COMPLEX-VALUED STABLE MEASURES AND THEIR DOMAINS OF ATTRACTION(1)

BY

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1. **Introduction and summary.** One of the old and basic problems of probability theory is the determination of limit distributions of sums of independent random variables. Very little seems to be known about the same analytic problem stated for the larger class of complex-valued measures rather than just for probabilities. The subject of this paper is the investigation of this more general case. It turns out that the possible limit measures are a substitute for idempotent measures on the real line.

Let $M(R)$ denote the Banach algebra of all complex-valued regular finite measures defined on the Borel sets of the real line R, where multiplication is defined by convolution, and

$$\|\mu\| = \sup \sum |\mu(R_i)|,$$

the supremum being taken over all finite collections of pairwise disjoint sets R_i whose union is R. Let $B(R)$ be the set of all Fourier transforms of measures in $M(R)$, i.e. all functions of the form

$$\hat{\mu}(t) = \int_{-\infty}^{\infty} \exp(itx)\, d\mu(x), \qquad \mu \in M(R).$$

Finally, let $A(R)$ denote the set of Fourier transforms of absolutely continuous measures. For more details see Rudin [12].

P. Lévy, B. W. Gnedenko, A. N. Kolmogorov and others (see [6], [7] or [11]) characterized all possible pointwise limits

$$\lim_{n\to\infty} (\hat{\nu}(t/B_n))^n \exp(itA_n) = \hat{\mu}(t), \tag{1.1}$$

where $A_n \in R$, $B_n > 0$, and ν and μ are probability measures. They found that μ is necessarily *stable*, i.e. satisfies the following condition: For all $a > 0$, $b > 0$ there exist $c > 0$ and $\gamma \in R$ such that

$$\hat{\mu}(at)\hat{\mu}(bt) = \hat{\mu}(ct)\exp(i\gamma t) \quad \text{for all } t \in R. \tag{1.2}$$

The positive definite solutions of this equation are

$$\hat{\mu}(t) = \exp(i\gamma t - c|t|^{\alpha}(1 + i\beta(t/|t|)\omega(t, \alpha))),$$

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where γ is real, $-1 \leqq \beta \leqq 1$, $0 < \alpha \leqq 2$, $c \geqq 0$, and $\omega(t, \alpha) = \tan(\pi\alpha/2)$ for $\alpha \neq 1$ and $= (2/\pi) \log |t|$ for $\alpha = 1$.

Little seems to be known about possible limits in (1.1) if $\nu \in B(R)$ is not necessarily a probability measure. The main difficulties in this case are that pointwise convergence of Fourier transforms does not automatically imply uniform convergence in every compact set, that the limit is not necessarily a Fourier transform even if it is continuous, and it is difficult to conclude how the corresponding measures behave.

H. Bergström [3] considered general convolution powers $\{\nu_n^{k_n}\}$, where $\nu_n \in M(R)$ and $\{k_n\}$ is a sequence of positive integers. His investigation is based on what he calls V-absolute convergence which is very strong and implies, for instance, that $\sup_n \|\nu_n^{k_n}\| < \infty$. As we shall see later, this is not a feasible condition in our case. However, Bergström does not give an explicit form for possible limit measures.

D. I. Ljalina and Ju. P. Studnev [10] generalized the central limit theorem of probability theory to arbitrary measures in $M(R)$ as follows: Given $\nu \in M(R)$ with $\int x^2\, d|\nu| < \infty$. Furthermore, let $\hat{\nu}(0) = \nu(R) = 1$, $\hat{\nu}'(0) = i \int x\, d\nu = 0$, and $\hat{\nu}''(0) = -\int x^2\, d\nu = -1$. (Every measure $\nu \in M(R)$ with $\int x^2\, d|\nu| < \infty$ can be put in this form by simple transformations, provided $|\hat{\nu}(t)|$ takes on its maximum at a finite point.) Then

$$\lim_n (\hat{\nu}(t/\sqrt{n}))^n = \exp(-t^2/2) = \hat{\mu}(t)$$

uniformly in $|t| \leqq a$, $a > 0$, from which it follows on putting $\nu((-\infty, x)) = \nu(x)$ for any $\nu \in M(R)$ that

$$\lim_n \int_{-\infty}^{\infty} |\nu^n(x\sqrt{n}) - \mu(x)|^2\, dx = 0$$

where μ is the Gaussian measure.

Our main results are as follows: Suppose $\mu, \nu \in M(R)$, where μ is nondegenerate, $\sup_t |\hat{\nu}(t)| \leqq 1$, $A_n \in R$, $B_n > 0$ and

$$0 < \liminf B_n/B_{n+1} \leqq \limsup B_n/B_{n+1} < \infty.$$

Then

$$\lim_n (\hat{\nu}(t/B_n))^n \exp(itA_n) = \hat{\mu}(t) \quad \text{for all } t \neq 0 \tag{1.3}$$

if and only only if μ is stable, i.e. satisfies (1.2) (Theorems 3.1 and 4.1). Moreover, this is the case if and only if $\hat{\mu}$ has the form

$$\begin{aligned} \hat{\mu}(t) &= \exp(-c|t|^\alpha + i\beta t), \qquad t \geqq 0, \\ &= \exp(-d|t|^\alpha + i\beta t), \qquad t < 0, \end{aligned}$$

or

$$\begin{aligned} \hat{\mu}(t) &= \exp(-c|t| + i\beta t \log |t|), \qquad t \geqq 0, \\ &= \exp(-d|t| + i\beta t \log |t|), \qquad t < 0, \end{aligned}$$

where $\alpha \in R$, $\alpha \neq 0$, and $\beta \in R$; c and d are complex numbers with $\operatorname{Re}(c) > 0$ and

Re $(d)>0$ (Theorem 4.2). The set of all measures ν which are attracted by a particular stable μ in the above sense is characterized (Theorems 5.1 and 5.2). Finally, if (1.3) holds for $A_n=0$ and if we put $\hat{\nu}(t/B_n)^n=\hat{\nu}_n(t)$, then

(i) for $\alpha>0$

$$\lim_n \int_{-\infty}^{\infty} |\nu_n(x)-\mu(x)|^p \, dx = 0$$

for all $p\geqq 2$ if $\alpha>\frac{1}{2}$ and $p>1/\alpha$ if $0<\alpha\leqq\frac{1}{2}$,

(ii) for $\alpha<0$

$$\lim_n \nu_n(I) = \mu(I)$$

for every compact interval $I\subset R$ (Theorem 6.1).

2. **A generalization of a lemma of Khintchine.** In the probability case, a lemma of Khintchine plays a crucial role for the proof that every limit measure is stable. This lemma can be generalized as follows:

LEMMA 2.1. *Let $\{f_n\}$ be a sequence of locally integrable functions on R. Let $a_n\in R$, $b_n>0$, and let the b_n's have at least one finite nonzero limit point. If there are two functions f (not vanishing a.e.) and f^* such that for all c, $d\in R$*

$$\lim_n \int_c^d |f_n(x)-f(x)| \, dx = 0$$

and

$$\lim_n \int_c^d |f_n(b_n x)\exp(ia_n x)-f^*(x)| \, dx = 0,$$

then

$$f^*(x) = f(bx)\exp(iax) \qquad a.e.,$$

where b is any finite nonzero limit point of $\{b_n\}$ and a is a suitably chosen limit point of $\{a_n\}$.

Proof. First take $a_n=0$ for all n. Then, for a subsequence $\{n_k\}$, $b_{n_k}\to b$, $0<b<\infty$. All we need show is that for all c, $d\in R$

$$\int_c^d |f^*(x)-f(bx)| \, dx = 0.$$

Clearly,

$$\int_c^d |f^*(x)-f(bx)| \, dx \leqq \int_c^d |f^*(x)-f_{n_k}(b_{n_k}x)| \, dx + \int_c^d |f_{n_k}(b_{n_k}x)-f(b_{n_k}x)| \, dx + \int_c^d |f(b_{n_k}x)-f(bx)| \, dx.$$

The first two terms tend to zero by assumption. That the third term tends to zero also can be proved in the same way as one shows that

$$\lim_{h\to 0} \int_{-\infty}^{\infty} |f(x+h)-f(x)| \, dx = 0 \quad \text{for all } f \text{ in } L_1(R)$$

(see e.g. [8, p. 199]).

Now let the a_n's be arbitrary. By the first step, $|f^*(x)|=|f(bx)|$ a.e., where $b_{n_k}\to b$. We show now that $\lim_k a_{n_k}=a$ (finite) and $f^*(x)=f(bx)\exp(iax)$ a.e.

Suppose first that $\{a_{n_k}\}$ has two finite limit points a and a^*, and $a_{n_k'}\to a$, $a_{n_k''}\to a^*$. By the same technique used before, we get for $k\to\infty$ and for all $c, d\in R$

$$\int_c^d |f_{n_k}(b_{n_k}x)-f(bx)|\,dx\to 0.$$

Therefore

$$\int_c^d |f_{n_k'}(b_{n_k'}x)\exp(ia_{n_k'}x)-f(bx)\exp(iax)|\,dx\to 0$$

and

$$\int_c^d |f_{n_k''}(b_{n_k''}x)\exp(ia_{n_k''}x)-f(bx)\exp(ia^*x)|\,dx\to 0,$$

whence

$$f^*(x)=f(bx)\exp(iax)=f(bx)\exp(ia^*x)\quad\text{a.e.}$$

Since f is not equal to zero a.e., we have $a=a^*$. There cannot be an infinite limit point of $\{a_{n_k}\}$ either. For, suppose $a_{n_k'}\to\infty$. Then

$$\int_A |\exp(ia_{n_k'}x)-f^*(x)/f(bx)|\,dx\to 0$$

for every set A in which $f(b\cdot)$ is bounded away from zero. This obviously is impossible, which completes the proof.

We shall make use of this lemma in a still different form:

LEMMA 2.2. (i) *The conclusion of Lemma* 2.1 *holds if* ess sup $|f_n(x)|\leqq C$ *for all* n, $f_n(x)\to f(x)$ *a.e. and* $f_n(b_nx)\exp(ia_nx)\to f^*(x)$ *a.e.*

(ii) *If in addition to the assumptions made in Lemma* 2.1 f *and* f^* *are continuous, with* $|f(x)|\not\equiv c$, *and* $0<\liminf b_n\leqq\limsup b_n<\infty$, *then the* b_n's *and* a_n's *actually converge, and*

$$f^*(x)=f(bx)\exp(iax)\quad\textit{for all } x\in R, \tag{2.1}$$

where $b=\lim_n b_n$ *and* $a=\lim_n a_n$.

Proof. Part (i) is an immediate consequence of the dominated convergence theorem. Equality in (2.1) follows since f and f^* are continuous. Suppose the b_n's do not converge. Then $b_{n_k'}\to b$ and $b_{n_k''}\to b^*$, $0<b$, $b^*<\infty$. Assume $b<b^*$, put $\alpha=b/b^*$. Lemma 2.1 yields

$$|f(x)|=|f(\alpha x)|=|f(\alpha^2x)|=\cdots=|f(0)|\quad\text{for all } x,$$

which contradicts $|f(x)|\not\equiv c$. Therefore $b_n\to b$. It then follows from the proof of Lemma 2.1 that $a_n\to a$, which completes the proof.

Examples show that Lemma 2.1 is false without the assumption imposed on the b_n's.

3. **Limits of convolution powers.** It turns out that we obtain the same class of limit functions in (1.1) if we allow the initial function $\hat{\nu}$ to be in $C(R)$, the space of continuous and bounded functions on R. The aim of this section is to establish the following

THEOREM 3.1. *Suppose* $f \in C(R)$, $\sup_t |f(t)| \leqq 1$, $A_n \in R$, $B_n > 0$,

$$0 < \liminf B_n/B_{n+1} \leqq \limsup B_n/B_{n+1} < \infty,$$

and

$$\lim_n (f(t/B_n))^n \exp(itA_n) = \varphi(t) \quad \textit{for all } t \neq 0, \tag{3.1}$$

where φ *is continuous, and* $|\varphi| \not\equiv c$. *Then there is an* $\alpha \in R$, $\alpha \neq 0$, *such that*

(i) $$\lim_n B_{rn}/B_n = r^{1/\alpha} \quad \textit{for all } r > 0(^2),$$

(ii) $$\begin{aligned} \lim_n (rA_n - rnA_{rn}B_{rn}/[rn]B_n) &= \gamma(r) = A(r^{1/\alpha} - r) \quad \textit{for } \alpha \neq 1, \\ &= Ar \log r \quad \textit{for } \alpha = 1, \end{aligned}$$

for all $r > 0$, *where* A *is a real constant.*

(iii) *The limit* φ *satisfies for all* $r > 0$ *the equation*

$$\varphi(t)^r = \varphi(r^{1/\alpha}t) \exp(i\gamma(r)t) \quad \textit{for all } t \in R. \tag{3.2}$$

The difficulties listed in §1 require a completely different proof from that in the probability case. We shall divide the proof into several lemmas, always assuming the conditions listed in Theorem 3.1.

LEMMA 3.1. *For* $n \to \infty$,

(i) $B_n/B_{n+1} \to 1$,

(ii) $nA_{n+1}/(n+1) - A_nB_n/B_{n+1} \to 0$,

(iii) $B_n \to 0$ *or* $B_n \to \infty$.

(iv) *The limit points of* $\{A_n\}$ *coincide with the interval* $[\liminf A_n, \limsup A_n]$.

Proof. (i), (ii): It is not hard to see that (3.1) implies

$$\lim_n (f(t/B_{n+1}))^n \exp(itnA_{n+1}/(n+1)) = \varphi(t) \quad \text{for all } t \neq 0.$$

We illustrate now how Lemma 2.2 can be applied. For this purpose, put

$$b_n = B_n/B_{n+1}, \qquad a_n = nA_{n+1}/(n+1) - A_nB_n/B_{n+1}, \qquad f_n(x) = f(x/B_n)^n \exp(ixA_n).$$

We have, for all $x \neq 0$, $f_n(x) \to \varphi(x)$ and $f_n(b_nx) \exp(ia_nx) \to \varphi(x)$. Lemma 2.2 yields

$$\varphi(x) = \varphi(bx) \exp(iax) \quad \text{for all } x \in R,$$

where $b = \lim b_n$ and $a = \lim a_n$. Since $|\varphi| \not\equiv c$ then $b = 1$ and $a = 0$.

(2) We define $a_x = a_{[x]}$ for all $x > 0$ and for any sequence $\{a_n\}$.

(iii) A limit point different from 0 and ∞ leads to $|\varphi| \equiv 0$ or 1, which is excluded. But having both 0 and ∞ as limit points at the same time is clearly not compatible with $B_n/B_{n+1} \to 1$.

(iv) An interval $(a, b) \subset [\liminf A_n, \limsup A_n]$ free of limit points would contradict either (i) or (ii), which completes the proof.

LEMMA 3.2. *If $B_n > 0$, $B_n \to 0$ or $B_n \to \infty$, and $B_n/B_{n+1} \to 1$ then $\{B_m/B_n\}_{m,n=1,2,\ldots}$ is dense in* $(0, \infty)$.

The proof is obvious.

LEMMA 3.3. *Given a sequence $\{f_n\}$ of continuous functions on R. If there is a $\delta > 0$ and if for every point x_0 in a set dense in an interval I there are sequences $\{x_i\}$, $x_i \to x_0$, and $\{y_i\}$, $y_i \to x_0$, such that for a subsequence $\{n_i\}$ (depending on x_0)*

$$|f_{n_i}(x_i) - f_{n_i}(y_i)| > \delta \quad \textit{for all } i,$$

then $\{f_n\}$ cannot converge to a continuous limit.

The proof is indirect: Suppose $\{f_n\}$ converges to a continuous function f. Then one can find a sequence $\{n'_k\}$, $n'_k \to \infty$, and a decreasing sequence of closed intervals I_k, so that for all k and for all $x \in I_k$

$$|f_{n'_k}(x) - f(x_0)| > \delta/2$$

and

$$|f(x) - f(x_0)| < \delta/4$$

which contradicts the convergence at $\xi \in \bigcap I_k \neq \varnothing$.

LEMMA 3.4.

$$\lim_n |f(t/B_n)|^n = |\varphi(t)|$$

uniformly in every interval (a, b), $0 < a < b < \infty$ or $-\infty < a < b < 0$.

Proof. Lemma 2.2 tells us that $\{B_{rn}/B_n\}$ cannot have 1 as limit point if $r \neq 1$. Otherwise $|\varphi(t)| = |\varphi(t)|^{1/r}$ would hold, which is impossible unless $|\varphi| \equiv 0$ or $|\varphi| \equiv 1$. This together with Lemma 3.2 enables us to show that the conditions of Lemma 3.3 are satisfied if the convergence is not uniform. This in turn would imply a discontinuous $|\varphi|$.

LEMMA 3.5. (i) *If $B_n \to \infty$ then $|\varphi(t)| \searrow$ $(t \to \pm\infty)$.* (ii) *If $B_n \to 0$ then $|\varphi(t)| \nearrow$ $(t \to \pm\infty)$.*

This is an application of Lemma 3.4 above.

LEMMA 3.6. (i) *If $B_n \to \infty$ then $|f(0)| = |\varphi(0)| = 1$, and*

$$\lim_n |f(t/B_n)|^n = |\varphi(t)| \quad \textit{uniformly in } |t| \leqq a, \quad a > 0.$$

(ii) *If* $B_n \to 0$ *then*

(a) *either* $\varphi(t)=0$ *for all* $t \geqq 0$ *or* $|f(\infty)|=|\varphi(\infty)|=1$ *and*

$$\lim_n |f(t/B_n)|^n = |\varphi(t)| \text{ uniformly for } t \geqq a, \quad a > 0,$$

(b) *either* $\varphi(t)=0$ *for all* $t \leqq 0$ *or* $|f(-\infty)|=|\varphi(-\infty)|=1$ *and*

$$\lim_n |f(t/B_n)|^n = |\varphi(t)| \text{ uniformly for } t \leqq a, \quad a < 0.$$

Proof. (i) Clearly $|f(0)|=\lim_n |f(t/B_n)|=1$. We claim that $|\varphi(0)|=a<1$ implies $|\varphi(t)| \equiv a$ which can be proved again by an appeal to Lemma 2.2, also using Lemma 3.5. Therefore $|\varphi(0)|=1$. Lemma 3.4 shows us that any nonuniformity of the convergence could take place only around $t=0$. But such nonuniformities could be transposed to a point $t_0 \neq 0$, by Lemma 3.2, which contradicts Lemma 3.4. The proof of (ii) is similar.

The object of this section up to this point has been to establish the uniform convergence formulated in the last lemma. This allows us to complete the proof of Theorem 3.1 very quickly.

A sequence $\{C_n\}$ of positive numbers is called *regularly varying*, if $C_{rn}/C_n \to c(r)$ (positive) for all $r>0$. According to J. Karamata [9], there is a $\rho \in R$ such that $c(r)=r^\rho$, and $C_n=n^\rho L_n$, with $\{L_n\}$ *slowly varying*, i.e. having the property $L_{rn}/L_n \to 1$ for all $r>0$.

Proof of Theorem 3.1. We prove first that for all $r>0$

$$0 < \liminf B_{rn}/B_n \leqq \limsup B_{rn}/B_n < \infty.$$

We prove only the case $B_n \to \infty$. Suppose $B_{rn_k}/B_{n_k} \to \infty$. Since, by Lemma 3.6, the convergence is uniform,

$$|\varphi(t)| = \lim_k |f(tB_{n_k}/B_{n_k}B_{rn_k})|^{rn_k} = |\varphi(0)|^r = 1 \quad \text{for all } t,$$

which is a contradiction.

We now apply Lemma 2.2 again to $f_n(t)=f^{rn}(t/B_{rn}) \exp(itrnA_{rn}/[rn])$, $b_n=B_{rn}/B_n$, $a_n=rA_n-rnB_nA_{rn}/B_n[rn]$, yielding

$$\varphi(t)^r = \varphi(c(r)t) \exp(it\gamma(r)),$$

where $c(r)=\lim b_n=\lim B_{rn}/B_n$, and $\gamma(r)=\lim a_n$. Thus $\{B_n\}$ is regularly varying, whence $c(r)=r^\rho$ for some $\rho \in R$. Since $c(r) \neq 1$ for $r \neq 1$, we have $\rho \neq 0$. Put, for further use, $\rho=1/\alpha$. It remains to be proved that $\gamma(r)$ has the form given in Theorem 3.1 (ii). In fact, (3.2) leads to the following functional equation:

$$\gamma(rs) = \gamma(s)r^{1/\alpha}+\gamma(r)s = \gamma(r)s^{1/\alpha}+\gamma(s)r,$$

which immediately gives $\gamma(r)=A(r^{1/\alpha}-r)$ if $\alpha \neq 1$ and $\gamma(r)=Ar \log r$ if $\alpha=1$, for some real A. The proof is completed.

We remark that, without the condition imposed on the B_n's, Theorem 3.1 is false.

4. **Stable measures.** As mentioned, we call a function $\varphi \in C(R)$ *stable* if it has the following property: For all $a>0$, $b>0$ there exist $c>0$ and $\gamma \in R$ such that

$$\varphi(at)\varphi(bt) = \varphi(ct)\exp(i\gamma t) \quad \text{for all } t \in R. \tag{4.1}$$

A *measure* $\mu \in M(R)$ *is* called *stable* if $\hat{\mu}$ is stable in the above sense. In terms of measures, (4.1) reads

$$\mu(\cdot/a) * \mu(\cdot/b) = \mu((\cdot-\gamma)/c). \tag{4.1$'$}$$

Thus, a stable measure convolved with itself reproduces itself after being properly shifted and scaled. Therefore, stable measures may be considered as substitutes for idempotent measures which do not exist (except for degenerate ones) on the real line. (4.1)′ implies

$$\|\mu * \mu\| = \|\mu\|,$$

hence $\|\mu\|=0$ or $\|\mu\| \geqq 1$ for every stable measure. A more detailed study of their norms can be found in [1] and [2].

THEOREM 4.1. *A function $\varphi \in C(R)$, $|\varphi| \not\equiv c$, is stable if and only if it appears as a limit in* (3.1).

Proof. Suppose φ is a limit in (3.1). Then clearly (3.2) implies (4.1). To see this, write $a=r_1^{1/\alpha}$, $b=r_2^{1/\alpha}$. Then $c=(r_1+r_2)^{1/\alpha}$ and $\gamma=\gamma(r_1+r_2)-\gamma(r_1)-\gamma(r_2)$. Conversely, let φ be stable. Then necessarily $\sup_t |\varphi(t)| \leqq 1$. By (4.1), there are constants $c_n>0$ and $\gamma_n \in R$ such that

$$\varphi(t/c_n)^n \exp(-it\gamma_n/c_n) = \varphi(t).$$

Thus, every solution of (4.1) with a nonconstant modulus can be obtained as a uniform limit of (3.1), hence (3.2) is satisfied. This completes the proof.

Our next goal is to solve equation (3.2) in $C(R)$ and to single out those solutions which are actually Fourier transforms, i.e. in $B(R)$. For the latter purpose, we need the following criterion:

LEMMA 4.1. *Let φ be an absolutely continuous function, and let φ and φ' be in $L_p(R)$ for some p, $1<p\leqq 2$. Then $\varphi=\hat{\mu}$ is in $A(R)$ (i.e. μ is absolutely continuous), and (putting $1/p+1/q=1$)*

$$\|\mu\| \leqq q^{1/pq}\pi^{-1/q}(\|\varphi\|_p)^{1/q}(\|\varphi'\|_p)^{1/p}. \tag{4.2}$$

Proof. The L_p-transforms of φ and φ',

$$f(x) = \lim_{A\to\infty} (1/2\pi)\int_{-A}^{A} \exp(-itx)\varphi(t)\,dt,$$

and

$$g(x) = \lim_{A\to\infty} (1/2\pi)\int_{-A}^{A} \exp(-itx)\varphi'(t)\,dt,$$

exist in the norm of $L_q(R)$, and $\|f\|_q \leqq (2\pi)^{-1/p}\|\varphi\|_p$, $\|g\|_q \leqq (2\pi)^{-1/p}\|\varphi'\|_p$. Moreover, $g(x)=ixf(x)$, which in turn implies that f is in $L_q(R)$ as well as in $L_1(R)$. Hence

$$\varphi(t) = \int \exp(itx)f(x)\,dx.$$

This means that $\varphi=\hat{\mu} \in A(R)$, μ having density f. Moreover, for all $a>0$

$$\begin{aligned}\|\mu\| &= \int |f(x)|\,dx = \int |f(x)|((a+|x|)/(a+|x|))\,dx \\ &\leqq \left(\int (a+|x|)^{-p}\,dx\right)^{1/p}\left(\int (|f(x)|(a+|x|))^q\,dx\right)^{1/q} \\ &\leqq (2/(p-1))^{1/p}a^{-1/q}(a\|f\|_q+\|xf(x)\|_q) \\ &\leqq (2/(p-1))^{1/p}a^{-1/q}(2\pi)^{-1/p}(a\|\varphi\|_p+\|\varphi'\|_p).\end{aligned}$$

The right-hand side takes on its minimum value, given by (4.2), if $a=p\|\varphi'\|_p/q\|\varphi\|_p$.

For $p=q=2$ Lemma 4.1 specializes to a result of Beurling [4], but (4.2) has now the constant $(\sqrt{2}/\pi)^{1/2}$ which is smaller than Beurling's 1.

THEOREM 4.2. *A measure $\mu \in M(R)$ is stable if and only if $\hat{\mu}$ has one of the following forms ($\beta \in R$; c and d are complex constants with* Re $(c)>0$ *and* Re $(d)>0$*):*

(i) $\hat{\mu}(t)=\exp(i\beta t)$ *or* $\hat{\mu}\equiv 0$.

(ii) *There is an $\alpha \in R$, $\alpha\neq 0$, $\alpha\neq 1$, such that*

$$\begin{aligned}\hat{\mu}(t) &= \exp(-c|t|^\alpha+i\beta t), \qquad t \geqq 0, \\ &= \exp(-d|t|^\alpha+i\beta t), \qquad t < 0,\end{aligned}$$

(iii) $(\alpha=1)$

$$\begin{aligned}\hat{\mu}(t) &= \exp(-c|t|+i\beta t\log|t|), \qquad t \geqq 0, \\ &= \exp(-d|t|+i\beta t\log|t|), \qquad t < 0.\end{aligned}$$

For $\alpha>0$, μ is absolutely continuous. For $\alpha<0$, $\delta_\beta-\mu$ is absolutely continuous (δ_β denotes the unit mass at $x=\beta$).

Proof. (a) We first solve (3.2) in $C(R)$. Either $\varphi\equiv 0$ or at least one of the values $\varphi(1)$ and $\varphi(-1)$ is not zero. Suppose $\varphi(1)\neq 0$. Then

$$\varphi(1)^r = \varphi(r^{1/\alpha})\exp(i\gamma(r)).$$

Putting $x=r^{1/\alpha}$, we get

$$\varphi(x) = \exp(x^\alpha\log\varphi(1)-i\gamma(x^\alpha)) \quad \text{for all } x > 0.$$

For $\varphi(-1)\neq 0$, we get

$$\varphi(-x) = \exp(x^\alpha\log\varphi(-1)+i\gamma(x^\alpha)) \quad \text{for } x > 0.$$

Since φ must be in $C(R)$, and since $\varphi(1)$ and $\varphi(-1)$ are arbitrary, we obtain the

functions listed in (ii) and (iii) as well as those for which Re (c) or Re (d) are zero if $\alpha>0$. In addition, for $\alpha<0$, we get

$$\begin{aligned} \varphi(t) &= 0, \quad \text{or} \quad \exp(-c|t|^{\alpha}+i\beta t), \qquad t \geqq 0, \\ &= \exp(-d|t|^{\alpha}+i\beta t), \quad \text{or} \quad 0, \qquad t < 0. \end{aligned} \tag{4.3}$$

(b) It can be easily verified that the functions listed in the theorem are stable. Conversely, if $\hat{\mu} \in B(R)$ is stable then either $|\hat{\mu}| \equiv C$ or $|\hat{\mu}| \not\equiv C$. By a result of Beurling and Helson [5], the first case implies $\hat{\mu} \equiv 0$ or $\hat{\mu}(t) = \exp(it\beta)$. In the second case, $\hat{\mu}$ must be one of the functions found in step (a).

The functions (4.3) and the case Re $(c)=0$ or Re $(d)=0$ for $\alpha>0$ can be excluded by the following argument: If φ were in $B(R)$ then $\varphi\bar{\varphi}=|\varphi|^2$ would also be. But $|\varphi(t)|^2 \to a$ $(t \to \infty)$, $|\varphi(t)|^2 \to b$ $(t \to -\infty)$, with $a \neq b$. This is impossible for a Fourier transform.

It remains to prove that the functions listed in (ii) and (iii) are in $B(R)$. For $\alpha>0$, Lemma 4.1 applies, and $\hat{\mu} \in A(R)$. Take now $\alpha<0$ and consider without restriction of generality the case $\beta=0$. Put $\psi(t)=1-\hat{\mu}(t)$. Again, Lemma 4.1 applies to ψ provided $\alpha<-\frac{1}{2}$. It remains to consider $-\frac{1}{2} \leqq \alpha < 0$. Write $\psi=\psi_1+\psi_2$, where ψ_1 is even and ψ_2 is odd. By Beurling's second criterion [4], $\psi_1 \in A(R)$. We have to show that there is an $f_2 \in L_1(R)$ such that

$$\psi_2(t) = \int \exp(itx) f_2(x)\, dx.$$

The integral

$$f_2(x) = (1/2\pi) \int \exp(-itx)\psi_2(t)\, dt$$

exists for all $x \neq 0$, since $\psi_2 \to 0$ $(|t| \to \infty)$, and $\psi_2' \in L_1(R)$. Integration by parts yields

$$h(x) = 2\pi i f_2(x) = \int \exp(-itx)\psi_2'(t)\, dt.$$

We have $h \in L_1(R)$ since $\psi_2' \in A(R)$, by Beurling's first criterion [4]. Exploiting the particular form of ψ_2', one can show that there is an $x_0>0$ and a $C>0$ such that

$$|f_2(x)| \leqq C|x|^{(\alpha+2)/(\alpha-2)} \quad \text{for } |x| \leqq x_0.$$

This and the fact that $xf_2(x) \in L_1(R)$ imply $f_2 \in L_1(R)$, so that

$$\psi_2^*(t) = \int \exp(itx) f_2(x)\, dx$$

exists. Differentiation shows that $\psi_2^* = \psi_2$, which completes the proof.

REMARK 4.1. A measure μ is said to be *stable in the restricted sense*, if for all $a>0$, $b>0$ there is a $c>0$ such that

$$\hat{\mu}(at)\hat{\mu}(bt) = \hat{\mu}(ct) \quad \text{for all } t \in R.$$

The solutions of this equation are $\hat{\mu}\equiv 0$, $\hat{\mu}\equiv 1$, and

$$\begin{aligned}\hat{\mu}(t) &= \exp(-c|t|^{\alpha}), \qquad t \geqq 0,\\ &= \exp(-d|t|^{\alpha}), \qquad t < 0,\end{aligned}$$

where $\alpha \in R$, $\alpha \neq 0$, $\operatorname{Re}(c)>0$, and $\operatorname{Re}(d)>0$. These functions are exactly the limits of (3.1) under the conditions of Theorem 3.1, if we put $A_n=0$ for all n.

5. **Domains of attraction.** A measure ν is said to belong to the *domain of attraction* $\mathscr{A}(\mu)$ of the measure μ if $\sup_t |\hat{\nu}(t)| \leqq 1$ and if there are constants $A_n \in R$ and $B_n>0$, satisfying $0<\liminf B_n/B_{n+1} \leqq \limsup B_n/B_{n+1}<\infty$, such that

$$\lim_n (\hat{\nu}(t/B_n))^n \exp(itA_n) = \hat{\mu}(t) \quad \text{for all } t \neq 0.$$

The *domain of attraction in the restricted sense*, $\mathscr{A}_r(\mu)$, is obtained if $A_n=0$ for all n. According to §4, *only stable* (*restrictedly stable*) *measures can possess a nonempty* $\mathscr{A}$ ($\mathscr{A}_r$).

The domains of attraction of stable probabilities are characterized either by the tail behavior of the corresponding distribution functions (see [6, pp. 543–547]) or by the behavior of the Fourier transforms at $t=0$ (see [7, p. 171 ff.]).

In the general case, we shall give a description of $\mathscr{A}(\mu)$ in terms of Fourier transforms, consisting of conditions on the behavior at $t=0$ for $\alpha>0$, and at $t=\pm\infty$ for $\alpha<0$. Since obviously

$$\mathscr{A}(\mu) = \mathscr{A}(\mu((\cdot - b)/a)),$$

we can normalize $\hat{\mu}$ to have $\operatorname{Re}(c)=1$, $\beta=0$ for $\alpha\neq 1$, and $\operatorname{Re}(c)=1$, $\operatorname{Im}(c)=0$ for $\alpha=1$, in the representation of Theorem 4.2.

According to Karamata [9], a measurable, positive function $L(s)$ is called *slowly varying* for $s\to\infty$ ($s\to +0$) if $L(st)/L(s)\to 1$ for $s\to\infty$ ($s\to +0$) and for all $t>0$.

THEOREM 5.1. (a) *Let* $\alpha>0$, $\alpha\neq 1$, *and write*

$$\hat{\nu}(t) = \exp(-|t|^{\alpha}(L_1(t)+iL_2(t)))$$

with real L_2 *and* (*necessarily*) *nonnegative* L_1. *Then* $\nu\in\mathscr{A}(\mu)$ *if and only if for* $s\to +0$

(i) $L_1(st)/L_1(s)\to 1$ *for all* $t>0$ (*i.e.* L_1 *is slowly varying*),

(ii) $L_1(-s)/L_1(s)\to \operatorname{Re}(d)$,

(iii) $(t^{\alpha}L_2(st)-tL_2(s))/L_1(s)\to \operatorname{Im}(c)(t^{\alpha}-t)$ *for all* $t>0$,

(iv) $(L_2(s)+L_2(-s)/L_1(s)\to \operatorname{Im}(c+d)$.

The constants can be chosen as follows: $1/B_n$ *is the smallest positive root of* $s^{\alpha}L_1(s)=1/n$, *and* $A_n=-\operatorname{Im}(c)+nL_2(1/B_n)/B_n^{\alpha}$.

(b) *For* $\alpha<0$, *part* (a) *remains valid if we replace* $s\to+0$ *by* $s\to\infty$ *and "smallest positive root" by "largest positive root."*

(c) *For* $\alpha=1$, *write*

$$\hat{\nu}(t) = \exp(-|t|L_1(t)+iL_2(t)).$$

Then $\nu\in\mathscr{A}(\mu)$ *if and only if for* $s\to+0$

(i) $L_1(st)/L_1(s)\to 1$ *for all* $t>0$,

(ii) $L_1(-s)/L_1(s)\to \mathrm{Re}\,(d)$,

(iii) $(L_2(st)-L_2(s))/L_1(s)\to\beta\log t$ *for all* $t>0$,

(iv) $(L_2(-s)-L_2(s))/L_1(s)\to \mathrm{Im}\,(d)$.

The constants can be chosen as follows: $1/B_n$ *is the smallest positive root of* $sL_1(s)=1/n$, *and* $A_n=nL_2(1/B_n)/B_n$.

Proof. We restrict ourselves to the proof of part (a). First suppose $\nu\in\mathscr{A}(\mu)$. In other words, there are A_n's and B_n's such that

$$\exp(-|t|^\alpha(L_1(t/B_n)+iL_2(t/B_n))n/B_n^\alpha+itA_n)\to\hat{\mu}(t).$$

This implies

$$\begin{aligned}|t|^\alpha nL_1(t/B_n)/B_n^\alpha &\to |t|^\alpha, && t\geqq 0,\\ &\to \mathrm{Re}\,(d)|t|^\alpha, && t<0,\end{aligned}\tag{5.1}$$

and

$$\begin{aligned}|t|^\alpha nL_2(t/B_n)/B_n^\alpha-itA_n &\to \mathrm{Im}\,(c)|t|^\alpha, && t\geqq 0,\\ &\to \mathrm{Im}\,(d)|t|^\alpha, && t<0.\end{aligned}\tag{5.2}$$

Putting $t=1$ in (5.1) and dividing yields for all $t>0$

$$L_1(t/B_n)/L_1(1/B_n)\to 1$$

and, similarly,

$$L_1(-t/B_n)/L_1(t/B_n)\to \mathrm{Re}\,(d).$$

Since, by Lemma 3.4, (5.1) actually holds uniformly in every interval bounded away from zero and infinity, and since $\{B_m/B_n\}$ is dense, (i) and (ii) follow.

Relation (5.2) leads easily to

$$(t^\alpha L_2(t/B_n)-tL_2(1/B_n))/L_1(1/B_n)\to \mathrm{Im}\,(c)(t^\alpha-t)\qquad (t>0)$$

and

$$(L_2(t/B_n)+L_2(-t/B_n))/L_1(t/B_n)\to \mathrm{Im}\,(c+d).$$

The proof would be completed if (5.2) were to hold uniformly in every interval bounded away from zero and infinity. This is indeed the case, as an argument similar to the one used in Lemma 3.4 shows.

Conversely, assume that (i) through (iv) are true. We have to show $\nu\in\mathscr{A}(\mu)$. The B_n's are well defined since $L_1(t)>0$ in some interval around $t=0$, and $s^\alpha L_1(s)\to 0$ $(s\to+0)$. An application of Lemma 6.1 shows that $B_n/B_{n+1}\to 1$ so that our fundamental condition on the B_n's is satisfied.

Consider the case $t>0$ and use the fact that $nL_1(1/B_n)=B_n^\alpha$. Then

$$\begin{aligned}(\hat{\nu}(t/B_n))^n \exp(itA_n) &= \exp(-|t|^\alpha(L_1(t/B_n)/L_1(1/B_n)) \\ &\qquad +i(L_2(t/B_n)-t^{1-\alpha}L_2(1/B_n))/L_1(1/B_n)+it^{1-\alpha}\operatorname{Im}(c)) \\ &\to \exp(-c|t|^\alpha),\end{aligned}$$

by (i) and (iii). For $t<0$, we have to use (ii) and (iii) to see that

$$(\hat{\nu}(t/B_n))^n \exp(itA_n) \to \exp(-d|t|^\alpha).$$

This completes the proof.

The set $\mathscr{A}_r(\mu)$ can be characterized even more simply.

THEOREM 5.2. *Let μ be a nondegenerate stable measure in the restricted sense (thus being of the form stated in Remark* 4.1), *and write*

$$\hat{\nu}(t) = \exp(-|t|^\alpha(L_1(t)+iL_2(t))).$$

Then $\nu \in \mathscr{A}_r(\mu)$ if and only if for $s \to +0$ (if $\alpha>0$) or for $s\to\infty$ (if $\alpha<0$)

(i) $L_1(st)/L_1(s) \to 1$ *for all* $t>0$,

(ii) $L_1(-s)/L_1(s) \to \operatorname{Re}(d)/\operatorname{Re}(c)$,

(iii) $L_2(s)/L_1(s) \to \operatorname{Im}(c)/\operatorname{Re}(c)$,

(iv) $L_2(-s)/L_1(s) \to \operatorname{Im}(d)/\operatorname{Re}(c)$.

The B_n's can be chosen as follows: B_n^{-1} is the smallest (if $\alpha>0$) or the largest (if $\alpha<0$) positive root of $s^\alpha L_1(s)=\operatorname{Re}(c)/n$.

Note that a measure ν can belong to (at most) one domain of attraction $\mathscr{A}(\mu_1)$, where μ_1 has a positive exponent α, and *at the same time* to (at most) one domain of attraction $\mathscr{A}(\mu_2)$, where μ_2 has a negative exponent α. An example is $\hat{\nu}(t) = \exp(-|t|^\alpha)+\exp(-|t|^{-\alpha})$ for $\alpha>0$.

6. **Convergence of the corresponding measures.** If $\sup_t |\hat{\nu}_n(t)| \leqq c<\infty$ and $\lim_n \hat{\nu}_n(t)=\hat{\mu}(t)$ for all $t\in R$ (for almost all t) then

$$\lim_n \int f(x)\,d\nu_n(x) = \int f(x)\,d\mu(x)$$

for all f in $B(R)$ ($A(R)$). This in turn implies convergence of $\{\nu_n\}$ to μ in the distribution sense, i.e. convergence for all f which are rapidly decreasing and infinitely differentiable. But we cannot hope for any stronger mode of convergence unless we impose norm-boundedness of $\{\nu_n\}$ which is unnatural in our case.

However, in the particular case of convolution powers considered here something can be said. The point is that pointwise convergence of the special sequence considered here automatically implies a much stronger form of convergence, as we shall see in Lemma 6.1.

In this section, we confine ourselves to restrictedly stable measures and $\mathscr{A}_r$. In other words, we omit the A_n's.

Everything which follows is based on

LEMMA 6.1. *Let ν be in $\mathscr{A}_r(\mu)$.*

(i) *If $\alpha>0$ then for any $\varepsilon>0$*

$$\lim_n ((\hat{\nu}(t/B_n))^n-\hat{\mu}(t))|t|^{-\alpha+\varepsilon} = 0$$

uniformly in every interval $|t|\leqq a$, $a<\infty$.

(ii) *If $\alpha<0$ then for any $\varepsilon>0$*

$$\lim_n ((\hat{\nu}(t/B_n))^n-\hat{\mu}(t))|t|^{-\alpha-\varepsilon} = 0$$

uniformly in every set $|t|\geqq a$, $a>0$.

Proof. (i) It suffices to treat the case $t\geqq 0$ and $\operatorname{Re}(c)=1$. Therefore let $\hat{\mu}(t)=\exp(-(1+ic_2)|t|^\alpha)$ and $\hat{\nu}(t)=\exp(-(L_1(t)+iL_2(t))|t|^\alpha)$, the L_k's satisfying the conditions of Theorem 5.2. It is sufficient to prove that

$$\lim_n (nL_1(t/B_n)/B_n^\alpha-1)t^\varepsilon = 0 \tag{6.1}$$

and

$$\lim_n (nL_2(t/B_n)/B_n^\alpha-c_2)t^\varepsilon = 0, \tag{6.2}$$

both holding uniformly in every interval $[0, a]$, $0<a<\infty$. We know that $nL_1(1/B_n)/B_n^\alpha=1+\varepsilon_n$, where $\varepsilon_n\to 0$. It is also known (see [9, p. 45]) that for every slowly varying function L_1

$$\lim_{s\to+0} L_1(st)/L_1(s) = 1 \quad \text{uniformly in } [\delta, a], \qquad 0<\delta\leqq a<\infty. \tag{6.3}$$

Therefore, (6.1) will be established if we can show that for every (for $s\to+0$) slowly varying function L_1

$$\lim_{s\to+0} t^\varepsilon L_1(st)/L_1(s) = t^\varepsilon \quad \text{uniformly for } 0\leqq t\leqq\delta, \qquad \delta>0. \tag{6.4}$$

We now prove this. Because of (6.3), it suffices to prove that for every $\varepsilon_1>0$ there is an interval $[0, \delta_1]$ and an $s_0(\varepsilon_1)$ such that for all $s\leqq s_0$,

$$t^\varepsilon L_1(st)/L_1(s) < \varepsilon_1 \quad \text{for all } t\in[0, \delta_1].$$

According to Karamata [9],

$$L_1(s) = c(s)\exp\left(\int_s^1 (\varepsilon(t)/t)\,dt\right),$$

where $c(s)\to c>0$ and $\varepsilon(s)\to 0$ $(s\to+0)$. Choose s_0 and τ_0 so that $c(ts)/c(s)<1+\varepsilon_2$ if $s\leqq s_0$ for all $t\in[0, a]$, and $|\varepsilon(\tau)|<\varepsilon_3$ for all $\tau\leqq\tau_0$. Then

$$\begin{aligned} t^\varepsilon L_1(st)/L_1(s) &= t^\varepsilon(c(st)/c(s))\exp\left(\int_{st}^s (\varepsilon(\tau)/\tau)\,d\tau\right) \\ &\leqq t^{\varepsilon+\varepsilon_3}(1+\varepsilon_2)<\varepsilon_1, \end{aligned}$$

for all $s \leqq s_0$ and $0 \leqq t \leqq \delta_1(\varepsilon_1)$. So (6.4) and, therefore, (6.1) are true. Taking into account property (iii) of Theorem 5.2, (6.2) reduces to (6.1).

(ii) The proof is completely analogous to that of (i) if one uses the fact that

$$\lim_{s \to \infty} t^{-\varepsilon} L_1(st)/L_1(s) = t^{-\varepsilon}$$

uniformly in $|t| \geqq a$, $a > 0$, if L_1 varies slowly for $s \to \infty$.

The choice of $\varepsilon = \alpha$ (if $\alpha > 0$) and $\varepsilon = -\alpha$ (if $\alpha < 0$) in Lemma 6.1 shows that in particular uniform convergence of $(\hat{\nu}(t/B_n))^n$ to $\hat{\mu}(t)$ holds.

THEOREM 6.1. *Let* $\nu \in \mathscr{A}_r(\mu)$. *Put* $(\hat{\nu}(t/B_n))^n = \hat{\nu}_n(t)$ *and* $\nu_n((-\infty, x)) = \nu_n(x)$, $\mu((-\infty, x)) = \mu(x)$.

(i) *If* $\alpha > 0$ *then*

$$\lim_n \int_{-\infty}^{\infty} |\nu_n(x) - \mu(x)|^p \, dx = 0$$

for all $p \geqq 2$ *if* $\alpha > \frac{1}{2}$ *and* $p > 1/\alpha$ *if* $0 < \alpha \leqq \frac{1}{2}$.

(ii) *If* $\alpha < 0$ *then* $\delta_0 - \nu_n$ *is continuous and*

$$\lim_n \nu_n(I) = \mu(I)$$

for every compact interval $I \subset R$.

Proof. (i) We use the fact that

$$\hat{\beta}_n(t) = (\hat{\nu}_n(t) - \hat{\mu}(t))/(-it) = \int \exp(itx)(\nu_n(x) - \mu(x)) \, dx.$$

By Lemma 6.1 (i), $\hat{\beta}_n$ is in $L_p(R)$ for all n, where $p > 1$ for $\alpha \geqq 1$, and $1 < p < 1/(1-\alpha)$ for $0 < \alpha < 1$. We conclude (see [13, p. 96]) that $\nu_n(x) - \mu(x) \in L_q(R)$, where $2 \leqq q < \infty$ for $\alpha > 1/2$ and $1/\alpha < q < \infty$ for $0 < \alpha \leqq 1/2$. Furthermore,

$$\int_{-\infty}^{\infty} |\nu_n(x) - \mu(x)|^q \, dx \leqq (2\pi)^{1-q} \left(\int_{-\infty}^{\infty} |\hat{\beta}_n(t)|^p \, dt \right)^{1/p-1}$$

With the help of Lemma 6.1 (i) it is now easy to show that the right-hand side tends to zero.

(ii) The measure $\delta_0 - \nu_n$ is continuous since $1 - \hat{\nu}_n$ vanishes at infinity. Because $\delta_0 - \mu$ is absolutely continuous, $\nu_n - \mu$ is continuous. Hence, for every x, y,

$$\nu_n((x, y)) - \mu((x, y)) = (2\pi)^{-1} \int_{-\infty}^{\infty} ((\exp(-ity) - \exp(-itx))/(-it))(\nu_n(t) - \hat{\mu}(t)) \, dt.$$

Again, it is a simple calculation that Lemma 6.1 (ii) implies that the right-hand side of the last equation tends to zero for $n \to \infty$, which completes the proof.

As far as we know, Theorem 6.1 (i) is new even for the probability case, except for $\alpha = 2$ (see [7, p. 250 ff.]).

It seems to us that Theorem 6.1 is quite compatible with the general Tauberian idea that local (global) behavior in $B(R)$ is reflected by global (local) behavior in

$M(R)$. For example, $\nu \in \mathscr{A}_r(\mu)$ is a local statement for $\alpha > 0$ but a global statement (on the tail) for $\alpha < 0$. Theorem 6.1 shows global convergence of the measures in the first case, but local convergence in the second case.

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