ON HANDLE DECOMPOSITIONS AND DIFFEOMORPHISMS

BY

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In this paper we are concerned with two questions which arise naturally in the study of manifolds. One has to do with whether certain types of handle decompositions of a manifold are unique; and the other, under what circumstances they are preserved by diffeomorphisms. The corollaries to Theorem 1 and 2 are partial answers to these questions. Corollary 2 may prove useful for the general study of diffeomorphisms. We shall begin with some notation.

$R^k$ shall denote real $k$-dimensional vector space with unit disc $D^k$ and unit sphere $S^{k-1}$. Let $I = [0, 1]$. All manifolds are assumed to be compact and $C^\infty$. Homology groups are taken with integer coefficients. $\cong$ means either diffeomorphic or isotopic. By a nice Morse function on a manifold $M^n$ (see [5, p. 44]) we shall mean a function $\eta: M^n \to R$ which has only nondegenerate critical points, and if $p$ is a critical point, then $\eta(p) = \text{index } p$. It is well known that in this case $M_k(\eta) \equiv \eta^{-1}[0, k+1/2]$ is a $k$-submanifold of $M$ for $0 \leq k \leq n$, and

$$M_k(\eta) = M_{k-1}(\eta) \cup \phi_1(D^k \times D^{n-k}) \cup \cdots \cup \phi_p(D^k \times D^{n-k}),$$

where $\phi_j: \partial D^k \times D^{n-k} \to \eta^{-1}[k-1/2]$ are imbeddings. Call $\eta$ minimal if the number of critical points of index $k$ is equal to $p(k) + q(k) + q(k-1)$, where $p(k)$ is the torsion-free rank of $H_kM$ and $q(k)$ is the rank of the torsion subgroup of $H_kM$ (see [6, §6]).

From now on $M^n$ will be a closed simply-connected manifold with $n \geq 6$.

**Theorem 1.** Let $\eta_j$ be minimal nice Morse functions on $M$, $j = 1, 2$. Assume that $H_{i_s}M$ is torsion-free for $0 = i_1 < i_2 < \cdots < i_t = n$. Then there is a diffeomorphism $H: M \to M$ satisfying

(i) $H \cong \text{identity},$

(ii) $H(M_s(\eta_1)) = M_s(\eta_2)$, for $1 \leq s \leq t$.

**Theorem 2.** Let $F: M \to M$ be a diffeomorphism and $\eta_1$ any minimal nice Morse function on $M$. Assume that $F_\ast |\text{torsion} (H_{i_t}M) = \text{identity}$ for $0 = i_1 < i_2 < \cdots < i_{t-1} < i_t = n$. Then $F$ is isotopic to a diffeomorphism $F'$ such that $F'(M_s(\eta_1)) = M_s(\eta_1)$, for $1 \leq s \leq t$.

**Remark.** The existence of minimal nice Morse functions on $M$ is guaranteed by [6, §6].

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As an immediate consequence of Theorems 1 and 2 we get the following corollaries.

**Corollary 1.** Assume that $H_\* M$ is torsion-free. Let $\eta_j$ be minimal nice Morse functions on $M$, $j = 1, 2$. Then $M_1(\eta_1) \approx M_2(\eta_2)$, for $0 \leq i \leq n$ (i.e., minimal handle decompositions of $M$ are unique up to diffeomorphisms).

**Corollary 2.** Let $F: M \to M$ be a diffeomorphism such that $F_\*|_{\text{torsion } H_\* M}$ = identity. Let $\eta$ be a minimal nice Morse function on $M$. Then $F \approx F'$, where $F'(M_1(\eta)) = M_1(\eta)$, for $0 \leq i \leq n$ (i.e., up to isotopy, most diffeomorphisms of $M$ are "level-preserving").

In order to prove Theorems 1 and 2 we begin with a lemma.

**Lemma.** Let $\eta_1$ and $\eta_2$ be minimal nice Morse functions on $M$. Define $M'_1 = \eta_1^{-1}[0, i + 1/2]$ and suppose that $M_k = M_k = A$ for some fixed $k$, $0 \leq k < n$. Let $k < l < n - 2$.

(a) If $H_\* M$ is torsion-free, then there is a diffeomorphism $H$ of $M$ such that

(i) $H \approx$ identity,
(ii) $H|A = \text{identity},$
(iii) $H(M'_1) = M'_1$.

(b) Assume $\eta_2 = \eta_1 F^{-1}$, where $F$ is a diffeomorphism of $M$ with $F_\*|_{\text{torsion } H_\* M}$ = identity. Then there is a diffeomorphism $H$ of $M$ satisfying (i)-(iii) in part (a).

**Proof.** (a) The following easy general position argument shows that there is a diffeomorphism $h_1$ of $M$ satisfying (i), (ii) and $h_1(M'_1) \subset \text{int } M'_1$: If $M'_2 \subset M'_1$, for $q > l$, write $M'_2 = M'_1 \cup \psi_i (D^q \times D^n)$, with respect to imbeddings $\psi_i: \partial D^q \times D^n \to \partial M'_1$. Now as a cell complex $M'_1$ has dimension at most $l$, so that $q > l$ implies that we may deform $M'_2$ so that $M'_2 \cap (0 \times D^n) = \varnothing$; but once we have this, it is easy to push $M'_2$ into int $M'_1$. Repeating this process for each $q > l$ we get $h_1$.

Let $B = h_1(M'_1)$ and let $f: B \to M$ be the inclusion map. Consider

\[ \cdots \to H_{j+1}(M, B) \xrightarrow{\partial_{j+1}} H_j B \to H_j M \to H_j(M, B) \to \cdots \]

\[ \downarrow \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \downarrow \]

\[ \cdots \to H_{j+1}(M, M'_1) \xrightarrow{\partial_{j+1}} H_j M'_1 \to H_j M \to H_j(M, M'_1) \to \cdots \]

Since $H_j(M, B) = 0 = H_j(M, M'_1)$ for $j \leq l$, $f_\*$ is an isomorphism for $j < l$. But $\partial_{l+1} = 0 = \partial_{l+1}^*$, because $H_j M$ is torsion-free. Thus $f_\*$ is an isomorphism for all $j$. By the Whitehead theorem (see [3, Theorem 10.1]), $f$ is a homotopy equivalence.

**Claim.** $\pi_1 Y = \pi_1 \partial M'_1 = \pi_1 \partial B = 0$, where $Y = \text{closure of } M'_1 - B$.

Assuming this for the moment, we get by [5, Theorem 9.1] that $Y \approx \partial B \times I$. Thus, using the $I$-coordinate it should be clear how to define a diffeomorphism $h_2$ of $M$ satisfying (i), (ii) and $h_2(B) = M'_1$. Let $H = h_2 h_1$. Then $H$ is the desired map.
To prove the claim, let \( g: S^1 \to Y \). Since \( \pi_1 M^1 = 0 = \pi_1 M \), there is a \( \tilde{g}: D^2 \to M^1 \) with \( \tilde{g}|S^1 = g \). But \( l + 2 < n \) and so we can push \( \tilde{g} \) into \( Y \) by a general position argument similar to the one at the beginning of this proof. Therefore, \( g \simeq 0 \) in \( Y \), i.e., \( \pi_1 Y = 0 \). To see that \( \pi_1 \partial M^1 = 0 \) we use induction (\( \pi_1 \partial B = 0 \) by an analogous argument). Clearly, \( \pi_1 \partial M^1 = \pi_1 \partial M^1 = \pi_1 S^{n-1} = 0 \), because \( n > 2 \) and \( \pi_1 M = 0 \). Now \( \partial M^1 \) is obtained from \( \partial M^1 \) by cutting out a certain number of disjoint \( S^{t-1} \times D^{n-t} \) and adding an equal number of \( D^t \times S^{n-t} \). If \( \pi_1 \partial M^1 = 0 \), for \( 1 < j < n \), then Van Kampen’s theorem \([2, \text{Theorem 6.4.3}]\) shows that \( \pi_1 \partial M^1 = 0 \). This proves that \( \pi_1 \partial M^1 = 0 \) for \( 0 \leq j \leq n \), and finishes the proof of part (a).

(b) The method of part (a) breaks down only at one point; namely, we do not know that \( \partial_{i+1} = 0 = \partial_{i+1} \). Hence we must give a special argument to make \( f \) into a homotopy equivalence. Specifically, assume that \( F(M^1) \subset M^1 \) and let \( f = F \mid M^1 : M^1 \to M^1 \). Again, \( f \) is an isomorphism for \( j < l \). Consider

\[
\begin{array}{cccccc}
\cdots & \cdots & H_{i+1}(M, M^1) & \xrightarrow{\delta_{i+1}} & H_i M^1 & \xrightarrow{i_*} & H_i M & \longrightarrow 0 \\
\downarrow & & F_* & & f_* & & F_* & & \\
\cdots & \cdots & H_{i+1}(M, M^1) & \xrightarrow{\delta_{i+1}} & H_i M^1 & \xrightarrow{i_*} & H_i M & \longrightarrow 0.
\end{array}
\]

Write

\[
H_i M^1 = \bigoplus_{s=1}^{q(l)} \{ x_s \} \bigoplus \bigoplus_{s=1}^{q(l)} \{ y_s \},
\]

\[
H_i M = \bigoplus_{s=1}^{p(l)} \{ x_s \} \bigoplus \bigoplus_{s=1}^{q(l)} \{ y_s \}, \quad k_s = \text{order of } y_s.
\]

Note that \( \ker f_* \subset \ker i_* \subset \sum_{s=1}^{q(l)} \{ y_s \} \) and \( f_*(\sum_{s=1}^{q(l)} \{ y_s \}) \subset \sum_{s=1}^{q(l)} \{ y_s \} \). Let \( f_* y_s = \sum_{s=1}^{q(l)} c_{sf} y_s \), and set \( C = \{ c_{sf} \} \). We may assume without loss of generality that \( i_* x_s = x_s' \) and \( i_* y_s = y_s' \). Then our hypothesis shows that \( y_s' = i_* y_s = f_* i_* y_s = f_* x_s' = f_* x_s = \sum_{s=1}^{q(l)} c_{sf} y_s \), i.e., \( c_{sf} = \delta_{sf}(k_s) \). Our task will be to modify \( F \) so that \( c_{sf} = \delta_{sf} \).

Put \( c_{sf} = \delta_{sf} + a_{sf} k_s \), and let

\[
M^1 = M^{1-1}_t \cup \Phi_s, (D^t \times D^{n-t}) \cup \cdots \cup \Phi_{s,l} (D^t \times D^{n-t}),
\]

\[
H_\delta(M^1, M^{1-1}_t) = \bigoplus_{s=1}^{t} \{ z_s \},
\]

where \( t = p(j) + q(j) + q(j-1) \) and \( z_s \) corresponds to \( (D^t \times D^{n-t}) \). Consider

\[
\begin{array}{cccc}
H_{i+1}(M^1, M^1) & \xrightarrow{i_*} & H_i M^1 & \\
\downarrow & & \downarrow & \\
H_\delta(M^1, M^{1-1}_t) & \xrightarrow{i_*} & H_\delta(M^1, M^{1-1}_t)
\end{array}
\]

We may assume (see \([6, \S 6]\)) that \( i_* x_s = z_s + i_* y_s = z_s^{l+1} + i_* y_s = k_s y_s \), and \( \delta' \delta^{l+1} + c_{sf} = k_s y_s \).
Let
\[ T^{s,t} = F[(D^1 \times 0)_{\theta s + p(l+1)}] \cap (0 \times D^{n-1})_{\theta t + p(l+1), t} \]
\[ T^{0,t}_s = (\partial D^{i+1} \times 0)_{\theta s + p(l+1) + q(l+1), i+1} \cap (0 \times \partial D^{n-1})_{\theta t + p(l+1), t} . \]

By a general position argument we can make \( T^{s,t}_s \) and \( T^{0,t}_s \) consist of only a finite number of points. In fact, using methods of Whitney (e.g., see [1, §7]), we may suppose that \(|T^{s,t}_s| = |c_{st}|, |T^{0,s}_s| = k_s, \) and \( T^{0,t}_0 = \emptyset \), for \( s \neq t \) (\(|X|\) denotes the cardinality of \( X \)).

We shall now describe isotopies \( h^s_\partial \), with \( h^0_\partial = \text{identity} = h^s_\partial |A \), so that if we let \( F' = h^s_\partial F \) and define \( C' = (c'_t) \) similarly to \( C \), then \( c'_{st1} = c_{st1} \), for \( s_1 \neq s, t_1 \neq t \), and \( c'_{st} = c_{st} \pm k_t \). Using such isotopies it should be clear how to deform \( F \) so that \( c_{st} = 0 \).

**The Construction of \( h^s_\partial \).** Let \( p \in T^{s,t}_s \) and \( q \in T^{0,t}_0 \). Let \( \alpha : D^n \to M^n \) be an imbedding with the following properties:

1. \((D^{i+1} \times 0)_{\theta s + p(l+1) + q(l+1), i+1} \subset \text{int } \alpha(D^{i+1})\), where
\[ D^{i+1}_0 = \{(x_1, \ldots, x_{i+1}) \in D^{i+1} | x_{i+1} \geq 0\}, \]

2. \( \alpha(D^{i+1}) \cap F[(D^1 \times 0)_{\theta s + p(l)}] = \alpha(D^1) \),

3. If \( U \equiv F(M_l) \cap \alpha(D^1) \), then \( U \) is a ball neighborhood of \( p \) with \( U \subset (D^1 \times D^{n-1})_{\theta s + p(l)}, \) and \( U \cap T^{s,t}_s = \{p\} \),

![Diagram](https://www.ams.org/journal-terms-of-use)
\( \alpha(D^n) \cap (D^j \times D^{n-j})_{|y_j+\pi H_t,i|=0} = \emptyset, \text{ for } j \neq t, \)
\( \alpha(D^{i+1}) \subseteq \text{int } M_i^x \).

\( h^* \) is then the isotopy which pushes \( \alpha(D^n) \) to \( \alpha(D^j) \), where \( D^j = D^j_0 \cap S^j \). \( h^- \) is defined in essentially the same manner, except that one must give \( \alpha(D^n) \) first a small twist in order to change the orientation.

Thus, assume now that \( c_{it} = \delta_{it} \). By previous observations we get that \( f_* \) is a monomorphism. A little diagram chasing shows that it is also onto, since it is already onto \( \sum_i \gamma_i \). Therefore \( f_* \) is an isomorphism and \( f \) is a homotopy equivalence. The rest of the proof goes as in part (a). This finishes the lemma.

We are now ready to prove our theorems, keeping the notation of the lemma.

**Proof of Theorem 1.** By [4] we can find a diffeomorphism \( H_{t_1} \) of \( M \) isotopic to the identity and \( H_{t_1}(M_{i_1}) = M_{i_1} \). Suppose inductively that we have defined a sequence of diffeomorphisms, \( \{H_{t_s} \mid s = 1, 2, \ldots, k \} \), of \( M \) satisfying

1. \( H_{t_1} \approx H_{t_{i_1} - 1} \),
2. \( H_{t_1} \mid M_{i_1} = H_{t_1} \mid M_{i_1} \),
3. \( H_{t_1}(M_{i_1}) = M_{i_1} \).

Let \( \gamma_3 = H_{t_1}(h_{i_1}^{-1}) \) and \( M_{i_1} = \gamma_3^{-1} \{0, i + 1/2 \} \). Then \( M_{i_1} = M_{i_1} \). Assume that \( 1 \leq i_k < i_{k+1} < n - 2 \). Applying part (a) of the lemma to \( \gamma_3 \) and \( \gamma_2 \) we obtain a diffeomorphism \( h \) of \( M \) with \( h \approx \text{identity}, h \mid M_{i_1} = \text{identity}, \) and \( h(M_{i_1}) = M_{i_1} \). Define \( H_{t_1}(M_{i_1}) = M_{i_1} \). It is easily checked that \( H_{t_1} \) satisfies (1)-(3).

If \( n - 2 \leq i_{k+1} \leq n \), let \( \eta_i = \eta_i - \eta_j, \) and \( M_{i_1} = \eta_i^{-1} \{0, i + 1/2 \} \). (Observe that \( \eta_i \) is also a minimal nice Morse function.) Using [4] we can find a diffeomorphism \( h_1 \) of \( M \) such that \( h_1 \approx \text{identity}, h_1 \mid M_{i_1} = \text{identity}, \) and \( h_1(M_{i_1}) = M_{i_1} \). Define \( H_{t_1}(M_{i_1}) = M_{i_1} \). If \( i_{k+1} = n - 1 \), define \( H_{t_1}(M_{i_1}) = H_{t_1} \). If \( i_{k+1} = n - 2 \), apply part (a) of the lemma to \( \gamma_3 \) and \( \gamma_2 \) with \( l = 1 \) to get a diffeomorphism \( h_2 \) of \( M \) with the property that \( h_2 \approx \text{identity}, h_2 \mid M_{i_1} = \text{identity}, \) and \( h_2(M_{i_1}) = M_{i_1} \). Furthermore, the proof of the lemma shows that we may suppose that \( h_2 \mid M_{i_1} = \text{identity} \). Define \( H_{t_1}(M_{i_1}) = H_{t_1} \). Finally, if \( i_{k+1} = n \), define \( H_{t_1}(M_{i_1}) = H_{t_1} \). In all cases, \( H_{t_1} \) satisfies (1)-(3). This finishes our inductive definition of the \( H_{t_s} \) and \( H \) satisfies the conclusion of the theorem.

**Proof of Theorem 2.** Let \( \eta_i = \eta_i - \eta_i \) and apply part (b) of the lemma in the proof of Theorem 1 instead of part (a) to obtain a diffeomorphism \( H: M \to M \) such that \( H \approx \text{identity} \) and \( H(M_{i_1}) = M_{i_1} \). Let \( F' = H^{-1} F \).

At first glance, one might be led to conjecture that the condition, \( F_\ast \mid \text{torsion } H \) = \text{identity}, in the lemma is unnecessary; however, the following example shows that it is not:

Let \( M^8 = \text{double of } (S^2 \times D^6 \cup_\alpha (D^3 \times D^5)), \) where \( \alpha: S^2 \times D^5 \to S^2 \times S^3 \) is an imbedding with \( [\alpha(S^2 \times 0)] = 5[S^2 \times 0] \in H_2(S^2 \times D^6) \). Let \( \eta: M \to R \) be a minimal nice Morse function such that \( M_2 = S^2 \times D^6 \) and \( M_3 = S^2 \times D^6 \cup_\alpha (D^3 \times D^5), \) where \( M_\alpha = M_\alpha(\eta) \). Construct an imbedding \( h: M_2 \to M_2 \) such that \( [h_\alpha(S^2 \times 0)] = 3[S^2 \times 0] \in H_2(M_2). \) One can extend \( h \) to an imbedding \( M_2 \cup_\alpha (D^3 \times 0) \to M_3. \) But \( \pi_2 SO_5 = 0, \) and so we can thicken the handle to get an imbedding \( h: M_2 \to M_3. \)
Now $H_2M_3 \approx H_2M \approx \langle a \rangle$, where $a$ has order 5, and $h_*(a) = 3a$ by construction. Therefore $h$ is a homotopy equivalence and we may assume $h$ is a diffeomorphism of $M_3$. Let $F: M \to M$ be the double of $h$. Then $F_* | H_2M \neq \text{identity}$. However, if $F \approx F'$ and $F'(M_2) = M_2$, then we must have $F_* | H_2M = F'_* | H_2M = \pm \text{identity}$ because $H_2M_2 \approx \mathbb{Z}$. Thus $(M, F)$ is the example that we wanted. In fact, this also shows that Theorem 1 is the best possible (set $\eta_1 = \eta$ and $\eta_2 = \eta F$).

Finally, although we just saw that Theorem 1 is the best possible, this still leaves open the interesting question whether, despite this, the manifolds $M_1(\eta_1)$ and $M_1(\eta_2)$ are diffeomorphic anyway.

**REFERENCES**


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