

COMPACTIFICATION OF STRONGLY COUNTABLE-DIMENSIONAL SPACES⁽¹⁾

BY
ARLO W. SCHURLE

1. **Introduction.** In dimension theory it is well known [2, p. 65] that a finite-dimensional separable metrizable space may be topologically embedded in a compact metrizable space of the same dimension. A natural question is whether an analogous statement holds for the various types of infinite-dimensional spaces defined in [6, pp. 161–162]. Our main result is that strongly countable-dimensional absolute G_δ -spaces have strongly countable-dimensional compactifications. (A space is strongly countable-dimensional if it is a countable union of closed finite-dimensional subsets.) We restrict ourselves exclusively to separable metrizable spaces, assuming a fixed compatible metric to be given whenever necessary. Notation and definitions will be as in [3] and [6].

E. G. Sklyarenko has shown [7, p. 40] that weakly infinite-dimensional spaces have weakly infinite-dimensional compactifications. He also gives an example [7, p. 42] of a strongly countable-dimensional space which has neither a countable-dimensional nor a strongly countable-dimensional compactification. Hence, an added condition is necessary to insure that a countable-dimensional or strongly countable-dimensional space has the desired compactification. Lelek has shown [5] that if X is countable-dimensional and an absolute G_δ , then X has a countable-dimensional compactification. In the same paper he raises the question which Theorem 2 answers affirmatively, namely, whether his result is true with countable-dimensional replaced by strongly countable-dimensional (weakly countable-dimensional in [5]).

2. **Some preliminary propositions.** Throughout this paper letters such as i, j, k will denote positive integers. Also, unless stated otherwise, notation such as $\{A_i\}$ and $\bigcup A_i$ will always denote $\{A_i \mid i=1, 2, \dots\}$ and $\bigcup_{i=1}^{\infty} A_i$. If there are double subscripts, then the variable one will be indicated. The following proposition may be proved easily by using the results and techniques of [2, pp. 53–55], and so the proof is omitted.

PROPOSITION 1. *Let $M \subset X$ with $\dim M \leq n$. Let $\{U_i\}$ be a collection of sets open in X and covering M . Then there is a collection $\{V_i\}$ of sets open in X and covering M*

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such that $\text{ord } \{V_i\} \leq n+1$ and such that $V_{k(n+1)+j} \subset U_{k+1}$ for $k=0, 1, 2, \dots$ and $j=1, 2, \dots, n+1$.

PROPOSITION 2. *Let G be an open subset of a totally bounded space Y and let M_1, M_2, \dots, M_r be relatively closed subsets of G with $\dim M_i = m_i < \infty$ for $i=1, 2, \dots, r$. Let $\epsilon > 0$. Then there is a collection $\{G_i\}$ such that*

- (i) $G = \bigcup G_i$,
- (ii) $\bar{G}_i \subset G$ for $i=1, 2, \dots$,
- (iii) G_i is open in Y for $i=1, 2, \dots$,
- (iv) $\text{diam } G_i < \epsilon$ for $i=1, 2, \dots$ and $\text{diam } G_i \rightarrow 0$ as $i \rightarrow \infty$,
- (v) $\{G_i\}$ is star-finite, and,
- (vi) $\text{ord } \{G_i \mid G_i \text{ meets } M_1 \cup M_2 \cup \dots \cup M_k\} \leq m_1 + 1 + m_2 + 1 + \dots + m_k + 1$ for $k=1, 2, \dots, r$.

Proof. By a very slight modification of [4, Proposition 5, p. 114] there exists a collection $\mathcal{G}_0 = \{G_{0,i}\}_i$ satisfying (i)–(v). Letting $M_0 = \emptyset$ and $m_0 = -1$, we see that \mathcal{G}_0 satisfies (vi) for $k=0$. We now inductively define collections satisfying (i)–(v) and (vi) for $k=1, 2, \dots, r$ successively. Suppose \mathcal{G}_n is a collection satisfying (i)–(v) and (vi) for $k=0, 1, \dots, n$. Let $C = M_0 \cup M_1 \cup \dots \cup M_n$. Then $\{G_{n,t} - C\}_t$ is an open cover of $G - C$, so by Proposition 1 there is an open collection $\{V_i\}$ covering $M_{n+1} - C$ such that $\text{ord } \{V_i\} \leq m_{n+1} + 1$ and such that

$$(a) \quad V_{t(m_{n+1}+1)+j} \subset G_{n,t+1} - C$$

for $t=0, 1, 2, \dots$ and $j=1, 2, \dots, m_{n+1} + 1$.

Let

$$\begin{aligned} \mathcal{G}_{n+1} &= \{G_{n,t} - (C \cup M_{n+1})\}_t \cup \{V_i\} \cup \{G_{n,t} \mid G_{n,t} \text{ meets } C\} \\ &= \{G_{n+1,i}\}_i, \end{aligned}$$

where in the last sequence we rename the sets in the union. Using (a) we easily see that \mathcal{G}_{n+1} satisfies (i)–(v), and it is clear that \mathcal{G}_{n+1} satisfies (vi) for $k=0, 1, \dots, n$. Let $\mathcal{H} = \mathcal{K} \cup \mathcal{L}$, where

$$\mathcal{H} = \{G_{n,t} \mid G_{n,t} \text{ meets } M_0 \cup M_1 \cup \dots \cup M_n\},$$

and

$$\mathcal{L} = \{V_i \mid V_i \text{ meets } M_{n+1}\}.$$

Let $s = m_1 + 1 + m_2 + 1 + \dots + m_{n+1} + 1$, and let \mathcal{P} be a set of $s+1$ distinct elements of \mathcal{H} . If $\mathcal{P} \cap \mathcal{K}$ contains $(m_0 + 1 + m_1 + 1 + \dots + m_n + 1) + 1$ or more elements, then $\bigcap \mathcal{P} = \emptyset$, since $\text{ord } \mathcal{K} \leq m_0 + 1 + m_1 + 1 + \dots + m_n + 1$. If this is not the case, then $\mathcal{P} \cap \mathcal{L}$ must contain $(m_{n+1} + 1) + 1$ or more elements. Then again $\bigcap \mathcal{P} = \emptyset$ since $\text{ord } \mathcal{L} \leq m_{n+1} + 1$. Therefore $\text{ord } \mathcal{H} \leq s$. The induction is complete, and the collection \mathcal{G}_r satisfies the requirements of the proposition.

In our terminology the term “simplex” always denotes a closed simplex with the Euclidean topology. I^ω denotes the Hilbert cube, and if $x \in I^\omega$, then the first coordinate of x will be called the abscissa of x . The following proposition is

essentially contained in [4, Proposition 2, p. 213] and is stated here for the sake of completeness.

PROPOSITION 3. *Let a sequence of points $\{g_i\}$ be given in I^ω such that the abscissa of each g_i is zero, and let a sequence of positive numbers $\{\varepsilon_i\}$ be given. Then there is a sequence of points $\{p_i\}$ in I^ω satisfying*

- (i) $d(p_i, g_i) < \varepsilon_i$ for $i = 1, 2, \dots$
- (ii) The abscissa of p_i is positive for $i = 1, 2, \dots$
- (iii) Each finite subset $\{p_{i_0}, p_{i_1}, \dots, p_{i_n}\}$ of $\{p_i\}$ spans an n -simplex.
- (iv) Each subspace of I^ω which is the union of a countable star-finite collection of simplexes spanned by finite subsets of $\{p_i\}$ is a polytope with the weak topology.

3. The main theorems. The term “map” will be reserved for continuous functions. The first theorem is analogous to [4, Theorem 1, p. 215], but the conditions concerning dimension are new and will be crucial in the following theorem.

THEOREM 1. *Let C be a closed subset of a compact space Y and let $\varepsilon > 0$. Let M_1, M_2, \dots, M_r be closed subsets of Y such that $\dim M_i = m_i < \infty$ for $i = 1, 2, \dots, r$. Then there is an ε -map f from Y into I^ω such that*

- (i) $f(C) \cap f(Y - C) = \emptyset$,
- (ii) $f|C$ is a homeomorphism,
- (iii) $f(Y - C)$ is a countable polytope P ,
- (iv) $P = \bigcup \Sigma_i$, where $\{\Sigma_i\}$ is the collection of simplexes of P and $\text{diam } \Sigma_i \rightarrow 0$ as $i \rightarrow \infty$,
- (v) $\dim f(M_i - C) \leq m_i + 1 + m_2 + 1 + \dots + m_i$ for $i = 1, 2, \dots, r$.

Proof. Without loss of generality we may assume that $Y \subset I^\omega$, and then that the abscissa of each point of Y is zero. If $C = Y$, then f may be taken to be the inclusion map of Y in I^ω , so we assume that $C \neq Y$. Let $G = Y - C$ and let $\mathcal{G} = \{G_i\}$ satisfy the conditions of Proposition 2 for G, Y , and $M_1 - C, M_2 - C, \dots, M_r - C$, with $\text{diam } G_i < \varepsilon/8$ for $i = 1, 2, \dots$. In the remainder of the proof we consider only those integers i for which $G_i \neq \emptyset$, even though we retain the same notation and continue to write $i = 1, 2, \dots$. For each $i = 1, 2, \dots$ choose a $g_i \in G_i$. Now choose points $p_i \in I^\omega$ such that

$$d(p_i, g_i) < \min \{1/i, \varepsilon/8\}$$

and such that the sequence $\{p_i\}$ satisfies the conditions of Proposition 3.

Let N be the union of the collection of simplexes in I^ω defined as follows. Each simplex of the collection is spanned by a finite subset of $\{p_i\}$ and $\{p_{i_0}, p_{i_1}, \dots, p_{i_n}\}$ spans a simplex in the collection if and only if $G_{i_0} \cap G_{i_1} \cap \dots \cap G_{i_n} \neq \emptyset$. By Proposition 3, N is a polytope with the weak topology.

Define a function $f': Y \rightarrow I^\omega$ by

$$f'(x) = \begin{cases} x & \text{if } x \in C, \\ \sum_{i=1}^\infty d(x, Y - G_i) p_i / \sum_{i=1}^\infty d(x, Y - G_i) & \text{if } x \in Y - C. \end{cases}$$

A standard argument shows that f' is continuous, and that $d(y, f'(y)) < \epsilon/4$ for each $y \in Y$. $f'|_C$ is the identity map on C and $f'(Y-C) \subset N$. Each vertex of N , and hence each point of N , has abscissa greater than zero, so $f'(C) \cap f'(Y-C) = \emptyset$.

If N is finite, then by repeating the empty simplex we may write $N = \bigcup \Sigma_i$, where $\text{diam } \Sigma_i \rightarrow 0$ as $i \rightarrow \infty$. Suppose N is infinite, and let $\delta > 0$. Let $N = \bigcup \Sigma_i$. There is an integer j_0 such that $i \geq j_0$ implies that $\text{diam } G_i < \delta/6$ and $1/i < \delta/6$. Since N is star-finite, there is an integer k_0 such that for $k > k_0$ none of the points p_1, p_2, \dots, p_{j_0} is a vertex of Σ_k . Suppose $k > k_0$ and let Σ_k be spanned by $p_{i_0}, p_{i_1}, \dots, p_{i_t}$. Then for $s=0, 1, \dots, t$

$$d(p_{i_s}, g_{i_s}) < 1/i_s < \delta/6$$

and

$$d(g_{i_s}, g_{i_0}) \leq \text{diam } G_{i_s} + \text{diam } G_{i_0} < \delta/3.$$

Hence $d(p_{i_s}, g_{i_0}) < \delta/2$, so that $\text{diam } \Sigma_k < \delta$. Therefore $\text{diam } \Sigma_j \rightarrow 0$ as $j \rightarrow \infty$.

As a subset of I^ω N is bounded, so by repeated barycentric subdivision N may be triangulated into simplexes of diameter less than $\epsilon/4$. Since each "old" simplex of N yields only finitely many "new" ones, we may write $N = \bigcup \Sigma_i$ where $\text{diam } \Sigma_i \rightarrow 0$ as $i \rightarrow \infty$ and $\{\Sigma_i\}$ is the collection of all "new" simplexes of N . We use the "new" simplexes in the next construction.

We now define a map F taking $f'(Y) \cap N$ onto a subpolytope P of N . Let \mathcal{F}_0 be the set of all simplexes Σ of N such that Σ is a face of no other simplex of N . Since N is star-finite, $N = \bigcup \mathcal{F}_0$. Let $\mathcal{F}_0 = \{\Sigma_i\}$. For each i define a map h_i from $f'(Y) \cap \Sigma_i$ into N as follows. If $\Sigma_i \subset f'(Y)$, then let h_i be the identity injection. If $\Sigma_i \not\subset f'(Y)$, then since $f'(Y)$ is compact, there is a point

$$q_i \in (\text{Int } \Sigma_i) - f'(Y);$$

in this case, let h_i project $f'(Y) \cap \Sigma_i$ into the boundary of Σ_i from q_i . Define a function f_0 from $f'(Y) \cap N$ into N by $f_0 = \bigcup h_i$. f_0 is clearly well defined, and since N has the weak topology, f_0 is a map.

Next, let $\mathcal{M}_1 = \{\Sigma \mid \Sigma \in \mathcal{F}_0 \text{ and } \Sigma \subset f'(Y), \text{ or } \Sigma \text{ is a face of a simplex } \Sigma' \text{ in } \mathcal{F}_0 \text{ such that } \Sigma' \not\subset f'(Y)\}$. Let $N_1 = \bigcup \mathcal{M}_1$ and let \mathcal{F}_1 be the set of all simplexes Σ of N_1 such that Σ is a face of no other simplex of N_1 . We again have that $N_1 = \bigcup \mathcal{F}_1$. Clearly $f_0(f'(Y) \cap N) \subset N_1$. Define a map f_1 from $f_0(f'(Y) \cap N)$ into N_1 in the same way that f_0 was defined, replacing $f'(Y)$ by $f_0(f'(Y) \cap N)$.

Continue in this manner to obtain a decreasing sequence $N_0 = N, N_1, N_2, \dots$ of subpolytopes of N and maps

$$\begin{aligned} f_0 &: f'(Y) \cap N \rightarrow N_1, \\ f_1 &: f_0(f'(Y) \cap N) \rightarrow N_2, \\ f_2 &: f_1(f_0(f'(Y) \cap N)) \rightarrow N_3, \\ &\dots \end{aligned}$$

These maps clearly have the property that for each $f'(y) \in N$ the sequence

$$\{f_n \circ f_{n-1} \circ \dots \circ f_0 \circ f'(y) \mid n = 0, 1, 2, \dots\}$$

is eventually constant. Define a function F from $f'(Y) \cap N$ into N by letting $F(f'(y))$ be the eventually constant value of the above sequence.

Let Σ be a simplex of N . It is clear that there is an integer n such that

$$f_m \circ f_{m-1} \circ \cdots \circ f_0 \circ f'(y) = f_n \circ f_{n-1} \circ \cdots \circ f_0 \circ f'(y)$$

for all $m \geq n$ and all $f'(y) \in \Sigma$. Hence

$$F | (f'(Y) \cap \Sigma) = f_n \circ f_{n-1} \circ \cdots \circ f_0 | (f'(Y) \cap \Sigma).$$

Since N has the weak topology, F is a map. It is easily seen that $F(f'(y))$ is in the same simplex as $f'(y)$ for each $f'(y) \in N$. Further, $F(f'(Y) \cap \Sigma)$ is the union of certain faces of Σ for each simplex Σ of N , so that $F(f'(Y) \cap N)$ is a subpolytope P of N .

Define a function f from Y into I^ω by

$$\begin{aligned} f(z) &= z && \text{if } z \in C, \\ &= Ff'(z) && \text{if } z \in Y - C. \end{aligned}$$

Since $f(z)$ and $f'(z)$ are in the same simplex for $z \in Y - C$, a standard argument shows that f is continuous. Properties (i)–(iv) certainly hold for f . N was triangulated into simplices of diameter less than $\varepsilon/4$, so that $d(f(y), f'(y)) < \varepsilon/4$ for each $y \in Y$. Therefore

$$\begin{aligned} d(f(y), y) &\leq d(f(y), f'(y)) + d(f'(y), y) \\ &< \varepsilon/4 + \varepsilon/4 = \varepsilon/2. \end{aligned}$$

It is then clear that f is an ε -map.

We need now only verify (v). Let $y \in M_i - C$, and let $n_i = m_1 + 1 + m_2 + 1 + \cdots + m_i$. By the properties of the open cover \mathcal{G} , y is in at most $n_i + 1$ elements of \mathcal{G} . Hence $f'(y)$ is in a simplex of dimension not greater than n_i . Since $f'(y)$ and $f(y)$ are in the same simplex and since subdividing a simplex does not raise its dimension, $f(y)$ is in a simplex of dimension not greater than n_i . Thus $f(M_i - C) \subset \bigcup \{\Sigma_j \mid \Sigma_j \text{ is a simplex of } P \text{ and } \dim \Sigma_j \leq n_i\}$. By the countable sum theorem [2, p. 30] and the subspace theorem [2, p. 26]

$$\dim f(M_i - C) \leq n_i$$

for $i = 1, 2, \dots, r$.

The goal of this paper is reached by the following theorem. In fact, the theorem yields the desired compactification in such a way that the remainder is a pseudopolytope. (A metric space is called a pseudopolytope if it is the union of a sequence $\{\Sigma_i\}$ of simplices such that for any positive integers i and j $\Sigma_i \cap \Sigma_j$ is either empty or a face of both Σ_i and Σ_j and such that $\text{diam } \Sigma_i \rightarrow 0$ as $i \rightarrow \infty$.)

THEOREM 2. *Let X be strongly countable-dimensional and an absolute G_δ . Then there is a strongly countable-dimensional compactification dX of X such that $dX - X$ is a pseudopolytope.*

Proof. Let $X = \bigcup F_i$, where F_i is closed and $\dim F_i = m_i < \infty$ for $i = 1, 2, \dots$. By a result of Hurewicz [1, p. 207] there is a compactification cX of X such that $\dim \bar{F}_i^{cX} = m_i$ for $i = 1, 2, \dots$. Since X is an absolute G_δ , there are compact subsets Y_1, Y_2, \dots of cX such that $Y_i \subset Y_{i+1}$ and $cX - X = \bigcup Y_i$. Let $Y_0 = \emptyset$, and let

$$n_k = m_1 + 1 + m_2 + 1 + \dots + m_k$$

for $k = 1, 2, \dots$. Let i be any positive integer. By Theorem 1 there is a $1/i$ -map f_i from Y_i into I^ω such that

- (i) $f_i(Y_{i-1}) \cap f_i(Y_i - Y_{i-1}) = \emptyset$,
- (ii) $f_i|Y_{i-1}$ is a homeomorphism,
- (iii) $f_i(Y_i - Y_{i-1})$ is a countable polytope N_i ,
- (iv) $N_i = \bigcup_j \Sigma_{ij}$ where $\text{diam } \Sigma_{ij} \rightarrow 0$ as $j \rightarrow \infty$,
- (v) $\dim f_i(\bar{F}_i^{cX} \cap (Y_i - Y_{i-1})) \leq n_k$ for $k = 1, 2, \dots, i$.

Decompose cX into sets $f_i^{-1}(z)$ for $z \in f_i(Y_i - Y_{i-1})$ and into individual points $x \in X$. Let the quotient space be dX and the quotient map be f .

We show that the decomposition of cX is upper semicontinuous. Let U be an open set in cX containing a point x of X . There is a positive ϵ such that $x \in S_\epsilon(x) \subset U$ and there is a positive integer i_0 such that $1/i_0 < \epsilon/2$. Then the set

$$\bigcup \{f^{-1}(z) \mid f^{-1}(z) \subset cX - Y_{i_0} \text{ and } f^{-1}(z) \text{ meets } S_{\epsilon/2}(x)\}$$

may easily be shown to be an inverse set which is a neighborhood of x contained in U .

Suppose that $f^{-1}(z_0) \subset Y_{i_0} - Y_{i_0-1}$ and that $f^{-1}(z_0) \subset U$, where U is open in cX . Since $f^{-1}(z_0)$ is compact, there is a positive ϵ such that $S_\epsilon(f^{-1}(z_0)) \subset U$, and then there is an integer $j_0 \geq i_0$ such that $1/j_0 < \epsilon/2$. Let

$$\begin{aligned} A &= \bigcup \{f^{-1}(z) \mid f^{-1}(z) \subset cX - Y_{j_0} \text{ and } f^{-1}(z) \text{ meets } S_{\epsilon/2}(f^{-1}(z_0))\}, \\ B &= \bigcup \{f^{-1}(z) \mid f^{-1}(z) \subset Y_{j_0} \cap S_\epsilon(f^{-1}(z_0))\}, \\ C &= A \cup B. \end{aligned}$$

Then C is clearly an inverse set containing $f^{-1}(z_0)$ and contained in U . If the decomposition of Y_k is upper semicontinuous for each k , then $B = V \cap Y_{j_0}$, where V is open in cX . Then $V \cap S_{\epsilon/2}(f^{-1}(z_0)) \subset C$, so that C is a neighborhood of $f^{-1}(z_0)$.

We show by induction that the decomposition of Y_k is upper semicontinuous for each k . f_i is a closed map on Y_i for $i = 1, 2, \dots$. Since the decompositions of Y_1 induced by f and f_1 are the same, the decomposition of Y_1 is upper semicontinuous. Suppose this is true of Y_{k-1} , and let U be open in Y_k with $f^{-1}(z_0) \subset U$. If $f^{-1}(z_0) \subset Y_k - Y_{k-1}$, then since f and f_k induce the same decompositions of $Y_k - Y_{k-1}$ and $f_k|Y_k - Y_{k-1}$ is clearly a closed map,

$$\bigcup \{f^{-1}(z) \mid f^{-1}(z) \subset U \cap (Y_k - Y_{k-1})\}$$

is a neighborhood of $f^{-1}(z_0)$. Suppose $f^{-1}(z_0) \subset Y_{k-1}$. Since the decomposition of Y_{k-1} is upper semicontinuous,

$$\bigcup \{f^{-1}(z) \mid f^{-1}(z) \subset U \cap Y_{k-1}\} = W \cap Y_{k-1},$$

where $W \subset U$ and W is open in Y_k . Since f_k is closed on Y_k ,

$$A = \bigcup \{f_k^{-1}(z) \mid f_k^{-1}(z) \subset W\}$$

is open in Y_k . But $A \cap Y_{k-1} = W \cap Y_{k-1}$, since $f_k|_{Y_{k-1}}$ is a homeomorphism. Therefore A is an inverse set with respect to f , and clearly

$$f^{-1}(z_0) \subset A \subset W \subset U.$$

Hence the decomposition of Y_k is upper semicontinuous.

Since the decomposition of cX is upper semicontinuous, $f: cX \rightarrow dX$ is a closed map, and dX is a compact metrizable space. X is an inverse set for f and $f|X$ is one-one, so that $f|X$ is a homeomorphism. Hence dX is a compactification of X . For each i define a function $g_i: f_i(Y_i) \rightarrow f(Y_i)$ by $g_i(f_i(y)) = f(y)$. It is easy to show that $g_i|f_i(Y_i - Y_{i-1})$ is a uniformly continuous homeomorphism.

$f_i(Y_i - Y_{i-1})$ is a polytope $N_i \subset I^\omega$, where $N_i = \bigcup_j \Sigma_{ij}$, with $\text{diam } \Sigma_{ij} \rightarrow 0$ as $j \rightarrow \infty$. Since g_i is uniformly continuous and N_i is bounded, we may triangulate N_i into simplexes which we also call Σ_{ij} in such a way that $\text{diam } g_i(\Sigma_{ij}) < 1/i$ for all j and $\text{diam } g_i(\Sigma_{ij}) \rightarrow 0$ as $j \rightarrow \infty$. Now

$$dX - X = \bigcup_i f(Y_i - Y_{i-1}) = \bigcup_i \left[\bigcup_j g_i(\Sigma_{ij}) \right].$$

If $i \neq k$, then $g_i(\Sigma_{ij})$ and $g_k(\Sigma_{ki})$ are clearly disjoint. Since $g_i|f_i(Y_i - Y_{i-1})$ is a homeomorphism, $f(Y_i - Y_{i-1}) = g_i(f_i(Y_i - Y_{i-1}))$ is a polytope. Hence by the property of the diameters of the $g_i(\Sigma_{ij})$ it is clear that $dX - X$ is a pseudopolytope.

Now $dX - X$ is a pseudopolytope and $X \subset \bigcup_i F_i \subset \bigcup_i \bar{F}_i^{dX}$. Hence we can finish the proof by showing that \bar{F}_k^{dX} is strongly countable-dimensional for all k . Let k be fixed.

Since f is a closed map, $\bar{F}_k^{dX} = f(\bar{F}_k^{cX})$. Since $f(Y_i - Y_{i-1})$ is homeomorphic to $f_i(Y_i - Y_{i-1})$ for all i ,

$$\dim f(\bar{F}_k^{cX} \cap (Y_i - Y_{i-1})) \leq n_k$$

for all $i \geq k$.

Let

$$E_i = \bigcup_{j=k}^i f(\bar{F}_k^{cX} \cap (Y_j - Y_{j-1}))$$

for $i \geq k$, and let $D_k = \bigcup_{i=k}^\infty E_i$. It is easily shown that E_i is closed in D_k for all $i \geq k$. Now $\dim E_k \leq n_k$, so suppose $\dim E_i \leq n_k$. Since

$$E_{i+1} = E_i \cup f(\bar{F}_k^{cX} \cap (Y_{i+1} - Y_i))$$

and

$$\dim f(\bar{F}_k^{cX} \cap (Y_{i+1} - Y_i)) \leq n_k,$$

$\dim E_{i+1} \leq n_k$ by [2, Corollary 1, p. 32]. By induction $\dim E_i \leq n_k$ for all $i \geq k$, so that $\dim D_k \leq n_k$ by the countable sum theorem [2, p. 30]. By [2, Proposition B, p. 28]

$$\dim D_k \cup F_k \leq n_k + m_k + 1.$$

Now $f(\bar{F}_k^{cX} \cap Y_{k-1})$ is closed in $dX - X$, so that $f(\bar{F}_k^{cX} \cap Y_{k-1})$ is a countable union of compact finite-dimensional sets. Since

$$D_k \cup F_k = \bar{F}_k^{dX} - f(\bar{F}_k^{cX} \cap Y_{k-1}),$$

$D_k \cup F_k$ is an open subset of the compact space \bar{F}_k^{dX} . By Proposition 2, $D_k \cup F_k = \bigcup_i G_{ki}$, where the closure H_{ki} of G_{ki} in \bar{F}_k^{dX} is contained in $D_k \cup F_k$ for all i . But

$$H_{ki} = \bar{G}_{ki}^{dX} \cap \bar{F}_k^{dX} = \bar{G}_{ki}^{dX}.$$

Therefore $\dim \bar{G}_{ki}^{dX} \leq n_k + m_k + 1$, so that $D_k \cup F_k$, and hence \bar{F}_k^{dX} , is a countable union of compact finite-dimensional sets.

Recall that a completion of a space X is an absolute G_δ containing X as a dense subspace. Theorem 2 then immediately yields the following.

COROLLARY 1. *A space X has a strongly countable-dimensional compactification if and only if it has a strongly countable-dimensional completion.*

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INDIANA UNIVERSITY,
BLOOMINGTON, INDIANA