ASYMPTOTIC BEHAVIOR OF THE CONVOLUTION OF A PAIR OF MEASURES

BY
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1. Introduction. Let the time parameter set $T$ be given either as the discrete set $T=\{0, 1, 2, \ldots\}$ or as the continuous set $T=[0, \infty)$. Consider random counting measures $N_t$, $t \in T$, defined on the Borel sets of $d$-dimensional Euclidean space $R^d$. Here $N_t(A)$ denotes the "number of particles" in $A$ at time $t$. Set $N=N_0$, let $\nu_t$, $t \in T$, be the measures defined by $\nu_t(A)=EN_t(A)$, and set $\nu=\nu_0$. The temporal structure of $N_t$ is determined by letting the particles in $R^d$ at time zero be translated independently according to stochastic processes isomorphic to a fixed stochastic process $Y_t$, $t \in T$. Let $\mu_t$ denote the distribution of $Y_t$. Then

\[ \nu_t(A) = EN_t(A) = (\nu * \mu_t)(A) \]

and

\[ E[N_t(A) | N] = (N * \mu_t)(A). \]

Extending and correcting results of Dobrushin [1], the author in [2] studied the asymptotic behavior as $t \to \infty$ of $(\nu * \mu_t)(A)$ and $(N * \mu_t)(A)$.

Let $dt$, $t \in T$, refer to counting measure and Lebesgue measure in the discrete and continuous cases respectively. Let the random measure $N^*$ be defined by

\[ N^* = \int_0^\infty N_t \, dt. \]

Then $N^*(A)$ denotes the total occupation time of $A$ by all the particles. Let $\mu$ denote the measure defined by

\[ \mu = \int_0^\infty \mu_t \, dt. \]

Then $EN^*$, the expected total occupation time measure, is given by $EN^* = \nu * \mu$, and $E[N^* | N]$ is given by $E[N^* | N] = N * \mu$.

In this paper we will present a general study of the asymptotic behavior of $(\nu * \mu)(x+A)$ and $(N * \mu)(x+A)$ as $x \to \infty$ appropriately. The intended application is to the case in which $\mu_{s+t} = \mu_s * \mu_t$, i.e. when $Y_t$ is a random walk or a process with independent increments. Then $\mu$ is the multidimensional renewal measure. This application will be presented in [3].

In §2 we give precise definitions and state our results. The proofs are given in §§3 and 4. Many of the arguments used here are refinements of those used in [2].

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framework discussed above was intended mainly for motivation, that of the rest of the paper being somewhat more general.

2. Definitions and statements of results. Let $X$ denote a $d$-dimensional closed subgroup of $R^d$. With no loss of generality, we can assume that $X$ is of the form

$$X = \{ x = (x_1, \ldots, x_d) \mid x^k \text{ are integers for } d_1 < k \leq d \}.$$ 

Set $Z^d = \{ x \mid x^k \text{ are integers for } 1 \leq k \leq d \}$. If $d_1 = d$, then $X = R^d$ and if $d_1 = 0$, then $X = Z^d$.

Set $\Delta = \{ x \in X \mid 0 \leq x^k < 1 \text{ for } 1 \leq k \leq d \}$. Set $U = \{ x \in Z^d \mid x^k = 1 \text{ for some } k \text{ and } x^j = 0 \text{ for } j \neq k \}$. Then $U$ consists of $d$ "unit vectors." For $a > 0$ and $x \in X$, set

$$a \cdot x = (ax_1, \ldots, ax^{d_1}, x^{d_1+1}, \ldots, x^d).$$

Let $| \cdot |$ denote Haar measure on $X$ defined as the product of Lebesgue measure on the first $d_1$ coordinates of $X$ and counting measure on the last $d - d_1$ coordinates.

Let $\mathscr{B}$ denote the collection of relatively compact Borel subsets of $X$. Let $\mathscr{A}$ denote the subcollection of $\mathscr{B}$ of sets $A$ such that $|\partial A| = 0$.

For $x, y \in R^d$ set

$$x \cdot y = x^1y^1 + \cdots + x^dy^d.$$ 

Then "\cdot" defines an inner product on $R^d$. Given any vector $v \in R^d$ of unit length, by applying the Gram-Schmidt Orthogonalization process to $v$, $(1, 0, \ldots, 0)$, $(0, 1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 0, 1)$, we get an orthonormal basis $v_1 = v, v_2, \ldots, v_d$ of $R^d$.

From now on $v$ will denote an arbitrary but fixed vector in $R^d$ of unit length. Whenever it can be done unambiguously, we will suppress the dependence on $v$ of various quantities defined below.

Let

$$Z^d_{v}^{-1} = \{ n_2v_2 + \cdots + n_dv_d \mid n_2, \ldots, n_d \text{ are integers} \}$$

and

$$\Gamma_m = \Gamma_{m,v} = \{ x_1v_1 + x_2v_2 + \cdots + x_dv_d \mid 0 \leq x_k < m \text{ for } 2 \leq k \leq d \}.$$ 

Let $\mathscr{F}_v$ denote the collection of sets

$$T = T_{a_1,a_2} = T_{v,a_1,a_2} = \{ x \in R^d \mid a_1 \leq x \cdot v \leq a_2 \},$$

where $a_1$ and $a_2$ denote finite real numbers.

All measures considered below will be positive measures on the Borel sets in $R^d$. Furthermore, it will always be assumed that these measures are supported by $X$.

For $0 \leq \lambda < \infty$, let $M_{v,\lambda}$ denote the collection of measures $\nu$ supported by $T$ for some $T \in \mathscr{F}_v$ (and also by $X$ as remarked above) and such that

$$\lim_{m \to \infty} \frac{\nu(x + \Gamma_m)}{m^{d-1}} = \lambda$$

uniformly for $x \in R^d$. 

For $0 \leq \kappa < \infty$, let $\mathcal{M}_{v, \kappa}$ denote the collection of measures $\mu$ (supported by $X$) such that

$$\lim_{a_1, a_2 \to \infty} \frac{\mu(T_{a_1, a_2})}{a_1 - a_2} = \kappa$$

and for each $0 < a < \infty$, $u \in U$, and $T \in \mathcal{T}_v$

$$\lim_{c \to \infty} \sum_{c + \Gamma \in T} |\mu(a \circ (n + u + \Delta)) - \mu(a \circ (n + \Delta))| = 0. \tag{2.2}$$

Here "$cv + T$" is used as an abbreviation for "$n \in \mathbb{Z}^d \cap (cv + T)$." Similar abbreviations will be used below without further comment.

Let $\mathcal{M}_{v, \kappa}^\ast$ consist of the subcollection of $\mathcal{M}_{v, \kappa}$ of measures such that for $0 < a < \infty$, $T \in \mathcal{T}_v$, compact subset $C$ of $X$, and $A \in \mathcal{B}$ with $|A| = 1$.

$$\lim_{c \to \infty} \sum_{c + \Gamma \in T} \sup_{x \in C} |\mu(a \circ (n + x + A)) - \mu(a \circ (n + \Delta))| = 0. \tag{2.3}$$

When the expression "$x \cdot v \to \infty$" is used below, it should be understood that $x \in X$.

**Theorem 1.** Let $v$ be in $\mathcal{N}_{v, \lambda}$. Let $\mu$ be in $\mathcal{M}_{v, \kappa}$ (or in $\mathcal{M}_{v, \kappa}^\ast$). Then for $A \in \mathcal{A}$ (or $A \in \mathcal{B}$)

$$\lim_{x \cdot v \to \infty} (v \ast \mu)(x + A) = \lambda \kappa |A|. \tag{2.5}$$

As a first converse, suppose $v$ is supported by some $T \in \mathcal{T}_v$, $\mu(T) < \infty$ for all $T \in \mathcal{T}_v$, and, for some $0 < a < \infty$ and $0 < a < \infty$,

$$\lim_{x \cdot v \to \infty} (v \ast \mu)(x + a \circ A) = a |a \circ A|. \tag{2.6}$$

Then there exist $\lambda > 0$ and $\kappa > 0$ such that $\lambda \kappa = \alpha$, $\nu \in \mathcal{N}_{v, \lambda}$, and $\mu$ satisfies (2.2).

As a second converse suppose $0 < \kappa < \infty$ and, for all $0 < a < \infty$, $0 < \lambda < \infty$, and $v \in \mathcal{N}_{v, \lambda}$, that (2.6) holds with $\alpha = \lambda \kappa$. Then $\mu \in \mathcal{M}_{v, \kappa}^\ast$.

The apparent difference between (2.3) and (2.4) is minimized by the following result which is useful in proving the first part of Theorem 1.

**Lemma 1.** Let $\mu \in \mathcal{M}_{v, \kappa}^\ast$. Then for $0 < a < \infty$, $T \in \mathcal{T}_v$, and compact subset $C$ of $X$.

$$\lim_{c \to \infty} \sum_{c + \Gamma \in T} \sup_{x \in C} |\mu(a \circ (n + x + \Delta)) - \mu(a \circ (n + \Delta))| = 0. \tag{2.7}$$

In Theorems 2 and 3, $N$ denotes a "random positive measure" on the Borel sets in $\mathbb{R}^d$ (supported by $X$ as usual).

**Theorem 2.** Let $0 \leq \kappa < \infty$ and $\mu \in \mathcal{M}_{v, \kappa}$ (or $\mathcal{M}_{v, \kappa}^\ast$). Let $0 \leq \lambda < \infty$ and let $N$ be such that $EN \in \mathcal{N}_{v, \lambda}$ and

$$\lim_{m \to \infty} E \left| \frac{N(x + \Gamma_m)}{m^{d-1}} - \lambda \right| = 0 \tag{2.8}$$

uniformly for $x \in \mathbb{R}^d$. Then for $A \in \mathcal{A}$ (or $A \in \mathcal{B}$)

$$\lim_{x \cdot v \to \infty} E |(N \ast \mu)(x + A) - \lambda \kappa |A| | = 0. \tag{2.9}$$
Conversely, suppose $0 < \kappa < \infty$, $0 < a < \infty$, $0 < \lambda < \infty$, $\mu$ satisfies (2.2), $EN \in \mathcal{N}_{\nu, \lambda}$, and

\[
\lim_{x, \nu \to \infty} E\left| (N * \mu)(x + a \cap \Delta) - \kappa |a \cap \Delta| \right| = 0.
\]

Then (2.8) holds.

It follows from (2.9) that

\[
\lim_{x, \nu \to \infty} (N * \mu)(x + A) = \lambda |A|.
\]

We next obtain analogous results when $\kappa = \infty$.

For $T \in \mathcal{F}_\nu$, we will say that $T$ is sufficiently large if for some $m \geq 1$

\[
(x + \Gamma_m) \cap \mathbb{Z}^d \cap T \neq \emptyset, \quad x \in \mathbb{R}^d.
\]

**Theorem 3.** Let $A$ be a Borel subset of $X$.

(i) Suppose that for sufficiently large $T \in \mathcal{F}_\nu$ and for all compact subsets $C$ of $X$

\[
\lim_{x, \nu \to \infty} \inf_{n \in C} \mu(n + x + y + A) = \infty.
\]

Then for $0 < \lambda < \infty$ and $\nu \in \mathcal{N}_{\nu, \lambda}$

\[
\lim_{x, \nu \to \infty} (\nu * \mu)(x + A) = \infty.
\]

(ii) In order that for all $0 < \lambda < \infty$ and $\nu \in \mathcal{N}_{\nu, \lambda}$

\[
(\nu * \mu)(x + A) = \infty, \quad x \in X,
\]

it is necessary and sufficient that for sufficiently large $T \in \mathcal{F}_\nu$ and for all compact subsets $C$ of $X$

\[
\sum_{T \in C} \inf_{y \in C} \mu(n + y + A) = \infty.
\]

**Theorem 4.** Let $N$ be such that for some $\lambda$, $0 < \lambda < \infty$, $EN \in \mathcal{N}_{\nu, \lambda}$ and (2.8) holds.

(i) If (2.13) holds for sufficiently large $T \in \mathcal{F}_\nu$ and for all compact subsets $C$ of $X$, then

\[
\lim_{x, \nu \to \infty} (N * \mu)(x + A) = \infty.
\]

(ii) If (2.16) holds for sufficiently large $T \in \mathcal{F}_\nu$ and all compact subsets $C$ of $X$, then

\[
P\{(N * \mu)(x + A) = \infty \text{ for } x \in X\} = 1.
\]

3. **Proof of Theorem 1.** Set

\[
Z_m^d = \{ n \in \mathbb{Z}^d \mid m^{-1} \cap n \in \Delta \}.
\]

We first prove Lemma 1, starting off with...
Lemma 2. Let \( \mu \) denote a probability measure on \( X \). Then for \( m \geq 1 \) there is a \( T_1 \in \mathcal{F}_v \) depending only on \( T \) and \( m \) such that

\[
\sum_{cv+T} \max_{keZ_m^n} \mu(m \circ n + k + \Delta) \leq m^{-a} \mu(cv + T_1)
\]

(3.1)

\[
+ \sum_{cv+T_1} \max_{ueU} |\mu(n + u + \Delta) - \mu(n + \Delta)|.
\]

Proof of Lemma 2. For \( n \in \mathbb{Z}^d \) choose \( b_n \in \mathbb{Z}_m^d \) such that

\[
\mu(m \circ n + b_n + \Delta) = \max_{keZ_m^n} \mu(m \circ n + k + \Delta).
\]

Then for \( k \in \mathbb{Z}_m^d \)

\[
\mu(m \circ n + b_n + \Delta) - \mu(m \circ n + k + \Delta)
\]

\[
\leq \sum_{i \in \mathbb{Z}_m^n} \max_{u \in U} |\mu(m \circ n + z + u + \Delta) - \mu(m \circ n + z + \Delta)|.
\]

Consequently

\[
\sum_{cv+T} \mu(m \circ n + b_n + \Delta) \leq \sum_{cv+T} \mu(m \circ n + k + \Delta)
\]

\[
+ \sum_{cv+T_1} \sum_{keZ_m^n} \max_{ueU} |\mu(m \circ n + z + u + \Delta) - \mu(m \circ n + z + \Delta)|
\]

\[
\leq \sum_{cv+T} \mu(m \circ n + k + \Delta) + \sum_{cv+T_2} \max_{ueU} |\mu(n + u + \Delta) - \mu(n + \Delta)|
\]

for suitable \( T_2 \in \mathcal{F}_v \) depending only on \( T \) and \( m \). Choose \( T_1 \equiv T_2 \) such that \( m \circ n + k + \Delta \subseteq T_1 \) for \( n \in cv + T \) and \( k \in \mathbb{Z}_m^d \). Then Lemma 2 follows by summing over \( k \in \mathbb{Z}_m^d \) and dividing by \( m^d \).

Proof of Lemma 1. Without loss of generality we can assume that \( a=1 \) and \( C=\Delta \). Then (2.7) becomes

\[
\lim_{c \to \infty} \sum_{cv \in T} \sup_{xeA} |\mu(n + x + \Delta) - \mu(n + \Delta)| = 0.
\]

Observe that

\[
\limsup_{c \to \infty} \sum_{cv \in T} \max_{keZ_m^n} |\mu(n + m^{-1} \circ k + \Delta) - \mu(n + \Delta)|
\]

\[
= \limsup_{c \to \infty} \sum_{cv + T} \max_{keZ_m^n} \mu(m^{-1} \circ (m \circ n + k + m \circ \Delta)) - \mu(m^{-1} \circ (m \circ n + m \circ \Delta))
\]

\[
\leq \limsup_{c \to \infty} \sum_{cv + T_1} \max_{keZ_m^n} \mu(m^{-1} \circ (m \circ n + k + u + m \circ \Delta))
\]

\[
- \mu(m^{-1} \circ (m \circ n + k + m \circ \Delta))
\]

for a suitable \( T_1 \in \mathcal{F}_v \). It now follows from (2.3) that

\[
\lim_{c \to \infty} \sum_{cv \in T_1} \max_{keZ_m^n} |\mu(n + m^{-1} \circ k + \Delta) - \mu(n + \Delta)| = 0.
\]
By Lemma 2

\[ \sum_{cv+T} \max_{k \in \mathbb{Z}^d_m} \mu(n+m^{-1} \circ k+m^{-1} \circ \Delta) \]

\[ = \sum_{cv+T} \max_{k \in \mathbb{Z}^d_m} \left( \frac{1}{m} \circ (m \circ n+k+\Delta) \right) \]

\[ \leq m^d \mu(cv+T_1) + \sum_{cv+T_1} \max_{u \in U} \left| \mu\left( \frac{1}{m} \circ (n+u+\Delta) \right) - \mu\left( \frac{1}{m} \circ (n+\Delta) \right) \right| \]

for some \( T_1 \) depending on \( T \), but which, as is easily seen, can be chosen independently of \( m \). It now follows from (2.2) and (2.3) that for some constant \( \beta_1, 0 < \beta_1 < \infty \), depending on \( T \)

\[ \limsup_{c \to \infty} \sum_{cv+T} \max_{k \in \mathbb{Z}^d_m} \mu(n+m^{-1} \circ k+m^{-1} \circ \Delta) \leq \beta_1 m^{-d_1}. \]

With no loss of generality, we can assume that \( d_1 \geq 1 \). Let \( W \) denote the set of \( 2^{d_1} \) points in \( \mathbb{Z}^d \) whose coordinates are all zero or one, the last \( d-d_1 \) of them all being zero. Choose \( x \in \Delta \). Choose \( k = (k_1, \ldots, k^d) \in \mathbb{Z}^d_m \) such that

\[ m^{-1}k^i \leq x^i < m^{-1}(k^i+1), \quad 1 \leq i \leq d. \]

Then

\[ |\mu(n+x+\Delta) - \mu(n+m \circ k+\Delta)| \]

\[ \leq 2^d(m+1)^{d_1-1} \sum_{u \in W} \max_{k \in \mathbb{Z}^d_m} \mu(n+w+m^{-1} \circ k+m^{-1} \circ \Delta). \]

Thus by (2.4), there is a constant \( \beta, 0 < \beta < \infty \), depending only on \( T \) and such that

\[ \limsup_{c \to \infty} \sum_{cv+T} \sup_{x \in X} \min_{k \in \mathbb{Z}^d_m} |\mu(n+x+\Delta) - \mu(n+m^{-1} \circ k+\Delta)| \leq \beta m^{-1}. \]

Choose \( \varepsilon > 0 \). Choose \( m \) such that

\[ \beta m^{-1} \leq \varepsilon/2. \]

By (3.3), (3.5), and (3.6), there is a \( c_0 \) such that for \( c \geq c_0 \)

\[ \sum_{cv+T} \max_{k \in \mathbb{Z}^d_m} |\mu(n+m^{-1} \circ k+\Delta) - \mu(n+\Delta)| \leq \frac{\varepsilon}{2} \]

and

\[ \sum_{cv+T} \sup_{x \in \Delta} \min_{k \in \mathbb{Z}^d_m} |\mu(n+x+\Delta) - \mu(n+m^{-1} \circ k+\Delta)| \leq \frac{\varepsilon}{2}. \]

Therefore for \( c \geq c_0 \)

\[ \sum_{cv+T} \sup_{x \in \Delta} |\mu(n+x+\Delta) - \mu(n+\Delta)| \leq \varepsilon. \]

Since \( \varepsilon \) can be made arbitrarily small, this completes the proof of (3.2) and hence also of Lemma 1.
Proof of first part of Theorem 1. With no loss of generality, we can assume that $\lambda = \kappa = 1$. Choose $\nu \in \mathcal{M}_{\nu,1}$ and $\mu \in \mathcal{M}_{\nu,1}$. Let $T_1 \in \mathcal{I}_\nu$ be such that $\nu$ is supported on $T_1$.

In proving that (2.5) holds for $A \in \mathcal{A}$, it suffices to consider the case $A = \Delta$. If $\mu \in \mathcal{M}_{\nu,1}^+$, then to prove that (2.5) holds for $A \in \mathcal{B}$, it suffices to consider $A$ such that $|A| = 1$. In either case, by Lemma 1, it suffices to consider $A \in \mathcal{B}$ such that $|A| = 1$ and (2.4) holds for $a = 1$, $T \in \mathcal{I}_\nu$, and $C$ a compact subset of $X$. From now on we let $A$ be such a set.

Choose $0 \leq \epsilon \leq 1$. Choose $m_1 > 0$ such that for $m \geq m_1$

\[
1 - \epsilon \leq m^{1-d}\nu(x - \Gamma_m) \leq 1 + \epsilon, \quad x \in R^d.
\]

Consider $T \in \mathcal{I}_\nu$ of the form

\[
T = \{x \in R^d \mid 0 \leq x \cdot v \leq a\}.
\]

By (2.2) there exist $a_1 > 0$ and $c_1 > 0$ such that if $a \geq a_1$ and $c \geq c_1$ then

\[
(1 - \epsilon)a \leq \mu(cv + T) \leq (1 + \epsilon)a.
\]

Let $b_m(T)$ denote the minimum (over $-\infty < c < \infty$ and $x \in R^d$) of the number of points $n \in Z^d$ such that

\[
n + \Delta \subseteq (cv + T) \cap (x + \Gamma_m).
\]

Let $b_m(T)$ denote the maximum (over $-\infty < c < \infty$ and $x \in R^d$) of the number of points $n \in Z^d$ such that

\[
(n + \Delta) \cap (cv + T) \cap (x + \Gamma_m) \neq \emptyset.
\]

Then clearly

\[
b_m(T) \leq am^{d-1} \leq b_m(T).
\]

There exist $m_2 \geq m_1$ and $a_2 \geq a_1$ such that if $m \geq m_2$ and $a \geq a_2$, then

\[
(1 - \epsilon)am^{d-1} \leq b_m(T)
\]

and

\[
(1 + \epsilon)am^{d-1} \geq b_m(T).
\]

Assume now that $a$ and $m$ are fixed with $a \geq a_2$ and $m \geq m_2$. By (2.4) there is a $c_2 \geq c_1$ such that whenever $c \geq c_2$ and the points $n_z \in Z^d$, $z \in Z_d^{d-1}$ are such that

\[
(n_z + \Delta) \cap (cv + T) \cap (mz + \Gamma_m) \neq \emptyset,
\]

then

\[
\sum_{z \in Z_d^{d-1}} \sup_{y \in Y_z} |\mu(y + A) - \mu(n_z + \Delta)| \leq \frac{\epsilon}{2m^{d-1}},
\]

where

\[
Y_z = X \cap (mz + \Gamma_m) \cap (cv - T_1).
\]
Choose \( x \in X \) such that \( x \cdot v = c \geq c_2 \). Then

\[
(3.13) \quad (v \ast \mu)(x + A) = \int_{X \cap (cv - T)} \nu(x - dy)\mu(y + A) = \sum_{z \in \mathbb{Z}^d} \int_{Y_z} \nu(x - dy)\mu(y + A).
\]

Thus by (3.7), (3.12), and (3.13), whenever \( n_z \in \mathbb{Z}^d \) satisfies (3.11), then

\[
(3.14) \quad (v \ast \mu)(x + A) - \sum_{z \in \mathbb{Z}^d} (v - m_z - \Gamma_m)\mu(n_z + \Delta) \leq e(1 + \varepsilon)/2 \leq \varepsilon.
\]

We can get \( b'_n(T) \) different choices of the sequence \( n_z, z \in \mathbb{Z}^d \), such that all

\[
\text{the } n_z\text{'s involved are distinct and, for each such } n_z,
\]

\[
n_z + \Delta \leq cv + T.
\]

Summing (3.14) and using (3.7), we get that

\[
(3.15) \quad b'_n(T)(v \ast \mu)(x + A) \leq (1 + \varepsilon)m^{d-1}\mu(cv + T) + eb'_n(T)
\]

and hence by (3.8) and (3.9) that

\[
(3.16) \quad (v \ast \mu)(x + A) \leq (1 + \varepsilon)^2/(1 - \varepsilon) + \varepsilon.
\]

Similarly we can get \( b''_n(T) \) different choices of the sequence \( n_z, z \in \mathbb{Z}^d \), such that every \( n_z \) satisfying

\[
(n_z + \Delta) \cap (cv + T) \neq \emptyset
\]

is included in at least one such sequence. Summing on (3.14) and using (3.7), we get that

\[
(3.17) \quad b''_n(T)(v \ast \mu)(x + A) \leq (1 - \varepsilon)m^{d-1}\mu(cv + T) - eb''_n(T)
\]

and hence by (3.8) and (3.10) that

\[
(3.18) \quad (v \ast \mu)(x + A) \geq (1 - \varepsilon)^2/(1 + \varepsilon) - \varepsilon.
\]

Since \( \varepsilon \) can be made arbitrarily small, (2.5) follows from (3.16) and (3.18).

**Proof of first converse of Theorem 1.** We can assume that \( a = a_1 = 1 \). Let \( \nu \) be supported by \( T_{ao, b_0} \in \mathcal{F}_v \), let \( \mu(T) < \infty \) for \( T \in \mathcal{F}_v \), and suppose that

\[
(3.19) \quad \lim_{x \cdot v \to \infty} (v \ast \mu)(x + \Delta) = 1.
\]

It follows easily from (3.19) that if \( a \) and \( b - a \) are sufficiently large, then \( \mu(T_{a, b}) > 0 \).

It also follows easily from (3.19) that

\[
(3.20) \quad \frac{(v \ast \mu)(x + \Gamma_m \cap T_{a, b})}{m^{d-1}(b - a)} \to 1
\]

as \( x \cdot v = 0, m \to \infty, a \to \infty, \) and \( b - a \to \infty \).
In order to prove the desired converse, we need only show that for every \( 0 \leq \varepsilon \leq \frac{1}{2} \) there exist \( a_1 \) and \( d_1 \) such that if \( a \geq a_1 \) and \( b - a \geq d_1 \), then

\[
\lim_{m \to \infty} \sup_{x \in \mathbb{R}^d} \frac{\nu(x + \Gamma_m)}{m^{d-1}} \leq \frac{(1+\varepsilon)(b-a)}{\mu(T_{a,b})}
\]

and

\[
\lim_{m \to \infty} \inf_{x \in \mathbb{R}^d} \frac{\nu(x + \Gamma_m)}{m^{d-1}} \geq \frac{(1-\varepsilon)(b-a)}{\mu(T_{a,b})}.
\]

Choose \( 0 < \varepsilon \leq \frac{1}{2} \). It follows from (3.20) that there exist positive numbers \( m_2, a_2, \) and \( d_2 \) such that

\[
1 - \frac{\varepsilon}{8} \leq \frac{(\nu * \mu)(x + \Gamma_m \cap T_{a,b})}{m^{d-1}(b-a)} < 1 + \frac{\varepsilon}{8}
\]

for \( x \cdot v = 0, m \geq m_2, a \geq a_2, \) and \( b - a \geq d_2 \). Choose \( a_1 \geq a_0 \) and \( d_1 \geq d_2 + b_0 - a_0 \) such that \( b_0 - a_0 \leq \varepsilon d_1/8 \) and \( \mu(T_{a,b}) > 0, a \geq a_1 \) and \( b - a \geq d_1 \).

We shall show that \( a_1 \) and \( d_1 \) are the numbers desired for (3.21) and (3.22). Choose \( a \) and \( b \) such that \( a \geq a_1 \) and \( b - a \geq d_1 \). There is an \( m_1 \) such that

\[
\mu(T_{a,b} \cap \Gamma_{m_1}) \leq (1-\varepsilon/8)\mu(T_{a,b}).
\]

Suppose (3.21) does not hold for this choice of \( a \) and \( b \). Then we can find \( m \geq m_2 \) and \( x_0 \in \mathbb{R}^d \) such that \( x_0 \cdot v = 0 \) and

\[
\nu(x_0 - y + \Gamma_m) \leq \left( 1 + \frac{\varepsilon}{2} \right) \frac{b-a}{\mu(T_{a,b})}, \quad y \in \Gamma_{m_1}.
\]

Consequently

\[
(\nu * \mu)(x_0 + \Gamma_m \cap T_{a+a_0,b+b_0}) = \int_{T_{a,b}} \nu(x_0 - y + \Gamma_m \cap T_{a+a_0,b+b_0}) \mu(dy)
\]

\[
\geq \int_{T_{a,b}} \nu(x_0 - y + \Gamma_m \cap T_{a+a_0,b+b_0}) \mu(dy)
\]

\[
= \int_{T_{a,b}} \nu(x_0 - y + \Gamma_m) \mu(dy)
\]

\[
\geq \int_{T_{a,b} \cap \Gamma_{m_1}} \nu(x_0 - y + \Gamma_m) \mu(dy)
\]

\[
\geq \left( 1 + \frac{\varepsilon}{2} \right) \frac{(b-a)\mu(T_{a,b} \cap \Gamma_{m_1})}{\mu(T_{a,b})}
\]

\[
\geq (1+\varepsilon/2)(1-\varepsilon/8)(1-\varepsilon/8)m^{d-1}(b+b_0-a-a_0)
\]

\[
\geq (1+\varepsilon/8)m^{d-1}(b+b_0-a-a_0),
\]

which contradicts (3.23). Thus we have verified that (3.21) holds. Suppose (3.22) does not hold. Then we can find \( x_0 \in \mathbb{R}^d \) and \( m \) such that \( x_0 \cdot v = 0, m \geq m_2 \),

\[
\frac{\nu(x_0 - y + \Gamma_m)}{m^{d-1}} \leq \frac{(1-\varepsilon/2)(b-a)}{\mu(T_{a,b})}, \quad y \in \Gamma_{m_1},
\]

and

\[
\frac{\nu(x_0 - y + \Gamma_m)}{m^{d-1}} \geq \frac{(1+3\varepsilon/2)(b-a)}{\mu(T_{a,b})}, \quad y \in \mathbb{R}^d.
\]
Then 

\[(
u \ast \mu)(x_0 + \Gamma_m \cap T_{a+b_0,b+a_0}) = \int \nu(x_0 - y + \Gamma_m \cap T_{a+b_0,b+a_0}) \mu(dy)\]

\[\leq \int_{T_{a,b}} \nu(x_0 - y + \Gamma_m) \mu(dy)\]

\[\leq (1 - \frac{e}{2})(b-a)m^{d-1} + \frac{e}{8} \left(1 + \frac{3e}{2}\right)(b-a)m^{d-1}\]

\[\leq \left(1 - \frac{e^2}{8}\right)(b-a)m^{d-1} \leq \left(1 - \frac{e}{8}\right)(b+a_0-a-b_0)m^{d-1},\]

which contradicts (3.23). This completes the proof of (3.22) and hence of the first converse of Theorem 1.

**Proof of second converse of Theorem 1.** It follows easily from the first converse of Theorem 1 and the assumptions of the second converse that \(\mu\) satisfies (2.2). We can assume that \(\kappa = 1\). By assumption we have that if \(0 < a < \infty, \nu \in \mathcal{N}_{v,1}\) and \(\nu\) is supported by \(a \bigcirc Z^d\), then (1.6) holds. We need to show that, for \(u \in U\) and \(T \in \mathcal{F}_v\),

\[(2.3) \lim_{c \to \infty} \sum_{c \bigcirc v + T} |\mu(a \bigcirc (n + u + \Delta)) - \mu(a \bigcirc (n + \Delta))| = 0.\]

With no loss of generality, we can assume that \(a = 1\). Then \(\nu\) is supported by \(Z^d\) and (2.3) becomes

\[(3.24) \lim_{c \to \infty} \sum_{c \bigcirc v + T} |\mu(n + u + \Delta) - \mu(n + \Delta)| = 0.\]

The general case now reduces to the lattice case \(d_1 = 0\) and \(X = Z^d\).

Assume now that \(d_1 = 0\) and \(X = Z^d\). For \(n \in Z^d\) set \(\nu(n) = \nu([n])\) and \(\mu(n) = \mu([n])\). We have that

\[(3.25) \lim_{a,b \to \infty} \frac{1}{b-a} \sum_{a \leq n \leq b} \mu(n) = 1.\]

We also have that whenever \(\nu \in \mathcal{N}_{v,1}\), then

\[(3.26) \lim_{k \to \infty} \sum_{k \bigcirc v} \nu(-n)\mu(k+n) = 1.\]

We want to prove that for \(u \in U\) and \(T \in \mathcal{F}_v\)

\[(3.27) \lim_{c \to \infty} \sum_{c \bigcirc v + T} |\mu(n + u) - \mu(n)| = 0.\]

It follows from (3.26) that

\[(3.28) \lim_{k \to \infty} \mu(k) = 0\]

and that

\[(3.29) \lim_{k \to \infty} \sum \nu(-n)(\mu(k+u+n) - \mu(k+n)) = 0.\]
In order to prove that (2.36) holds, we will suppose that for some \( u_0 \in U \) and \( T_0 = T_{a_0, b_0} \in \mathcal{F}_v \)

\[
\limsup_{c \to \infty} \sum_{c \in T_0} |\mu(n + u_0) - \mu(n)| > 8\alpha > 0,
\]

but (3.25) and (3.28) hold. We will then find a \( \nu \in \mathcal{M}_{v,1} \) such that (3.29) does not hold.

If we modify \( T_0 \) by making it bigger, then (3.30) still remains valid. Thus we can assume that \( T_0 \) is large enough so that there exists \( m_0 \) such that for \( m \geq m_0 \) and \( x \in \mathbb{R}^d \) the number of points in \( Z^d \cap (x + T_0 \cap \Gamma_m) \) is bounded above by

\[ 2m^{d-1}(b_0 - a_0), \]

or in other words

\[
\sum_{Z^d \cap (x + T_0 \cap \Gamma_m)} 1 \leq 2m^{d-1}(b_0 - a_0).
\]

Also by (3.25) we can find \( T_1 = T_{a_1, b_1} \in \mathcal{F}_v \) such that \( T_1 \supseteq T_0 \) and, for some \( c_1 \),

\[
\mu(cv + T_1) \leq 2(b_1 - a_1), \quad c \geq c_1.
\]

An elementary argument shows that \( T_1 \) can be made large enough so that there exists \( m_1 \geq m_0 \) such that for \( m \geq m_1 \), \( x \in \mathbb{R}^d \), \( u \in U \), and \(-\infty < \beta < 0\), if

\[
\mu(n + u) - \mu(n) \leq \beta, \quad n \in x + T_1 \cap \Gamma_m,
\]

then

\[
\mu(x + T_1 \cap \Gamma_m) \geq 2\alpha^{-1}(-\beta)(b_0 - a_0)m^{d-1}(b_1 - a_1).
\]

Set \( C = \{c \mid -\infty < c < \infty\} \). Let \( C_1 \) denote the subset of \( c \in C \) such that

\[
\sum_{c \in T_0} |\mu(n + u_0) - \mu(n)| \geq 8\alpha.
\]

Set

\[ P = \{n \in \mathbb{Z}^d \mid \mu(n + u_0) - \mu(n) > 0\}. \]

Let \( C_2 \) denote the subset of \( C_1 \) on which

\[
\sum_{c \in T_0 \cap (cv + T_0)} (\mu(n + u_0) - \mu(n)) \geq 4\alpha.
\]

With no loss of generality we can assume that \( C_2 \) is unbounded from above.

Choose \( m \geq m_1 \). For \( c \in C \) and \( z \in \mathbb{Z}^{d-1}_c \), let \( n_z(c) \) be in \( cv + mz + \Gamma_m \cap T_1 \) and such that \( \mu(n_z(c) + u_0) - \mu(n_z(c)) \) maximizes \( \mu(n + u_0) - \mu(n) \) over all \( n \in cv + mz + \Gamma_m \cap T_1 \). Let

\[
[cv] = ([cv^1], \ldots, [cv^d]),
\]

where \([cv^k]\) denotes the greatest integer in \( cv^k \). Set \( S_c = \{n_z(c) \mid z \in \mathbb{Z}^{d-1}_c\} \) and let \( v_c \) be the measure supported on \([cv] - S_c\) and defined by

\[
v_c(([cv] - n_z(c))) = m^{d-1}.
\]
Then $\nu_c \in \mathcal{M}_{\nu,1}$. Set $\nu_c(n) = \nu_c([n])$. For $c \in C$

\[(3.36) \sum \nu_c(-n)(\mu(n + [cv] + u_0) - \mu(n + [cv])) = m^{d-1} \sum_{n \in S_c} (\mu(n + u_0) - \mu(n)).\]

For $n = n_c(c) \in P \cap S_c$ we have by (3.31) that

\[m^{d-1}(\mu(n + u_0) - \mu(n)) \geq \frac{1}{2(b_0 - a_0)} \sum_{x \in P \cap F} (\mu(n + u_0) - \mu(n)).\]

Thus by (3.34) for $c \in C_2$

\[(3.37) \sum_{x \in P \cap F} (\mu(n + u_0) - \mu(n)) \geq \frac{2\alpha}{b_0 - a_0}.\]

For $n = n_c(c) \in (Z^d - P) \cap S_c$ (i.e. in $S_c$ and the complement of $P$), we have by (3.33) that

\[m^{d-1}(\mu(n + u_0) - \mu(n)) \geq \frac{\alpha(\mu + mz + \Gamma_m \cap T)}{2(b_0 - a_0)(b_1 - a_1)}.\]

Therefore

\[(3.38) \sum_{x \in Z^d - P \cap S_c} (\mu(n + u_0) - \mu(n)) \geq \frac{\alpha(\mu + \Gamma_1)}{2(b_0 - a_0)(b_1 - a_1)} \geq \frac{\alpha}{b_0 - a_0}.\]

by (3.32). By (3.36), (3.37), and (3.38), we have that for $c \in C_2$

\[(3.39) \sum \nu_c(-n)(\mu(n + [cv] + u_0) - \mu(n + [cv])) \geq \frac{\alpha}{b_0 - a_0}.\]

It is also clear that there is a set $T_2 \in \mathcal{T}_v$ such that $\nu_c$ is supported by $T_2$ for $c \in C_2$.

For $n \in Z^d$ set

\[|n| = ((n_1)^2 + \cdots + (n_d)^2)^{1/2}.\]

We next define an increasing sequence of points $c_1, c_2, \ldots$ in $C_2$. Choose $c_1 \in C_2$ and set $a_0 = 0$. Once $a_{i-1}$ and $t_i$ are chosen, choose $a_i \geq a_{i-1} + t_i$ such that

\[(3.40) \sum_{-T_3 \cap \{|n| < a_i\}} |\mu(n + [ci] + u_0) - \mu(n + [ci])| \leq \frac{\alpha}{8(b_0 - a_0)}.\]

Once $c_i$ and $a_i$ are chosen, choose $c_{i+1} \in C_2$ such that $c_{i+1} \geq c_i + 1$ and

\[(3.41) \sum_{-T_3 \cap \{|n| < a_i\}} |\mu(n + [c_{i+1}] + u_0) - \mu(n + [c_{i+1}])| \leq \frac{\alpha}{8(b_0 - a_0)}.\]

This is possible by (3.28). Clearly

\[(3.42) \lim_{i \to \infty} c_i = \infty.\]
Define $\nu$ by $\nu((n)) = v_c((n))$, $a_i \leq |n| < a_i$, and set $\nu(n) = \nu((n))$. Then $\nu \in \mathcal{N}_{v,1}$ and for $i \geq 1$,

$$
\sum_{n} \nu(-n)(\mu(n + [c,v] + u_0) - \mu(n + [c,v]))
\geq \sum_{n} v_c((n))(-n)(\mu(n + [c,v] + u_0) - \mu(n + [c,v]))
- 2m^{d-1} \sum_{|n| < a_i} |\mu(n + [c,v] + u_0) - \mu(n + [c,v])|
- 2m^{d-1} \sum_{|n| \geq a_i} |\mu(n + [c,v] + u_0) - \mu(n + [c,v])|
\geq \frac{\alpha}{(b_0 - a_0)(1 - \frac{1}{4} - \frac{1}{4})} \geq \frac{\alpha}{2(b_0 - a_0)}
$$

by (3.39), (3.40), and (3.41).

In summary, $\nu \in \mathcal{N}_{v,1}$, $c_i \to \infty$, and

$$
\sum_{n} \nu(-n)(\mu(n + [c,v] + u_0) - \mu(n + [c,v])) \geq \frac{\alpha}{2(b_0 - a_0)}.
$$

This contradicts (3.29) and completes the proof of the theorem.

4. Proof of Theorems 2 and 3. Proof of first part of Theorem 2. It suffices to consider $\kappa = \lambda = 1$ and $A \in \mathcal{A}$ such that $|A| = 1$ and for $T \in \mathcal{T}$ and compact subset $C$ of $X$

$$
\lim_{c \to \infty} \sup_{(n+T) \in B} |\mu(n+y+A) - \mu(n+\Delta)| = 0.
$$

Set $\nu = E_{n}$. Choose $0 \leq \varepsilon \leq 1$ and recall the definitions used in the first part of Theorem 1, starting with (3.7). By (2.8) we can also assume that

$$
E|N(x-\Gamma_m) - P(x-\Gamma_m)| \leq e^{m^{d-1}}, \quad x \in \mathbb{R}^d.
$$

Now (3.4) can be replaced by

$$
E\left|(N*\mu)(x+A) - \sum_{n \in \mathbb{Z}_d^{d-1}} N(x-mz-\Gamma_m)\mu(n_z+\Delta)\right| \leq \varepsilon.
$$

It follows from (4.2) and (4.3) that

$$
E\left|(N*\mu)(x+A) - \sum_{n \in \mathbb{Z}_d^{d-1}} \nu(x-mz-\Gamma_m)\mu(n_z+\Delta)\right| \leq \varepsilon + e^{m^{d-1}} \sum_{n \in \mathbb{Z}_d^{d-1}} \mu(n_z+\Delta).
$$

Let $S_1$ denote the sum of

$$
\sum_{n \in \mathbb{Z}_d^{d-1}} \nu(x-mz-\Gamma_m)\mu(n_z+\Delta)
$$

over $b'_m(T)$ different choices of the sequence $n_z$, $z \in \mathbb{Z}_d^{d-1}$, satisfying (3.11) and such that every $n_z$ satisfying

$$
(n_z+\Delta) \cap (cv+T) \neq \emptyset
$$
is included in at least one such sequence. Let $S_2$ denote the sum of
$$\sum_{n^2 \geq e} \mu(n^2 + \Delta)$$
over the same choice of sequences. It follows from (2.2) and (4.1) that, by changing
slightly the construction used in the proof of the first part of Theorem 1, we can assume that

\[(4.5) \quad (1-e)a \leq S_2 \leq (1+e)a.\]

Summing (4.4) on the $b^a_n(T)$ different sequences and using (4.5), we get that

$$E|b^a_n(T)(N * \mu)(x + A) - S_1| \leq \epsilon b^a_n(T) + \epsilon(1+\epsilon)m^d - 1 \cdot a.$$ 

Division by $b^a_n(T)$ yields

\[(4.6) \quad E|(N * \mu)(x + A) - S_1| b^a_n(T) \leq 3\epsilon.\]

By (3.7) and (4.5)

\[(4.7) \quad (1-\epsilon)^2 \leq S_1 \leq (1+\epsilon)^2.\]

Consequently

\[(4.8) \quad (1-\epsilon)^2/(1+\epsilon) \leq \epsilon b^a_n(T) \leq (1+\epsilon)^2.\]

Since $\epsilon$ can be made arbitrarily small, (4.6) and (4.7) yield the conclusion of the
first part of Theorem 2.

**Proof of first converse of Theorem 2.** With no loss of generality, we can assume
that $\kappa = \lambda = a = 1$. Then (2.9) becomes

\[(4.9) \quad \lim_{x \to \infty} E[(N * \mu)(x + \Delta) - 1] = 0.\]

It follows easily from this that

\[(4.10) \quad \int_{T_{a,b}} N(x - y - \Gamma_m)\mu(dy) \leq (N * \mu)(x + \Gamma_m \cap T_{a+b_0+b}) \]

and

\[(4.11) \quad (N * \mu)(x + \Gamma_m \cap T_{a+b_0+b}) \leq \int_{T_{a,b}} N(x - y + \Gamma_m)\mu(dy).\]

For $-\infty < x < \infty$ let $x^+ = x$ and $x^- = 0$ or $x^+ = 0$ and $x^- = x$ according as $x \geq 0$ or $x < 0$. In order to obtain (2.8) we need only prove that

\[(4.12) \quad \lim_{m \to \infty} E[(N(x + \Gamma_m)/m^d - 1)^+] = 0.\]
and

\begin{equation}
(4.13) \quad \lim_{m \to \infty} E[(N(x + \Gamma_m)/m^{d-1} - 1)^+] = 0,
\end{equation}

both limits holding uniformly for \( x \in \mathbb{R}^d \).

We first prove (4.12), supposing it is false and obtaining a contradiction. If it is false, we can choose 0 < \( \varepsilon < 1 \), \( m, m_1, m_2, a, b, \) and \( x_0 \in \mathbb{R}^d \) such that

\begin{equation}
(4.14) \quad N(x_0 - y + \Gamma_m) \geq N(x_0 + \Gamma_m), \quad y \in \Gamma_{m_2},
\end{equation}

\begin{equation}
(4.15) \quad E[((1-\varepsilon)N(x_0 + \Gamma_m)/m^{d-1} - 1)^+] \geq \varepsilon,
\end{equation}

\begin{equation}
(4.16) \quad \mu(T_{a,b} \cap \Gamma_{m_2}) \geq (1-\varepsilon)(b-a),
\end{equation}

and

\begin{equation}
(4.17) \quad E \left[ \left( \frac{(N*\mu)(x_0 + \Gamma_m \cap T_{a+a_0,b+b_0})}{m^{d-1}(b-a)} - 1 \right)^+ \right] < \varepsilon.
\end{equation}

By (4.10), (4.14), and (4.16)

\begin{equation}
(N*\mu)(x_0 + \Gamma_m \cap T_{a+a_0,b+b_0}) \geq \int_{T_{a,b} \cap \Gamma_{m_2}} N(x_0 - y + \Gamma_m) \mu(dy)
\end{equation}

\begin{equation}
\geq (1-\varepsilon)N(x_0 + \Gamma_m)(b-a).
\end{equation}

Thus by (4.15)

\begin{equation}
E \left[ \left( \frac{(N*\mu)(x_0 + \Gamma_m \cap T_{a+a_0,b+b_0})}{m^{d-1}(b-a)} - 1 \right)^+ \right] \geq E \left[ \left( \frac{(1-\varepsilon)N(x_0 + \Gamma_m)}{m^{d-1}} - 1 \right)^+ \right] \geq \varepsilon,
\end{equation}

which contradicts (4.17). This completes the proof of (4.12).

We next prove (4.13), again supposing it is false and obtaining a contradiction. If it is false we can choose 0 < \( \varepsilon < 1 \), \( m, m_1, m_2, a, b, \) and \( x_0 \in \mathbb{R}^d \) such that

\begin{equation}
(4.18) \quad N(x_0 - y + \Gamma_m) \leq N(x_0 + \Gamma_{m_2}), \quad y \in \Gamma_{m_2},
\end{equation}

\begin{equation}
(4.19) \quad E \left[ \left( \frac{(1+\varepsilon)N(x_0 + \Gamma_m)}{m^{d-1}} - 1 \right)^- \right] \leq 2\varepsilon,
\end{equation}

\begin{equation}
(4.20) \quad \mu(T_{a,b}) \leq (1+\varepsilon)(b-a),
\end{equation}

\begin{equation}
(4.21) \quad \mu(T_{a,b}) - \mu(T_{a,b} \cap \Gamma_{m_2}) \leq \varepsilon(b-a)/2,
\end{equation}

\begin{equation}
(4.22) \quad E\nu(x + \Gamma_{m_1}) \leq 2m^d, \quad x \in \mathbb{R}^d,
\end{equation}

and

\begin{equation}
(4.23) \quad E \left[ \left( \frac{(N*\mu)(x_0 + \Gamma_m \cap T_{a+a_0,b+b_0})}{m^{d-1}(b-a)} - 1 \right)^- \right] < \varepsilon.
\end{equation}
By (4.11), (4.18), and (4.20)
\[(N * \mu)(x_0 + \Gamma_m \cap T_{a+b_0,b+a_0}) \leq (1 + \varepsilon)N(x_0 + \Gamma_m)(b-a) + \int_{T \cap \Gamma_m^e} N(x_0 - y + \Gamma_m)\mu(dy)\] 
and hence
\[\left(\frac{(N * \mu)(x_0 + \Gamma_m \cap T_{a+b_0,b+a_0})}{m^{d-1}(b-a)} - 1\right) \leq \left(\frac{(1 + \varepsilon)N(x_0 + \Gamma_m)}{m^{d-1}} - 1\right) - \frac{1}{m^{d-1}(b-a)} \int_{T \cap \Gamma_m^e} N(x_0 - y - \Gamma_m)\mu(dy).\]

Thus by (4.19), (4.21), and (4.22), we get that
\[E\left[\left(\frac{(N * \mu)(x_0 + \Gamma_m \cap T_{a+b_0,b+a_0})}{m^{d-1}(b-a)} - 1\right)^2\right] > 2\varepsilon - \varepsilon \geq \varepsilon,\]
which contradicts (4.23). This completes the proof of Theorem 2.

**Proof of Theorem 3(i).** Let \(T \in \mathcal{V}_y\) be sufficiently large and let \(m \geq 1\) be such that (2.12) holds. Then there is a compact subset \(C\) of \(X\) such that
\[n + C \supseteq (mz + \Gamma_m) \cap T \cap X, \quad z \in \mathbb{Z}_d^{-1} \quad \text{and} \quad n \in (mz + \Gamma_m) \cap T \cap \mathbb{Z}_d.\]

Let \(0 < \lambda < \infty\) and \(\nu \in \mathcal{N}_{\nu,\lambda}\). We can assume that \(m\) is such that
\[\nu(-(mz + \Gamma_m)) \geq \lambda m^{d-1}/2, \quad z \in \mathbb{Z}_d^{-1},\]
and \(T\) is such that \(\nu\) is supported by \(-T\). Let \(A\) be a Borel subset of \(X\). Then for \(x \in X\)
\[\nu * \mu)(x + A) = \sum_{z \in \mathbb{Z}_d^{-1}} \int_{(mz + \Gamma_m) \cap T} \nu(-dy)\mu(x + y + A)\]
\[\geq \frac{\lambda m^{d-1}}{2} \sum_{z \in \mathbb{Z}_d^{-1}} \inf_{y \in (mz + \Gamma_m) \cap T \cap X} \mu(x + y + A).\]

Suppose (2.13) holds for this \(A\). Then
\[\lim_{x \to \infty} \sum_{z \in \mathbb{Z}_d^{-1}} \inf_{y \in C} \mu(n + x + y + A) = \infty.\]

Consequently
\[\lim_{x \to \infty} \sum_{z \in \mathbb{Z}_d^{-1}} \max_{y \in C} \inf_{y \in C} \mu(n + x + y + A) = \infty\]
and hence
\[\lim_{x \to \infty} \sum_{z \in \mathbb{Z}_d^{-1}} \inf_{y \in (mz + \Gamma_m) \cap T \cap X} \mu(x + y + A) = \infty.\]

Therefore (2.14) holds for this \(A\).
If $A$ satisfies (2.16) then it follows by almost the same argument that (2.15) holds.

Next we consider the results in the converse direction. Let $A$ be a Borel subset of $X$. Suppose there is a sufficiently large $T \in T_\nu$ and a compact subset $C$ of $X$ such that

$$\sum_{x \in Z_{m+1}^d} \inf_{y \in C} \mu(x+y+A) < \infty.$$ 

Then

$$\sum_{x \in Z_{m+1}^d} \sum_{y \in (mz+\Gamma_m) \cap T} \inf_{y \in C} \mu(x+y+A) < \infty.$$ 

Thus there exist $n_z \in (mz+\Gamma_m) \cap T \cap Z^d$ and $y_z \in C$ such that

$$\sum_{z \in Z_{m+1}^d} \mu(n_z+y_z+A) < \infty.$$ 

Let $\nu$ be defined by

$$\nu(B) = \sum_{z \in Z_{m+1}^d} 1_B(-n_z-y_z).$$

Then $\nu \in \mathcal{M}_{\nu,\lambda}$ for some $0 < \lambda < \infty$ and

$$(\nu * \mu)(A) = \sum_{z \in Z_{m+1}^d} \mu(n_z+y_z+A) < \infty.$$ 

This completes the proof of Theorem 3.

**Proof of Theorem 4(i).** As in the proof of Theorem 3(i), it follows from the assumptions that

$$\lim_{x \to \infty} \sum_{z \in Z_{m+1}^d} \inf_{y \in (mz+\Gamma_m) \cap \Lambda} \mu(x+y+A) = \infty.$$ 

Let $a_z(x), z \in Z_{m+1}^d$ and $x \in X$, be such that

$$0 \leq a_z(x) \leq \inf_{y \in (mz+\Gamma_m) \cap \Lambda} \mu(x+y+A),$$

$$\sum_{z \in Z_{m+1}^d} a_z(x) < \infty,$$

and

$$\lim_{x \to \infty} \sum_{z \in Z_{m+1}^d} a_z(x) = \infty.$$ 

We can assume that $N$ is supported by $T$. Then

$$(N * \mu)(x+A) = \sum_{z \in Z_{m+1}^d} \int_{mz+\Gamma_m} N(dy) \mu(x+y+A)$$

$$\geq \sum_{z \in Z_{m+1}^d} a_z(x) N(-(mz+\Gamma_m)).$$

Choose $\varepsilon > 0$. By (2.8) we can find $M > 0$ and $m \geq 1$ such that

$$E[(N(-(mz+\Gamma_m))-M)\varepsilon M,]$$
where \( r^- = -r \) if \( r < 0 \) and \( r^- = 0 \) if \( r \geq 0 \). It follows that

\[
E \left[ \left( \sum_{z \in Z^d_v} a_z(x)N(- (mz + \Gamma_m)) - M \sum_{z \in Z^d_v} a_z(x) \right)^2 \right] \leq \varepsilon M \sum_{z \in Z^d_v} a_z(x).
\]

Thus for \( 0 < y < \infty \)

\[
P\left\{ \sum_{z \in Z^d_v} a_z(x)N(- (mz + \Gamma_m)) \leq y \right\} \leq \frac{\varepsilon M \sum_{z \in Z^d_v} a_z(x)}{M \sum_{z \in Z^d_v} a_z(x) - y}.
\]

Consequently

\[
\lim_{x \to \infty} P\left\{ \sum_{z \in Z^d_v} a_z(x)N(- (mz + \Gamma_m)) \leq y \right\} = 0
\]

and hence

\[
\lim_{x \to \infty} \sum_{z \in Z^d_v} a_z(x)N(- (mz + \Gamma_m)) = \infty.
\]

Therefore

\[
\lim_{x \to \infty} (N * \mu)(x + A) = \infty.
\]

**Proof of Theorem 4(ii).** The proof of (ii) is similar to that of (i), but we give the details since it is (ii) which is needed in Stone [3].

Let \( C \) be a compact subset of \( X \). It follows from the assumptions of (ii) that

\[
\sum_{z \in Z^d_v} \inf_{y \in X \cap (mz + \Gamma_m) \cap I \cap X} \inf_{x \in C} \mu(x + y + A) = \infty.
\]

Set

\[
a_z = \inf_{y \in X \cap (mz + \Gamma_m) \cap I \cap X} \inf_{x \in C} \mu(x + y + A).
\]

Then

\[
(N * \mu)(x + A) \geq \sum_{z \in Z^d_v} a_z N(-(mz + \Gamma_m)), \quad x \in C.
\]

Choose \( a_z^{(n)}, z \in Z^d_v \) and \( n \geq 1 \), such that

\[
0 \leq a_z^{(n)} \leq a_z, \quad \sum_{z \in Z^d_v} a_z^{(n)} < \infty,
\]

and

\[
\lim_{n \to \infty} \sum_{z \in Z^d_v} a_z^{(n)} = \infty.
\]

Choose \( \varepsilon > 0 \) and let \( M > 0 \) and \( m \geq 1 \) be as in the proof of (i). Then, as in (i), we observe that

\[
P\left\{ \sum_{z \in Z^d_v} a_z N(-(mz + \Gamma_m)) \leq y \right\} \leq \frac{\varepsilon M \sum_{z \in Z^d_v} a_z^{(n)}}{M \sum_{z \in Z^d_v} a_z^{(n)} - y}
\]

and hence

\[
P\left\{ \sum_{z \in Z^d_v} a_z N(-(mz + \Gamma_m)) \leq y \right\} \leq \varepsilon.
\]
Since $\epsilon$ can be made arbitrarily small and $y$ can be made arbitrarily large, it follows that

$$P\left\{ \sum_{\alpha \in \mathbb{Z}^d} a_\alpha N(-(mz + \Gamma_\alpha)) = \infty \right\} = 1$$

and therefore that

$$P\{(N*\mu)(x+A) = \infty \text{ for } x \in C\} = 1.$$ 

Since $X$ is $\sigma$-compact, we have (2.17) as desired.

**REFERENCES**


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