ON THE POTENTIAL OPERATOR FOR ONE-DIMENSIONAL RECURRENT RANDOM WALKS(1)

BY
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1. **Introduction.** Let μ denote a nondegenerate probability measure on R^d which generates a recurrent random walk on R^d. Then

$$\sum_{n=1}^{\infty} \mu^{(n)}(A) = \infty, \qquad A \subseteq R^d \text{ open and nonempty,} \tag{1.1}$$

where $\mu^{(n)}$ denotes the n-fold convolution of μ with itself. Let $\hat{\mu}(\theta)$, $\theta \in R^d$, denote the characteristic function of μ. Let X denote the closed subgroup of R^d generated by the support of μ. Without loss of generality, we can assume that X is d-dimensional. Then $d=1$ or $d=2$. If $d=1$, let σ^2 denote the variance of μ.

In [3] S. Port and the author showed that there is always a potential operator associated with μ that has useful properties. In the two-dimensional case our results were self-contained. In the one-dimensional case, however, the proof consisted of a reduction to the nonsingular case which had been treated earlier by Ornstein [1] and [2]. All of these results are, of course, extensions of results of Spitzer (see [5]) for lattice random walks.

The main purpose of this paper is to present a new and simpler proof of those consequences of Ornstein's work which were used in [3]. The result of [3] which depended on Ornstein's work will be stated here as

THEOREM 1. *Let $d=1$ and let a be a sufficiently small positive number such that $\hat{\mu}(\theta) \neq 1$ for $0 < |\theta| \leqq a$. Then*

$$\lim_{\lambda \uparrow 1} \frac{1}{2\pi} \int_{-a}^{a} \frac{i\theta}{1-\lambda\hat{\mu}(\theta)}\, d\theta \tag{1.2}$$

exists and is finite and

$$\lim_{y \to \pm\infty} \lim_{\lambda \uparrow 1} \frac{1}{2\pi} \int_{-a}^{a} \frac{i\theta e^{iy\theta}}{1-\lambda\hat{\mu}(\theta)}\, d\theta = \mp \sigma^{-2}. \tag{1.3}$$

As a related result we have

THEOREM 2. *Let $d=1$ and let a be a sufficiently small positive number such that $\hat{\mu}(\theta) \neq 1$ for $0 < |\theta| \leqq a$. Then*

$$\int_{|\theta| \leqq a} \Re\left(\frac{1}{1-\hat{\mu}(\theta)}\right) d\theta = \infty. \tag{1.4}$$

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Theorem 2 was proven by Spitzer [5] in the lattice case and by Ornstein [1] and [2] in general. It extends a classical result of Chung and Fuchs. Theorem 2 was not used in [3]. It is stated and proved here only because its proof follows almost immediately from that of Theorem 1. Theorem 2 is also valid in the two-dimensional case. The proof is rather similar, but easier since the analogy of Theorem 1 in the two-dimensional case is trivial.

In the proofs below we will use without further mention the facts that

$$|1-\lambda\hat{\mu}(\theta)| \geqq \lambda|1-\hat{\mu}(\theta)|, \qquad 0 \leqq \lambda \leqq 1 \text{ and } \theta \in R, \tag{1.5}$$

and that there exist positive numbers a and c such that

$$\begin{aligned} |1-\lambda\hat{\mu}(\theta)| &\geqq \Re(1-\lambda\hat{\mu}(\theta)) \\ &\geqq \Re(1-\hat{\mu}(\theta)) \geqq c|\theta|^2, \qquad 0 \leqq \lambda \leqq 1 \text{ and } |\theta| \leqq a. \end{aligned} \tag{1.6}$$

As a first step in the proof of Theorems 1 and 2 we clearly have

LEMMA 1. *Let ν denote a nondegenerate probability measure on R and let $\hat{\nu}$ denote its characteristic function. Suppose that*

$$\int_{-\infty}^{\infty} \left| \frac{\hat{\nu}(\theta)-\hat{\mu}(\theta)}{\theta^4} \right| d\theta < \infty. \tag{1.7}$$

If (1.2), (1.3), *and* (1.4) *hold when μ is replaced by μ then Theorem* 1 *holds.*

We say that a probability measure μ on R^d is *nonsingular* if it is nonsingular with respect to Lebesgue measure on R^d. We say that μ is *strongly nonlattice* if

$$\limsup_{|\theta|\to\infty} |\hat{\mu}(\theta)| < 1. \tag{1.8}$$

A sufficient condition for $\hat{\mu}$ to be strongly nonlattice is that μ be nonsingular. If $\hat{\nu}$ is strongly nonlattice, then

$$\hat{\mu}(\theta) \neq 1, \qquad \theta \neq 0. \tag{1.9}$$

A second observation, essentially made in [3], can be stated as

LEMMA 2. *There is a nonsingular* (*and hence strongly nonlattice*) *probability measure ν on R which defines a recurrent random walk on R and is such that its characteristic function $\hat{\nu}$ satisfies* (1.7).

By Lemmas 1 and 2, Theorem 1 reduces to

THEOREM 1′. *Theorem* 1 *holds under the additional restriction that μ be strongly nonlattice* (*or nonsingular*).

Similarly Theorem 2 reduces to

THEOREM 2′. *Theorem* 2 *holds under the additional restriction that μ be strongly nonlattice* (*or nonsingular*).

Theorems 1′ and 2′ were proven by Ornstein in [1] (μ nonsingular). We will present new and simpler proofs of these results below. The proofs will be patterned in part on methods of Port and Stone [3] and Spitzer [5].

In §2 we will introduce more notation, reformulate Theorems 1′ and 2′ and obtain some elementary results. The proof of Theorems 1′ and 2′ will be completed in §3.

In §4 we will discuss the modifications necessary to sharpen the first part of Theorem 1 by showing that under the conditions of the theorem

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_{-a}^{a} \frac{i\theta(1-\hat{\mu}^n(\theta))}{(1-\hat{\mu}(\theta))}\,d\theta$$

exists and is finite. This corresponds to proving the ordinary convergence of various sums arising in [3] instead of the slightly weaker Abel summability. In applications it does not seem to matter whether convergence or Abel summability is used.

2. **Preliminaries.** We assume in this section that $d=1$ and that μ is strongly nonlattice.

For an integrable function f on $(-\infty,\infty)$ let $\hat{f}$ denote its Fourier transform, defined by

$$\hat{f}(\theta) = \int_{-\infty}^{\infty} e^{ix\theta}f(x)\,dx, \qquad \theta\in R.$$

Let $\mathfrak{F}$ denote the collection of all continuous real-valued functions f on $(-\infty,\infty)$ which have compact support and are such that $\hat{f}$ is absolutely integrable on $(-\infty,\infty)$. For $f\in\mathfrak{F}$ we have the inversion formula

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\theta}\hat{f}(\theta)\,d\theta, \qquad x\in R. \tag{2.1}$$

For $f\in\mathfrak{F}$ set

$$J(f) = \int_{-\infty}^{\infty} f(x)\,dx \quad\text{and}\quad K(f) = \int_{-\infty}^{\infty} xf(x)\,dx.$$

Then $\hat{f}(0)=J(f)$ and $\hat{f}'(0)=iK(f)$. Also

$$\hat{f}(\theta) = J(f)+i\theta K(f)+O(\theta^2), \qquad \theta\to 0. \tag{2.2}$$

Let $\mathfrak{F}^+$ denote the collection of nonnegative functions in $\mathfrak{F}$. For any constants $0\leqq a<\infty$ and $-\infty<b<\infty$, we can find an $f\in\mathfrak{F}^+$ such that $J(f)=a$ and $K(f)=b$.

For $0\leqq\lambda<1$ let U^λ be the operator defined on $\mathfrak{F}$ by

$$U^\lambda f(x) = \sum_{n=1}^{\infty} \lambda^n \int f(x+y)\mu^{(n)}(dy), \qquad f\in\mathfrak{F} \text{ and } x\in R.$$

Then by (2.1)

$$U^\lambda f(x) = \frac{\lambda}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ix\theta}\hat{f}(-\theta)\hat{\mu}(\theta)}{1-\lambda\hat{\mu}(\theta)}\,d\theta. \tag{2.3}$$

Let g be a fixed element of $\mathfrak{F}^+$ with $J(g)=1$ and $K(g)=0$. Set $c^\lambda = U^\lambda g(0)$, $0 \leqq \lambda < 1$. For $0 \leqq \lambda < 1$ let A^λ be the operator defined on $\mathfrak{F}$ by

$$A^\lambda f(x) = c^\lambda J(f) - U^\lambda f(x), \qquad f \in \mathfrak{F} \text{ and } x \in R.$$

It follows from (2.3) that

$$A^\lambda f(x) = \frac{\lambda}{2\pi} \int_{-\infty}^{\infty} \frac{\hat{g}(-\theta)J(f) - e^{ix\theta}\hat{f}(-\theta)}{1-\lambda\hat{\mu}(\theta)} \hat{\mu}(\theta)\, d\theta \tag{2.4}$$

and

$$A^\lambda f(x+y) - A^\lambda f(y) = \frac{\lambda}{2\pi} \int_{-\infty}^{\infty} \frac{e^{iy\theta}(1-e^{ix\theta})\hat{f}(-\theta)}{1-\lambda\hat{\mu}(\theta)} \hat{\mu}(\theta)\, d\theta. \tag{2.5}$$

For $f \in \mathfrak{F}$ and $y \in R$ let $f_y \in \mathfrak{F}$ be defined by $f_y(x) = f(x-y)$, $x \in R$. Then

$$J(f_y) = J(f), \tag{2.6}$$

$$K(f_y) = K(f) + yJ(f), \tag{2.7}$$

$$U^\lambda f_y(x) = U^\lambda f(x-y), \qquad 0 \leqq \lambda < 1 \text{ and } x \in R, \tag{2.8}$$

and

$$A^\lambda f_y(x) = A^\lambda f(x-y), \qquad 0 \leqq \lambda < 1 \text{ and } x \in R. \tag{2.9}$$

Also if $f \in \mathfrak{F}^+$, then $f_y \in \mathfrak{F}^+$.

It follows from (2.4) and (2.5) that Theorem 1 is equivalent to

THEOREM 3. *Let $d=1$ and let μ be strongly nonlattice. For $f \in \mathfrak{F}$ and $-\infty < x < \infty$*

$$\lim_{\lambda \uparrow 1} A^\lambda f(x) = Af(x) \tag{2.10}$$

exists and is finite. Also

$$\lim_{y \to \pm\infty} (Af(x+y) - Af(y)) = \pm x\sigma^{-2}J(f). \tag{2.11}$$

The convergence in these limits is uniform on compacts.

REMARK. In order to prove Theorems 1′ and 3 it suffices to prove Theorem 3 for $f \in \mathfrak{F}^+$.

We can easily prove as a special case of Theorem 3

LEMMA 3. *For $f \in \mathfrak{F}$ with $K(f)=0$*

$$\lim_{\lambda \uparrow 1} (A^\lambda f(y) + A^\lambda f(-y)) \tag{2.12}$$

exists and is finite. Moreover

$$\lim_{y \to \infty} y^{-1} \lim_{\lambda \uparrow 1} (A^\lambda f(y) + A^\lambda f(-y)) = 2\sigma^{-2}J(f). \tag{2.13}$$

Proof of Lemma 3. From (2.4) we obtain

$$A^\lambda f(y) + A^\lambda f(-y) = \frac{\lambda}{\pi} \int_{-\infty}^{\infty} \frac{\hat{g}(-\theta)J(f) - \cos y\theta \hat{f}(-\theta)}{1-\lambda\hat{\mu}(\theta)} \hat{\mu}(\theta)\, d\theta. \tag{2.14}$$

The remainder of the proof follows that of Spitzer [5, pp. 345–346].

It follows from (2.12) and (2.14) that Theorem 2′ is implied by

THEOREM 4. *Let $d=1$ and let μ be strongly nonlattice. For $f \in \mathfrak{F}^+$ such that $J(f)>0$ (and $K(f)=0$)*

$$\lim_{y\to\infty} \lim_{\lambda\uparrow 1} (A^\lambda f(y)+A^\lambda f(-y)) = \infty. \tag{2.15}$$

We can also obtain directly

LEMMA 4. *For $f \in \mathfrak{F}$ and $x \in R$*

$$\lim_{\lambda\uparrow 1} (A^\lambda f(y+2x)-2A^\lambda f(y+x)+A^\lambda f(y)) \tag{2.16}$$

exists and is finite. Moreover

$$\lim_{|x|\to\infty} \lim_{\lambda\uparrow 1} (A^\lambda f(y+2x)-2A^\lambda f(y+x)+A^\lambda f(y)) = 0. \tag{2.17}$$

The convergence in these limits is uniform on compacts.

Proof of Lemma 4. From (2.4) or (2.5) we obtain

$$A^\lambda f(y+2x)-2A^\lambda f(y+x)+A^\lambda f(y) = \frac{\lambda}{\pi}\int_{-\infty}^{\infty} \frac{e^{iy\theta}e^{ix\theta}(1-\cos x\theta)\hat{f}(-\theta)\hat{\mu}(\theta)}{1-\lambda\hat{\mu}(\theta)}\,d\theta, \tag{2.18}$$

from which the lemma follows by dominated convergence and the Riemann-Lebesgue Lemma.

LEMMA 5. *Suppose $\sigma^2<\infty$. Then for $f \in \mathfrak{F}$*

$$\lim_{\lambda\uparrow 1} A^\lambda f(y) = Af(y) \tag{2.19}$$

exists and is finite and the convergence is uniform on compacts. Moreover

$$\lim_{y\to\pm\infty} y^{-1}Af(y) = \pm\sigma^{-2}J(f). \tag{2.20}$$

Proof of Lemma 5. If $\sigma^2<\infty$, then as is well known

$$\int_{-\infty}^{\infty} \left|\frac{\Im\hat{\mu}(\theta)}{\theta^3}\right| d\theta < \infty. \tag{2.21}$$

It follows easily that

$$\lim_{\lambda\uparrow 1} A^\lambda f(y) = Af(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Re\left(\frac{(\hat{g}(-\theta)J(f)-e^{iy\theta}\hat{f}(-\theta))\hat{\mu}(\theta)}{1-\hat{\mu}(\theta)}\right) d\theta \tag{2.22}$$

and the convergence is uniform on compacts. Thus

$$Af(y)-Af(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Re\left(\frac{(1-e^{iy\theta})\hat{f}(-\theta)\hat{\mu}(\theta)}{1-\hat{\mu}(\theta)}\right) d\theta. \tag{2.23}$$

It is easily seen that

$$\lim_{|y|\to\infty} \left[\frac{Af(y)-Af(0)}{y}-\frac{J(f)}{2\pi y}\int_{-1}^{1} \Re\left(\frac{1-e^{iy\theta}}{1-\hat{\mu}(\theta)}\right) d\theta\right] = 0. \tag{2.24}$$

From (2.21) we obtain easily

$$\lim_{|y|\to\infty} \frac{1}{2\pi y} \int_{-1}^{1} \frac{\sin y\theta \Im\hat{\mu}(\theta)}{|1-\hat{\mu}(\theta)|^2}\, d\theta = 0. \tag{2.25}$$

We also have that

$$\begin{aligned} \lim_{y\to\pm\infty} \frac{1}{2\pi y} \int_{-1}^{1} \frac{(1-\cos y\theta)\Re(1-\hat{\mu}(\theta))}{|1-\hat{\mu}(\theta)|^2}\, d\theta &= \lim_{y\pm\infty} \frac{1}{\pi y\sigma^2} \int_{-1}^{1} \frac{1-\cos y\theta}{\theta^2}\, d\theta \\ &= \frac{\pm 1}{\pi\sigma^2} \int_{-\infty}^{\infty} \frac{1-\cos\theta}{\theta^2}\, d\theta = \pm\sigma^{-2}. \end{aligned} \tag{2.26}$$

Lemma 5 now follows from (2.24), (2.25), and (2.26).

Let S_n denote the random walk with transition distribution μ. Then the S_n's are real-valued random variables; $S_0, S_1-S_0, S_2-S_1, \ldots$ are independent; and $S_1-S_0, S_2-S_1, \ldots$ have distribution μ. Let E_x denote expectation when $S_0 \equiv x$.

Let B be a fixed Borel subset of R which is relatively compact and has a non-empty interior. Let $1_B(x)$, $x \in R$, denote the indicator function of B. Set

$$T_B = \min\{n > 0 : S_n \in B\}.$$

For $0 \leqq \lambda \leqq 1$ let G_B^λ and Π_B^λ be operators on $\mathfrak{F}$ defined by

$$G_B^\lambda f(x) = E_x \sum_1^{T_B} \lambda^n f(S_n)$$

and

$$\Pi_B^\lambda f(x) = E_x \lambda^{T_B} f(S_{T_B}).$$

Set $G_B = G_B^1$ and $\Pi_B = \Pi_B^1$. Then

$$\lim_{\lambda\uparrow 1} G_B^\lambda f(x) = G_B f(x) \tag{2.27}$$

and

$$\lim_{\lambda\uparrow 1} \Pi_B^\lambda f(x) = \Pi_B f(x), \tag{2.28}$$

both limits being uniform on compacts. (Actually the definition of Π_B^λ and (2.28) are valid for any continuous function f.) Note also that

$$U^\lambda f(x) = E_x \sum_1^{\infty} \lambda^n f(S_n). \tag{2.29}$$

As in [3] we have the important identity that for $f \in \mathfrak{F}$, $x \in R$, and $0 \leqq \lambda < 1$

$$U^\lambda f(x) = G_B^\lambda f(x) + \Pi_B^\lambda U^\lambda f(x). \tag{2.30}$$

Set

$$L_B^\lambda(x) = c^\lambda(1 - \Pi_B^\lambda 1_X(x)).$$

Then $L_B^\lambda(x) \geqq 0$. It follows from (2.30) that

$$A^\lambda f(x) - \Pi_B^\lambda A^\lambda f(x) = -G_B^\lambda f(x) + L_B^\lambda(x) J(f). \tag{2.31}$$

Note finally that if C is a compact subset of R having a nonempty interior and $f \in \mathfrak{F}$ is such that $f(x)=0$, $x \notin C$, then for $x \in R$ and $0 \leqq \lambda < 1$

$$U^\lambda f(x) = E_x \lambda^{T_C}(f(S_{T_C}) + U^\lambda f(S_{T_C})). \tag{2.32}$$

3. **Proofs of Theorems 3 and 4.** We assume again in this section that $d=1$ and μ is strongly nonlattice. We will prove Theorems 3 and 4, from which the other theorems follow immediately.

Set

$$d^\lambda = \frac{\lambda}{2\pi} \int_{-1}^{1} \frac{i\theta}{1-\lambda\hat{\mu}(\theta)}\, d\theta, \qquad 0 \leqq \lambda < 1,$$

and let D^λ, $0 \leqq \lambda < 1$, be the operator on $\mathfrak{F}$ defined by

$$\begin{aligned} D^\lambda f(x) &= A^\lambda f(x) + d^\lambda(xJ(f) - K(f)) \\ &= c^\lambda J(f) - U^\lambda f(x) + d^\lambda(xJ(f) - K(f)). \end{aligned}$$

LEMMA 6. *For $f \in \mathfrak{F}$ and $x \in R$*

$$\lim_{\lambda \uparrow 1} D^\lambda f(x) = Df(x) \tag{3.1}$$

exists and is finite and the convergence is uniform on compacts.

Proof of Lemma 6.

$$\begin{aligned} D^\lambda f(x) = \frac{\lambda}{2\pi} \int_{-1}^{1} & \frac{(\hat{g}(-\theta)J(f) - e^{ix\theta}\hat{f}(-\theta)\hat{\mu}(\theta) + i\theta(xJ(f) - K(f))}{1-\lambda\hat{\mu}(\theta)}\, d\theta \\ & + \frac{\lambda}{2\pi} \int_{|\theta|>1} \frac{(\hat{g}(-\theta)J(f) - e^{ix\theta}\hat{f}(-\theta))}{1-\lambda\hat{\mu}(\theta)}\, \hat{\mu}(\theta)\, d\theta. \end{aligned}$$

The second term causes no problem. In studying the first term, it suffices to study

$$\frac{\lambda}{2\pi} \int_{-1}^{1} \frac{\hat{g}(-\theta)J(f) - e^{ix\theta}\hat{f}(-\theta) + ix\theta J(f) - i\theta K(f)}{1-\lambda\hat{\mu}(\theta)}\, d\theta.$$

Recall that $J(g)=1$ and $K(g)=0$. Thus the numerator of the integrand is of the form $O(\theta^2)$, and Lemma 6 now follows easily.

From (2.31) we get the identity

$$\begin{aligned} D^\lambda f(x) - \Pi_B^\lambda D^\lambda f(x) = & -G_B^\lambda f(x) + J(f)\left(L_B^\lambda(x) + xd^\lambda - d^\lambda \int_B z\Pi_B^\lambda(x, dz)\right) \\ & + K(f)d^\lambda(\Pi_B^\lambda(x, B) - 1), \end{aligned} \tag{3.2}$$

where $\Pi_B^\lambda(x, dz)$ is the measure defined by

$$\Pi_B^\lambda f(x) = \int \Pi_B^\lambda(x, dz) f(z), \qquad f \in \mathfrak{F} \text{ and } x \in R.$$

The quantities $D^\lambda f(x)$, $\Pi_B^\lambda D^\lambda f(x)$, and $G_B^\lambda f(x)$ have limits as $\lambda \uparrow 1$ which exist uniformly on compacts (by (2.27) and (2.28) and Lemma 6). Since $J(f)$ and $K(f)$ can be chosen arbitrarily we obtain from (3.2)

LEMMA 7. *For $x \in R$*

$$\lim_{\lambda \uparrow 1} \left(L_B^\lambda(x) + x d^\lambda \int_B z \Pi_B^\lambda(x, dz) \right) \tag{3.3}$$

exists and is finite and the convergence is uniform on compacts.

From this result we will get

LEMMA 8. $\limsup_{\lambda \uparrow 1} |d^\lambda| < \infty$.

Proof of Lemma 8. By definition $L_B^\lambda(x) \geqq 0$ for $0 \leqq \lambda < 1$ and $x \in R$. Also there is a finite constant M such that

$$\left| \int_B z \Pi_B^\lambda(x, dz) \right| \leqq M, \qquad 0 \leqq \lambda \leqq 1 \text{ and } x \in R.$$

Suppose $\limsup_{\lambda \uparrow 1} d^\lambda = \infty$. Then by choosing $x > M$ and using (3.3) we obtain a contradiction.

Suppose instead that $\liminf_{\lambda \uparrow 1} d^\lambda = -\infty$. Then by choosing $x < -M$ and using (3.3) we again get a contradiction. This proves the lemma.

Let λ_n, $n \geqq 1$, be any sequence of nonnegative numbers such that $\lambda_n \uparrow 1$ as $n \to \infty$ and

$$\lim_{n \to \infty} d^{\lambda_n} = d \tag{3.4}$$

exists. By Lemma 8 such sequences exist and d is necessarily finite.

By (3.4) and Lemma 7 we see that

$$\lim_{n \to \infty} L_B^{\lambda_n}(x) = L_B(x) \tag{3.5}$$

exists and is finite and the convergence is uniform on compacts. Clearly $L_B(x) \geqq 0$, $x \in R$. We cannot say yet, however, that $L_B(x)$ is independent of the choice of the sequence λ_n, $n \geqq 1$.

LEMMA 9. *For $f \in \mathfrak{F}$ and $x \in R$*

$$\lim_{n \to \infty} A^{\lambda_n} f(x) = Af(x) \tag{3.6}$$

exists and is finite and the convergence is uniform on compacts.

Proof of Lemma 9. This result follows immediately from Lemma 6 and equation (3.4). Again we cannot say yet that A is independent of the choice of the sequence λ_n, $n \geqq 1$.

It now follows from equations (2.31), (2.27), (2.28), (3.5), and (3.6) that

$$Af(x) - \Pi_B Af(x) = -G_B f(x) + L_B(x) J(f). \tag{3.7}$$

LEMMA 10. *For $f \in \mathfrak{F}$ and compact subset C_1 of R there is a finite constant M such that*

$$|Af(y+z) - Af(y)| \leqq M, \qquad y \in R \text{ and } z \in C_1. \tag{3.8}$$

Proof of Lemma 10. Let C be a compact subset of R having a nonempty interior and such that $f(y)=0$, $y \notin C$, and

$$f_{-z}(y) = f(y+z) = 0, \qquad z \in C_1 \text{ and } y \notin C.$$

Then for $z \in C_1$ by (2.32)

$$\begin{aligned} U^\lambda f(y+z) - U^\lambda f(y) &= U^\lambda f_{-z}(y) - U^\lambda f(y) \\ &= E_y \lambda^{T_C}(f_{-z}(S_{T_C}) - f(S_{T_C})) + E_y \lambda^{T_C}(U^\lambda f_{-z}(S_{T_C}) - U^\lambda f(S_{T_C})) \\ &= E_y \lambda^{T_C}(f(z+S_{T_C}) - f(S_{T_C})) + E_y \lambda^{T_C}(A^\lambda f(S_{T_C}) - A^\lambda f(z+S_{T_C})). \end{aligned}$$

The desired result now follows from Lemma 9.

LEMMA 11. *If $f \in \mathfrak{F}$, then*

$$\Pi_B A f_y(x) - A f_y(0)$$

is uniformly bounded for $x \in R$ and $y \in R$.

Proof of Lemma 11. Note that

$$\begin{aligned} \Pi_B A f_y(x) - A f_y(0) &= \int \Pi_B(x, dz)(A f_y(z) - A f_y(0)) \\ &= \int \Pi_B(x, dz)(A f(z-y) - A f(-y)), \end{aligned}$$

and the result follows from Lemma 10.

From (3.7) we see that if $f \in \mathfrak{F}$, $x \in R$ and $y \in R$, then

$$\begin{aligned} A f(x+y) - A f(x) - A f(y) &= A f_{-y}(x) - A f(x) - A f_{-y}(0) \\ &= (\Pi_B A f_{-y}(x) - A f_{-y}(0)) \\ &\quad - \Pi_B A f(x) - G_B f_{-y}(x) + G_B f(x). \end{aligned} \tag{3.9}$$

We study next the right side of (3.9). It follows from Lemma 11 that

$$(\Pi_B A f_{-y}(x) - A f_{-y}(0))$$

is bounded uniformly in x and y. Also $\Pi_B A f(x)$ is bounded for $x \in R$, since B is relatively compact and Af is bounded on compacts. Clearly $G_B f(x)$ is bounded for $x \in R$.

LEMMA 12. *If $f \in \mathfrak{F}^+$, then $Af(x+y) - Af(x) - Af(y)$ is bounded from above uniformly for $x \in R$ and $y \in R$.*

Proof of Lemma 12. The result follows from (3.9), the observations which follow, and the fact that if $f \in \mathfrak{F}^+$, then $G_B f_{-y}(x) \geqq 0$ and hence $-G_B f_{-y}(x)$ is bounded from above uniformly in x and y.

LEMMA 13. *If $f \in \mathfrak{F}^+$ and $J(f) > 0$, then*

$$\lim_{|y| \to \infty} G_B f_y(y) = \infty. \tag{3.10}$$

Proof of Lemma 13. Note that

$$G_B f_y(y) = E_y \sum_1^{T_B} f_y(S_n) = E_y \sum_1^{T_B} f(S_n - y) = E \sum_1^{T_{B-y}} f(S_n). \tag{3.11}$$

With probability one

$$\lim_{|y| \to \infty} T_{B-y} = \infty. \tag{3.12}$$

If $f \in \mathfrak{F}^+$ and $J(f) > 0$, then with probability one

$$\sum_1^{\infty} f(S_n) = \infty. \tag{3.13}$$

Lemma 13 follows from (3.11)–(3.13).

Proof of Theorem 4. Choose $f \in \mathfrak{F}^+$ with $J(f) > 0$. From (3.9), the observations which follow it, and from Lemma 13 it follows that

$$\lim_{y \to \infty} (Af(y) + Af(-y)) = \infty. \tag{3.14}$$

Theorem 4 now follows from Lemma 9 and the first part of Lemma 3.

LEMMA 14. *If $f \in \mathfrak{F}^+$, then $Af(x)$, $x \in R$, is bounded from below.*

Proof of Lemma 14. Let C be a compact set having a nonempty interior and containing the support of f. Then by Lemma 9 there is a finite constant M such that

$$U^{\lambda_n} f(0) - U^{\lambda_n} f(y) \geqq -M, \qquad n \geqq 1 \text{ and } y \in C. \tag{3.15}$$

We can also assume that

$$f(y) \leqq M, \qquad y \in R. \tag{3.16}$$

Then by (2.32), (3.15), and (3.16) for $x \in R$

$$\begin{aligned} U^{\lambda_n} f(0) - U^{\lambda_n} f(x) &= U^{\lambda_n} f(0) - E_x \lambda_n^{T_C} f(S_{T_C}) - E_x \lambda_n^{T_C} U^{\lambda_n} f(S_{T_C}) \\ &\geqq E_x \lambda_n^{T_C} (U^{\lambda_n} f(0) - U^{\lambda_n} f(S_{T_C})) - M \geqq -2M. \end{aligned}$$

Thus by Lemma 9, $Af(x) - Af(0) \geqq -2M$, $x \in R$, from which the desired result follows.

LEMMA 15. *If $\sigma^2 = \infty$ and $f \in \mathfrak{F}^+$, then*

$$\lim_{|y| \to \infty} y^{-1} Af(y) = 0. \tag{3.17}$$

Proof of Lemma 15. The result follows immediately from Lemmas 3, 9, and 14. This result complements Lemma 5.

Set $\Delta_x(y) = Af(y) - Af(y-x)$, $x \in R$ and $y \in R$. Then by Lemma 4

$$\lim_{|y| \to \infty} |\Delta_x(y+x) - \Delta_x(y)| = 0, \qquad x \in R, \tag{3.18}$$

and the convergence is uniform on compacts.

LEMMA 16. *If* $\sigma^2=\infty$ *and* $f\in\mathfrak{F}^+$, *then*

$$\lim_{|y|\to\infty}(Af(x+y)-Af(y))=0,\qquad x\in R, \tag{3.19}$$

and the convergence is uniform on compacts.

Proof of Lemma 16. By Lemma 12 there is a finite constant M such that, for $x\in R$, $y\in R$, and $n\geqq 1$

$$Af(y+nx)-Af(y)\leqq Af(nx)-Af(0)+M$$

and hence that

$$\Delta_x(y+x)+\cdots+\Delta_x(y+nx)\leqq Af(nx)-Af(0)+M.$$

Thus by (3.18) for fixed $n\geqq 1$

$$\limsup_{|y|\to\infty} n\Delta_x(y+x)\leqq Af(nx)-Af(0)+M \tag{3.20}$$

uniformly on compacts. Thus by Lemma 15

$$\limsup_{|y|\to\infty}\Delta_x(y+x)\leqq 0 \tag{3.21}$$

uniformly on compacts. It follows from (3.21) that

$$\limsup_{|y|\to\infty}(Af(y+x)-Af(y))\leqq 0 \tag{3.22}$$

and

$$\limsup_{|y|\to\infty}(Af(y)-Af(y+x))\leqq 0, \tag{3.23}$$

both limits being uniform on compacts. Equation (3.19) follows from (3.22) and (3.23).

LEMMA 17. *If* $\sigma^2<\infty$ *and* $f\in\mathfrak{F}$, *then*

$$\lim_{y\to\pm\infty}(Af(x+y)-Af(y))=\pm x\sigma^{-2}J(f) \tag{3.24}$$

and the convergence is uniform on compacts.

Proof of Lemma 17. Let C_1 be a compact subset of R. Let C be a compact interval of R having a nonempty interior and such that $f(z)=f_x(z)=0$ for $x\in C_1$ and $z\notin C$. Then from (2.32) we get that for $0\leqq\lambda<1$, $x\in C_1$ and $y\in R$

$$\begin{aligned}U^\lambda f_x(y)-U^\lambda f(y)&=E_y\lambda^{T_C}(f_x(S_{T_C})-f(S_{T_C}))+E_y\lambda^{T_C}(U^\lambda f_x(S_{T_C})-U^\lambda f(S_{T_C}))\\&=E_y\lambda^{T_C}(f_x(S_{T_C})-f(S_{T_C}))+E_y\lambda^{T_C}(A^\lambda f(S_{T_C})-A^\lambda f_x(S_{T_C})).\end{aligned}$$

It now follows from Lemma 9 that

$$Af_x(y)-Af(y)=E_y(Af_x(S_{T_C})-Af(S_{T_C}))+E_y(f(S_{T_C})-f_x(S_{T_C})). \tag{3.25}$$

It also follows from Lemma 9 that Af is a continuous function.

It follows from Theorem 3.4 of Spitzer [4] that the left and right ladder processes associated with the random walk S_n have finite means. It is now a consequence of

the Blackwell renewal theorem that $P_y(S_{T_C} \in dz)$ have limiting distributions as $y \to \pm\infty$.

Thus by (3.25)

$$\lim_{y \to \pm\infty} (Af(x+y) - Af(y))$$

exists and is finite and the convergence is uniform on compacts. Lemma 17 now follows from Lemma 5.

From (3.7) we get that for $f \in \mathfrak{F}$, $x \in R$, and

$$(Af_y(x) - Af_y(0)) - \Pi_B(Af_y - Af_y(0))(x) = -G_B f_y(x) + L_B(x)J(f). \tag{3.26}$$

Thus from Lemmas 16 and 17 we get

LEMMA 18. *If $f \in \mathfrak{F}^+$ or $\sigma^2 < \infty$ and $f \in \mathfrak{F}$, then*

$$\lim_{y \to \pm\infty} G_B f_y(x) = J(f)\left(L_B(x) \pm \sigma^{-2} \int (x-z)\Pi_B(x, dz)\right) \tag{3.27}$$

and the convergence is uniform on compacts.

Proof of Theorem 3. It follows from Lemma 18, by choosing $f \in \mathfrak{F}^+$ with $J(f) > 0$, that $L_B(x)$ is independent of the choice of the sequence λ_n, $n \geqq 1$. Thus by (3.3), the d in (3.4) is independent of the choice of the sequence λ_n, $n \geqq 1$. Therefore by Lemma 8

$$\lim_{\lambda \uparrow 1} d^\lambda = d \tag{3.24}$$

exists and is finite. Consequently (2.10) holds. It follows from Lemmas 16 and 17 that if $f \in \mathfrak{F}^+$ or $\sigma < \infty$ and $f \in \mathfrak{F}$, then (2.11) holds. By the remark following Theorem 3, this completes the proof of Theorem 3.

4. **On replacing Abel summability by convergence.** Let $P^n f$ be defined by

$$P^n f(x) = \int f(x+y)\mu^{(n)}(dy)$$

whenever f is continuous, bounded, and integrable. Set

$$G_n = \sum_{k=1}^{n} P^k,$$

set $c_n = G_n g(0)$, where g is as in §2 and set

$$A_n f = c_n J(f) - G_n f = \sum_{k=1}^{n} (J(f)P^k g(0) - P^k f).$$

If

$$\lim_{n \to \infty} A_n f(x) = Af(x), \tag{4.1}$$

then

$$\lim_{\lambda \uparrow 1} A^\lambda f(x) = Af(x). \tag{4.2}$$

Equation (4.1) states that the series

$$\sum_{k=1}^{n} (J(f)P^k g(0) - P^k f(x))$$

is convergent, whereas (4.2) states the slightly weaker result that the series is Abel summable.

Although (4.1) and (4.2) seem to be equally useful in applications it is still of interest to know that (4.1) holds for suitable f. Following the arguments of [3] we see that this is equivalent to replacing the first part of Theorem 1 by

THEOREM 5. *Under the conditions of Theorem* 1

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_{-a}^{a} \frac{i\theta(1-\hat{\mu}^n(\theta))}{1-\hat{\mu}(\theta)}\, d\theta$$

exists and is finite.

Note that Theorem 5 states that the series

$$\sum_{k=1}^{n} \frac{1}{2\pi} \int_{-a}^{a} i\theta\hat{\mu}^k(\theta)\, d\theta$$

is convergent, whereas the first part of Theorem 1 states only that the series is Abel summable.

As was the case with Theorem 1, Theorem 5 can be reduced to

THEOREM 5′. *Theorem* 5 *holds under the added restriction that* μ *be strongly nonlattice* (*or nonsingular*).

To reduce Theorem 5 to Theorem 5′, we find a strongly nonlattice probability measure ν defining a recurrent random walk and such that $\hat{\nu}(\theta)-\hat{\mu}(\theta)=O(\theta^4)$ as $\theta\to 0$. Clearly

$$\frac{1}{2\pi}\int_{-a}^{a} \left| \frac{i\theta}{1-\hat{\mu}(\theta)} - \frac{i\theta}{1-\hat{\nu}(\theta)} \right| d\theta < \infty.$$

Thus it suffices to show that

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_{-a}^{a} \left| \frac{i\theta}{1-\hat{\nu}(\theta)} \right| |\hat{\mu}^{n+1}(\theta)-\hat{\nu}^{n+1}(\theta)|\, d\theta = 0. \tag{4.3}$$

We can assume that a is such that for some $c>0$

$$|\hat{\mu}(\theta)| \leqq 1-c\theta^2, \qquad |\theta| \leqq a, \tag{4.4}$$

and

$$|\hat{\nu}(\theta)| \leqq 1-c\theta^2, \qquad |\theta| \leqq a. \tag{4.5}$$

This follows easily by truncating μ and ν and using the fact that these measures are nondegenerate, By (4.4) and (4.5) we have that for $|\theta| \leqq a$

$$\begin{aligned}|\hat{\mu}^{n+1}(\theta)-\hat{\nu}^{n+1}(\theta)| &\leqq (n+1)|\hat{\mu}(\theta)-\hat{\nu}(\theta)|(1-c\theta^2)^n \\ &\leqq (n+1)|\hat{\mu}(\theta)-\hat{\nu}(\theta)| \exp(-cn\theta^2) \\ &\leqq K_1(n+1)\theta^4 \exp(-cn\theta^2)\end{aligned}$$

for some $K_1<\infty$. Thus

$$\frac{1}{2\pi}\left|\frac{i\theta}{1-\hat{\nu}(\theta)}\right| |\hat{\mu}^{n+1}(\theta)-\hat{\nu}^{n+1}(\theta)| \leqq K(n+1)|\theta|^3 \exp(-cn\theta^2), \qquad |\theta| \leqq a,$$

for some $K<\infty$. Since

$$(n+1)\int_{-a}^{a} |\theta|^3 \exp(-cn\theta^2)\, d\theta \leqq \frac{n+1}{n^2}\int_{-\infty}^{\infty} |\theta|^3 \exp(-c\theta^2)\, d\theta \to 0$$

as $n \to \infty$, (4.3) is valid.

Let ${}_B\Pi^n$, ${}_BP^n$, and ${}_BG_n$ be defined by

$${}_B\Pi^n f(x) = E_x[f(S_n); T_B = n], \qquad {}_BP^n f(x) = E_x[f(S_n); T_B \geqq n],$$

and

$${}_BG_n = \sum_{k=1}^{n} {}_BP^k.$$

The proof of Theorem 5′ is similar to that of Theorem 1′. The main difference is that instead of (2.30) we start with

$$P^n = \sum_{k=1}^{n} {}_B\Pi^k P^{n-k} + {}_BP^n$$

and sum on n to get

$$G_n = \sum_{k=1}^{n} {}_B\Pi^k G_{n-k} + {}_BG_n. \tag{4.6}$$

The identity (4.6) plays the same role in proving Theorem 5′ as (2.30) does in proving Theorem 1′.

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