all points on the sphere \( S_r \) are regular for this sphere for the process \( X(t) \). The following fact ensues from Corollary 4.3 and Theorem 3.1 of [2]. For completeness, we sketch below an alternate proof which avoids the use of Hunt’s capacity theory.

**Proposition 2.1.** For the radial process \( Z_\alpha(t) \), \( r \) is regular for \( \{r\} \) provided \( \alpha > 1 \). If \( \alpha \leq 1 \), then \( \{r\} \) is polar.

**Proof.** Let \( A_n = \{x \in R^1 : |x - r| < 1/n\} \) and let \( \tau_n \) be the first hitting time of \( A_n \). Set

\[
H^*_n(x, B) = E_x(\exp (-\lambda\tau_n)1_B(x(\tau_n))); \tau_n < \infty,
\]
and let $u^\alpha(x, y)$ be the Laplace transform of $f_\alpha(t, x, y)$. Then the usual first passage arguments show that

$$u^\alpha(x, r) = \int_{A_n} H^\alpha_n(x, dz) u^\alpha(z, r).$$

Simple computations (see the proof of Corollary 4.3 of [2] for details) show that if $r > 0$, then

$$f_\alpha(t, r, r) \sim kt^{-1/\alpha}, \quad t \to 0,$$

where $k$ is some constant (dependent on $r > 0$). Thus $u^\alpha(x, r) \to \infty$ as $x \to r$ when $\alpha \leq 1$, while for $\alpha > 1$, $u^\alpha(x, r)$ is bounded and continuous in $x$ in a neighborhood of $r$.

Suppose $\alpha \leq 1$. Then (2.3) shows that

$$\infty > u^\alpha(x, r) \geq E_x(\exp(-\lambda \tau_r); \tau_r < \infty) \inf_{z \in A_n} u^\alpha(z, r),$$

and it follows that $P_x(\tau_r < \infty) = 0$ for all $x \neq r$. Since

$$P_x(\tau_r < \infty) = \lim_{t \to 0} \int_{A_n} f_\alpha(t, r, y) P_y(\tau_r < \infty) dy,$$

we see that $\{r\}$ is polar for all $r$.

Now suppose $\alpha > 1$. Since $Z_\alpha(t)$ is a standard process it is quasi-left continuous (see [1], §9 for a definition) and thus $\tau_n \uparrow \tau_r$ and $X(\tau_n) \to X(\tau_r)$, a.s. $P_x, x \neq r$. It then follows from (2.3) that for $x \neq r$ and $r > 0$,

$$u^\alpha(x, r)/u^\alpha(r, r) = E_x(\exp(-\lambda \tau_r), \tau_r < \infty).$$

Hence

$$\lim_{x \to r} E_x(\exp(-\lambda \tau_r), \tau_r < \infty) = 1,$$

and it follows easily from this that $r$ is regular for $\{r\}$ whenever $r > 0$. This completes the proof.

In view of the above result we shall henceforth only consider the processes with $\alpha > 1$. If the processes are recurrent, then $\Phi(x) \equiv 1$, so we need only consider transient processes (i.e., $\alpha < N$).

**Theorem 2.1.** Assume $1 < \alpha < N$. Then for $r > 0$,

$$\Phi_r(x) = \frac{\pi^{1/2} \Gamma\left(\frac{\alpha + N}{2}\right)}{\Gamma\left(\frac{\alpha - 1}{2}\right)} \frac{2^{\alpha - 1}}{r^{N - \alpha}} \left(\frac{|x|^2 + r^2}{|x|^2 - r^2} \right)^{N/2} \left(\frac{P_x^{1/2}}{P_x^{1/2}} - \frac{N/2}{\alpha - 1} \right) \left(\frac{|x|^2 + r^2}{|x|^2 - r^2} \right)^{N/2},$$

where $P_x^n$ is the usual Legendre function of the first kind.
Proof. Since the processes are transient, \( u'(x, y) \to u(x, y), \lambda \downarrow 0 \), where \( u(x, y) \) is the potential kernel of \( Z_\lambda(t) \). From (2.4) we see that
\[
(2.7) \quad P_x(\tau_r < \infty) = u(x, r)/u(r, r),
\]
so to establish (2.6) we need to compute the right-hand side of (2.7). This will be done in the following two lemmas.

**Lemma 2.1.** If \( 1 < \alpha < N \), then
\[
(2.8) \quad u_\alpha(x, y) = \frac{\Gamma \left( \frac{N}{2} \right) \Gamma \left( \frac{N-\alpha}{2} \right) 2^{(N/2) - \alpha}}{\Gamma(\alpha/2)} \frac{(xy)^{1-N/2} |x^2 - y^2|^{(\alpha/2) - 1} P^{1-N/2}_{\alpha/2} \left( \frac{x^2 + y^2}{|x^2 - y^2|} \right)}{
\]
where \( P_\alpha \) is the usual Legendre function of the first kind.

**Proof.** The stable subordinator \( T_{\alpha/2} \) of exponent \( \alpha/2 \) is the unique stable process on \((0, \infty)\) whose transition density has Laplace transform
\[
\int_0^\infty h_{\alpha/2}(t, u) e^{-tu} \, du = \exp \left( -t\gamma_{\alpha/2} \right),
\]
and thus
\[
\int_0^\infty \int_0^\infty h_{\alpha/2}(t, u) e^{-tu} \, du \, dt = \gamma^{-\alpha/2}.
\]
Hence the potential kernel of \( T_{\alpha/2} \) is \( \Gamma(\alpha/2)^{-1} \gamma^{(\alpha/2) - 1} \). From (2.2) we then see that
\[
u_\alpha(x, y) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty \gamma^{(\alpha/2) - 1} f_\alpha(u, x, y) \, du.
\]
Using the explicit formula for \( f_\alpha(u, x, y) \) and formula 8, p. 196 of [4], we obtain (2.8).

**Lemma 2.2.** If \( 1 < \alpha < N \), then for \( r > 0 \),
\[
(2.9) \quad u_\alpha(r, r) = \frac{\Gamma \left( \frac{N}{2} \right) \Gamma \left( \frac{N-\alpha}{2} \right) 2^{(N/2) - \alpha}}{\Gamma(\alpha/2)} \frac{(xy)^{1-N/2} |x^2 - y^2|^{(\alpha/2) - 1} P^{1-N/2}_{\alpha/2} \left( \frac{x^2 + y^2}{|x^2 - y^2|} \right)}{
\]

**Proof.** This follows from (2.8) and the asymptotic relation (see formula 20, p. 164 of [5]),
\[
P^{1-N/2}_{\alpha/2} \left( \frac{|x^2 + y^2|}{|x^2 - y^2|} \right) \sim \frac{\Gamma \left( \frac{\alpha-N/2}{2} \right) \Gamma \left( \frac{\alpha-1}{2} \right)}{\Gamma \left( \frac{\alpha+N}{2} - 1 \right)} \gamma^{\alpha-2} |x^2 - y^2|^{1-\alpha/2}, \quad x \to y.
\]

**Corollary 2.1.** Assume \( 1 < \alpha < N \). Then for any \( r > 0 \),
\[
(2.10) \quad \Phi_r(0) = \frac{\Gamma \left( \frac{\alpha+N}{2} - 1 \right) \Gamma \left( \frac{\alpha-N/2}{2} - \alpha \right)}{\Gamma(N/2) \Gamma \left( \frac{\alpha-1}{2} \right)}.
\]
Proof. This follows from (2.6) and the asymptotic formula (see formula 3 of p. 163 of [5]) that for \( y > 0 \),
\[
P_{1-N/2}^{-N} \left( \frac{x^2 + y^2}{|x|^2 - y^2} \right) \sim \frac{y^{(1-N/2)}}{\Gamma(N/2)} x^{-(1-N/2)}, \quad x \to 0.
\]

Corollary 2.2. Assume \( 1 < \alpha < N \). Then the capacity of the sphere of radius \( r \) is \( C_r \) where
\[
C_r = \frac{4\Gamma \left( \frac{\alpha + N}{2} - 1 \right) \pi^{(N+1)/2} \Gamma(\alpha/2)}{\Gamma(N/2) \Gamma \left( \frac{\alpha - 1}{2} \right) \Gamma \left( \frac{N - \alpha}{2} \right)} r^{N-\alpha}.
\]

Proof. Let \( \pi_r(dx) \) be the capacitory measure of the sphere \( S_r = \{ x : |x| = r \} \). The potential kernel of \( X(t) \) is just the Riesz kernel \( K|y-x|^{\alpha-N} \), where
\[
K = \frac{\Gamma((N-\alpha)/2)}{4\pi^{N/2} \Gamma(\alpha/2)}
\]
and it is a basic fact that (see Chapter 6 of [1]) the capacitory potential of \( S_r \) is just \( \Phi_r(x) \), i.e.,
\[
\Phi_r(x) = \int_{S_r} K|y-x|^{\alpha-N} \pi_r(dy).
\]

Thus \( \Phi_r(0) = r^{\alpha-N} K\pi(S_r) \). Since \( C_r = \pi(S_r) \), (2.11) follows from (2.10) and the above relation.

For \( \alpha = 2 \) (i.e., Brownian motion) and \( N > 2 \) the formula for \( u(x, y) \) is considerably simpler.
\[
u(x, y) = 2^{(N/2)-2} \Gamma(N/2)(N/2-1)^{-1} [\text{Max}(x, y)]^{2-N}.
\]
Using this, it is easily seen that Theorem 2.1 yields the well-known result
\[
\Phi_r(x) = 1, \quad x \leq r,
\]
\[
= (x/r)^{2-N}, \quad x > r.
\]

By the same type of arguments we may compute more elaborate hitting probabilities for the processes with \( 1 < \alpha < N \). Let \( B = \{ r_1, r_2, \ldots, r_n \} \) where \( r_1 < r_2 < \cdots < r_n \). Since potential \( \sum_{i=1}^n u(x, r_i) \mu_i \) on \( N \) uniquely determines the numbers \( \mu_i \), we see that the matrix \( U_{ij} = u(r_i, r_j) \) is invertible. Denote its inverse by \( K_B(i, j) \).

If \( \tau_B \) is the first hitting time of \( B \) by the radial process, and if
\[
T_B = \inf \{ t > 0 : |X(t)| \in B \}
\]
then, of course, \( P_x(T_B < \infty) = P_{|x|}(\tau_B < \infty) \). However, it is a fundamental fact in the theory of Markov processes (see [1, Chapter 6]) that there is a bounded measure \( \pi \) having support on \( B \) such that
\[
P_a(\tau_B < \infty) = \sum_{j=1}^n u(a, r_j) \pi_j.
\]

This fact may also be proved directly for the \( Z_a(t) \) process by an argument very similar to that used to deduce (2.7).] Since every point of \( B \) is regular for \( B \), we
see that \( \pi \) is the unique measure on \( B \) such that \( 1 = (U\pi)_j, 1 \leq j \leq n \), and thus \( \pi_j = \sum_{i=1}^{n} K_B(i, j) \). Consequently

\[
(2.14) \quad P_x(\tau_B < \infty) = \sum_{i=1}^{n} \sum_{j=1}^{n} u(a, r_j)K_B(i, j).
\]

In a similar manner we see that

\[
P_x(\min_{1 \leq i \leq n} (T_{r_i}) = T_{r_j}, T_B < \infty) = P_x|Z_a(\tau_B) = r_j, \tau_B < \infty).
\]

Set

\[
H_B(a, r_i) = P_x(Z_a(\tau_B) = r_i, \tau_B < \infty).
\]

Then the \( H_B(a, r_i) \) are uniquely determined by the equations

\[
u(a, r_j) = \sum_{i=1}^{n} H_B(a, r_i)u(r_i, r_j), \quad 1 \leq j \leq n,
\]

and thus

\[
(2.15) \quad H_B(a, r_j) = \sum_{i=1}^{n} u(a, r_i)K_B(r_i, r_j).
\]

For a two point set \( B=\{r_1, r_2\} \)

\[
K_B = \frac{1}{\Delta} \begin{pmatrix} U_{22} & -U_{12} \\ -U_{12} & U_{11} \end{pmatrix},
\]

where \( \Delta = U_{11}U_{22}-(U_{12})^2 \). Equations (2.14) and (2.15) then yield

\[
(2.16) \quad P_x(T_B < \infty) = \frac{u(|x|, r_1)u(r_2, r_2)+u(|x|, r_2)u(r_1, r_1)}{u(r_1, r_1)u(r_2, r_2)-u(r_1, r_2)^2} - \frac{u(r_1, r_2)[u(|x|, r_1)+u(|x|, r_2)]}{u(r_1, r_1)u(r_2, r_2)-u(r_1, r_2)^2},
\]

and

\[
(2.17a) \quad P_x(T_{r_1} < T_{r_2}) = \frac{u(|x|, r_1)u(r_2, r_2)-u(|x|, r_2)u(r_1, r_1)}{u(r_1, r_1)u(r_2, r_2)-u(r_1, r_2)^2},
\]

\[
(2.17b) \quad P_x(T_{r_2} < T_{r_1}) = \frac{u(|x|, r_2)u(r_1, r_1)-u(|x|, r_1)u(r_2, r_2)}{u(r_1, r_1)u(r_2, r_2)-u(r_1, r_2)^2}.
\]

In particular, for \( \alpha=2 \) we obtain the following well-known results for Brownian motion in dimension \( N \geq 3 \).

\[
P_x(T_{r_1} < T_{r_2}) = 1, \quad |x| \leq r_1,
\]

\[
= 0, \quad |x| \geq r_2,
\]

\[
= \frac{|x|^2-N-r_2^2-N}{r_1^2-N-r_2^2-N}, \quad r_1 \leq |x| \leq r_2,
\]

and

\[
P_x(T_{r_2} < T_{r_1}) = 0, \quad |x| \leq r_1,
\]

\[
= \frac{|x|/r_2|^{2-N}}{r_1^2-N-r_2^2-N}, \quad |x| \geq r_2,
\]

\[
= \frac{r_1^2-N-|x|^2-N}{r_1^2-N-r_2^2-N}, \quad r_1 \leq |x| \leq r_2.
\]
In the above discussion we omitted those processes with $\alpha \geq N$. We will now fill in this detail. If $N = 1$, then since $\alpha > 1$, the processes are point recurrent, and the above methods are not directly applicable since $u(x, y) = \infty$. However, in this case a sphere consists of two points, and explicit formulas for the hitting distribution of finite sets were given in [7] (see §3). Alternately, it is easily seen that the recurrent potential kernel of the $Z_a(t)$ process is given by $u(x, y) = a(y - x) + a(y + x)$ where $a(x)$ is the recurrent potential kernel of $X(t)$ given in [7]. With this $u$, the hitting distribution is again given by (2.15). The remaining process is $\alpha = N = 2$, i.e., planar Brownian motion. Owing to the continuity of the paths, the hitting probabilities for a finite $B$ can be reduced to that of a two point set. But for such a set the result is well known. (See, e.g., [6, p. 62].)

3. The hitting measure of $S_r$. Assume $1 < \alpha < N$. It is intuitively clear that the capacitory measure $\pi_r(\xi)$ of $S_r$ is $C_r \sigma_r(\xi)$, where here and in the following, $\sigma_r$ is the uniform measure on $S_r$, and $C_r$ is the capacity of $S_r$ given in (2.11). To establish this fact rigorously we note that since every point of $S_r$ is regular for $S_r$, the measure $\pi_r$ is the unique bounded measure having support on $S_r$ such that for all $x \in S_r$,

$$1 = K \int_{S_r} |\xi - x|^{\alpha - N} \pi_r(d\xi),$$

where here and in the following,

$$K = \frac{\Gamma((N - \alpha)/2)}{4^{\alpha/2} \pi^{\alpha/2} N^{N/2} \Gamma(\alpha/2)}$$

A change to spherical coordinates now easily shows that $C_r \sigma_r(\xi)$ satisfies (3.1).

The main result of this section is the following

**Theorem 3.1.** Assume $1 < \alpha < N$. Then the hitting measure $H_r(x, d\xi)$ of $S_r$ is given by

$$H_r(x, d\xi) = \frac{\Gamma\left(\frac{\alpha + N}{2} - 1\right) \pi^{1/2} a^{\alpha - 1} \sigma_r}{\Gamma(N/2) \Gamma\left(\frac{\alpha - 1}{2}\right)} |x|^2 - r^2|a - 1| |\xi - x|^{\alpha - N} \sigma_r(\xi), \ |x| \neq r,$$

while $H_r(x, d\xi)$ is the unit mass at $x$ if $|x| = r$.

**Remark.** The basis of the computation of $H_r(x, d\xi)$ which we shall use here is that of inversion in an appropriate sphere orthogonal to $S_r$. This idea was first used by M. Riesz in [8] to compute (in probabilistic terminology) the hitting measure of the solid ball. Later, Blumenthal, Getoor, and Ray [3] extended these computations to complete the story for the ball.
Proof. Consider first the case when $|x| > r$. The inversion in the sphere \( \{ y : |y - x| = a \} \) is the change of variable \( y' = y + a^2 (y - x)/|y - x|^2 \). Choose \( a^2 = |x|^2 - r^2 \). Then the sphere \( S_r \) and the inverting sphere are orthogonal, and thus the transformation maps \( S_r \) onto \( S_r \). If \( y', z' \) are the images of \( y, z \) under this inversion, then

\[
|y' - z'| = \frac{a^2 |y - z|}{|y - x||z - x|}.
\]

Define a measure \( \mu_x(d\xi) \) on \( S_r \) by

\[
\mu_x(d\xi)|\xi - x|^{a-N} = \pi_x(d\xi'),
\]

where \( \xi' \) is the image of \( \xi \), and \( \pi_r \) is the capacitory measure of \( S_r \). If \( z \in S_r \), then so does \( z' \), and (3.1), (3.4), and (3.5) now show that if \( z \in S_r \)

\[
1 = K \int_{S_r} |z' - y'|^{a-N} \pi_x(d\xi') = K \int_{S_r} |z - y|^{a-N} \mu_x(d\xi).
\]

Thus if \( z \in S_r \),

\[
K |z - x|^{a-N} = K(a^2)^{a-N} \int_{S_r} K |z - y|^{a-N} \mu_x(d\xi).
\]

But since every point of \( S_r \) is regular for \( S_r \), the measure \( H_r(x, d\xi) \) is the unique measure supported on \( S_r \), such that

\[
K |z - x|^{a-N} = \int_{S_r} H_r(x, d\xi) K |z - \xi|^{a-N}, \quad z \in S_r.
\]

Thus

\[
H_r(x, d\xi) = K(a^2)^{a-N} \mu_x(d\xi).
\]

Suppose \( \mu_x(d\xi) = k_x(\xi) d\sigma_\alpha(\xi) \). Then (3.5) shows that

\[
k_x(\xi) = C_r |\xi - x|^{N-a} d\sigma(\xi')/d\sigma(\xi).
\]

However, it is clear from the geometry that

\[
\frac{d\sigma(\xi')}{|\xi' - x|^{N-1}} = \frac{d\sigma(\xi)}{|\xi - x|^{N-1}},
\]

and thus, using (3.4), we find that

\[
k_x(\xi) = C_r (a^2)^{N-1} |\xi - x|^{2-a-N},
\]

and thus by (3.8),

\[
H_r(x, d\xi) = K C_r |x|^2 - r^2 |x|^{a-1} |\xi - x|^{2-a-N} d\sigma_\alpha(\xi), \quad |x| > r.
\]

Suppose now that \( |x| < r \). An inversion in the sphere \( S_r \), sends \( x \) to \( x' = r^2 x/|x|^2 \),
and by what has just been shown above we know that $H_r(x', d\xi)$ satisfies the equation

$$K|y-x'|^{\alpha-N} = \int_{S_r} K^2 C_r |x|^2 - r^2|^{\alpha-1}|\xi-x'|^{2-\alpha-N}|\xi-y|^{\alpha-N} d\sigma_r(\xi), \quad y \in S_r.$$ 

But

$$|y-x'| = r|x-y|/|x|, \quad [x^2 - r^2] = r^2[|x|^2 - r^2]/|x|^2,$$

and thus we find that

$$H_r(x, d\xi) = K C_r |x|^2 - r^2|^{\alpha-1}|\xi-x|^2 - r^2|^{\alpha-N} d\sigma_r(\xi)$$

satisfies (3.7). This completes the proof.

We note that for $\alpha = 2$, i.e., Brownian motion, the kernel in (3.3) becomes the classical Poisson kernel (as it should), and that for a general $\alpha$, the kernel is a very close analogue of this classical kernel.

We conclude this section with a comment on the quantity $H_r(x, d\xi)$ in the case of a recurrent stable process, $\alpha \geq N$. For $\alpha = N = 2$, i.e., planar Brownian motion, it is well known that (3.3) still gives the correct result. For $\alpha > 1 = N$, the sphere is a two point set, and an explicit formula for $H_r(x, d\xi)$ was computed in [7, §3].

4. The Green's function. Again we consider the case when $1 < \alpha < N$. The Green's function $g_r(x, y)$ for the sphere $S_r = \{y : |y| = r\}$ is uniquely defined by

$$g_r(x, y) = K|y-x|^{\alpha-N} - K^2 C_r \int_{S_r} |x^2 - r^2|^{\alpha-1}|\xi-x|^{2-\alpha-N}|\xi-y|^{\alpha-N} d\sigma_r(\xi)$$

where $K$ is given in (3.2) and $C_r$ is the capacity given in (2.11). Set

$$I = \int_{S_r} |\xi-x|^{2-\alpha-N}|\xi-y|^{\alpha-N} d\sigma_r(\xi).$$

Consider the case when $|x| > r$. An inversion in the sphere $\{y : |y-x|^2 = |x|^2 - r^2\}$ sends $\xi \rightarrow \xi' \in S_r$ and $y \rightarrow y'$. Performing this change of variable we find that

$$I = (|x|^2 - r^2)^{1-N}|y'-x|^{N-\alpha} \int_{S_r} |\xi'-y'|^{\alpha-N} d\sigma_r(\xi')$$

$$= (|x|^2 - r^2)^{1-N}|y'-x|^{N-\alpha} \Phi_r(y')(KC_r)^{-1}$$

$$= (KC_r)^{-1}(|x|^2 - r^2)^{1-\alpha} |y-x|^{\alpha-N} \Phi_r(y').$$

Substituting this expression for $I$ into (4.1) shows that

$$g_r(x, y) = K|y-x|^{\alpha-N}[1 - \Phi_r(y')], \quad |x| > r.$$ 

Now a simple computation shows that

$$|y'|^2 - |y|^2 = |y|^2 - r^4 - 2r^2(x \cdot y) = |y|^2|\frac{y}{y-x}||y|^2, \quad |y|^2,$$

and thus for $|x| > r$

$$g_r(x, y) = K|y-x|^{\alpha-N} \left(1 - \Phi_r\left(\frac{y}{y-x} \right) |x-r^2y/|y|^2\right).$$
To compute $g_r(x, y)$ for $|x| < r$, note that an inversion in the sphere $S_r$ sends $x \to \tilde{x} = r^2 x/|x|^2$, $y \to \tilde{y} = r^2 y/|y|^2$ and that $|\tilde{x}| > r$. Using (4.1) and some simple computations we easily obtain that

$$g_r(x, y)(r^2/|x| |y|)^{N-N} = g_r(\tilde{x}, \tilde{y}).$$

Thus for $|x| < r$,

$$g_r(x, y) = K|y-x|^{a-N}\left(1 - \Phi_r(|y|)\right)$$

where the last equality follows from the symmetry of $g_r(x, y)$ and the fact that $\Phi_r(t)$ is a function of $|t|$. Combining the above results we obtain

**Theorem 4.1.** The Green's function of the sphere is given by

$$g_r(x, y) = K|y-x|^{a-N}\left(1 - \Phi_r\left(\frac{|y|}{|y-x|}\right)\right),$$

where $\Phi_r$ is the hitting probability given in (2.6).

For $\alpha=2, N>2$, the above Green's function is the classical one for the Laplacian. To see this, note that the first and second equality in 4.3 and a little computation shows that for $|x| > r$,

$$[|y|^2 - r^2]|y-x|^2 = [|x|^2 - r^2][|y|^2 - r^2].$$

Hence for $|x| > r$, $|y| > r$ if either $|x| > r$, $|y| \leq r$ or $|x| \leq r$, $|y| > r$. It follows that $g_r(x, y) = 0$ if either $|x| > r$, $|y| \leq r$ or $|x| \leq r$, $|y| > r$, while for $|x| < r$, $|y| < r$ or $|x| > r$, $|y| > r$,

$$g_r(x, y) = K|y-x|^{a-N} - K|y/r|^{a-N}|x-r^2 y/|y|^2|^{2-N}.$$

For $\alpha = N=2$, i.e., planar Brownian motion, the Green's function of the circle is just the classical Green's function for the Laplacian, and may be found in all books on partial differential equations. For $\alpha > 1 = N$, the Green's function of the sphere was computed in [7].

**Bibliography**


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