IDEALS IN CHEVALLEY ALGEBRAS

BY

JAMES F. HURLEY(*)

1. Introduction. In Chevalley's fundamental paper [2], a procedure is given for obtaining a very special canonical basis for a finite dimensional split semisimple Lie algebra $L$ over a field of characteristic zero. With respect to this basis, all the constants of structure of $L$ turn out to be integers, thus enabling the original field to be replaced by an arbitrary field. In this paper we replace the original field by an arbitrary commutative ring $R$ with identity, and we call the resulting structure the Chevalley algebra $L_R$ of $L$ over $R$. Our attention focuses on the structure of $L_R$ and we obtain theorems characterizing its ideals, under the assumption that 2 and 3 are not zero divisors in $R$.

The main result is that for simple algebras $L$ a necessary and sufficient condition for every ideal in $L_R$ to arise from an ideal of $R$ is that the Cartan matrix of $L$ be invertible over $R$ and that the integer $m$ be invertible in $R$, where $m$ is defined as the ratio of the square of the length of any long root of $L$ to the square of the length of any short root of $L$. Along the way toward the proof we obtain results characterizing the ideals of $L_R$ when that condition does not necessarily hold in $R$. A special case of note occurs when $R$ is a field of finite characteristic not equal to 2 or 3. Then our main theorem contains certain results of Dieudonné [3] for exceptional algebras, as well as implicit results alluded to in that paper for the classical algebras.

In §2 we give definitions of the principal concepts and list for future reference certain computational properties. (For a detailed discussion of the structure theory of semisimple Lie algebras, the reader is referred to [4], [6], or [7]. An excellent summary of the basic results used here can be found in [1].) §3 contains statements of the principal theorems. In §4 and §5 the general results needed for the sufficiency part of the main theorem are proved, including the decomposition lemmas which are also used in proving necessity. §§6, 7, and 8 contain discussions of the general structure of $L_R$ for algebras with one root length, nonsymplectic algebras with two root lengths, and symplectic algebras respectively, as well as the proof of the necessity part of the main theorem for those respective cases. In §9 we give a general theorem on generators of ideals in $L_R$ together with some consequences for principal ideal rings $R$. We note in §8 that our main theorem specialized to the algebra of

Presented to the Society, January 25, 1967 under the title Scalar replacement in Lie algebras; received by the editors April 15, 1967 and, in revised form, February 8, 1968.

(*) This research was supported by a National Science Foundation Fellowship. The results published here are contained in the author's doctoral dissertation submitted to the University of California, Los Angeles, in 1966. The author wishes to express his gratitude to Professor Robert Steinberg for his guidance throughout the course of this research.

245
type $C_n$ yields a recent result of W. Jehne [5] which he obtained using the concrete model of $C_n$ as a Lie algebra of matrices. The abstract approach we employ throughout is that of [2] and [9].

2. The Chevalley algebras. Let $L$ be a finite dimensional split semisimple Lie algebra over a field of characteristic zero. Let $H$ be an $n$-dimensional Cartan subalgebra, $\Sigma$ the (ordered) set of nonzero roots determined by $H$, and $\Pi$ the set of simple roots. If $r$ and $s$ are nonzero roots, then we define $c(r, s)$ to be $2(r, s)/(s, s)$, where the inner product is that derived from the Killing form of $L$ in the usual way. We define $p_{rs}$ to be 0 if $r+s$ is not a root and to be the greatest integer $i$ such that $s - ir$ is a root in case $r+s$ is a root. We define $N_{rs} = p_{rs} = 0$ if $r+s$ is not a root, and $N_{rs} = p_{rs} + 1$ if $r+s$ is a root. By the length of a root $r$, we simply mean $(r, r)^{1/2}$. This is a departure from the terminology of Chevalley [2, p. 17]. It is known (see [7, p. V-9]) that if two roots $r$ and $s$ of a simple algebra $L$ have unequal lengths, then the lengths are in the ratio $\sqrt{3}$ to 1 or $\sqrt{2}$ to 1.

Chevalley [2, Theorem 1] constructs a basis for $L$ consisting of certain elements $e_r$ from each one-dimensional root space $L_r$ as $r$ varies through $\Sigma$, together with $n$ elements $h_1, h_2, \ldots, h_n$ of $H$ suitably obtained from the corresponding simple roots $r_1, \ldots, r_n$ in $\Pi$. This basis has the property that if $h_r$ is obtained from $r$ in $\Sigma$ in the same way that $h_i$ is obtained from $r_i$ in $\Pi$, then $h_r$ is an integral linear combination of the $h_i$’s. With respect to this basis, the equations of structure of $L$ are:

1. $e_r e_{-r} = h_r$,
2. $h_i^2 = 0$,
3. $e_r e_s = \pm N_{rs} e_{r+s}$ if $r+s \in \Sigma$ or if $r+s$ is not a root,
4. $h_r e_s = c(s, r) e_s$.

These equations reveal the premier property of a Chevalley basis, namely, the product of any two of its elements is an integral linear combination of basis elements. Now let us define $L_R$ to be the $R$-module generated by the elements of a fixed Chevalley basis of $L$, if $R$ is any commutative ring with identity. Equations (1) through (4) show that $L_R$ is closed under multiplication, as we need only interpret the integers of structure as integers in $R$. We call $L_R$ with this multiplication the Chevalley algebra of $L$ over $R$, and it is our purpose here to study its structure. We mention that in [2] the specific Chevalley algebra $L_{Z}$ over the ring of integers arises for the first time. Before proceeding with our study of $L_R$, we give the following result which enables us to reduce our considerations to Chevalley algebras arising from simple algebras.

2.1. If $L$ decomposes into the direct sum of simple ideals $I_1, I_2, \ldots, I_k$, then $L_R$ decomposes into the direct sum of the Chevalley algebras generated over $R$ by $I_1, I_2, \ldots, I_k$.

Proof. Clearly $L_R$ contains this direct sum. Furthermore, $\Sigma$ can be partitioned into $k$ mutually disjoint orthogonal systems comprising the nonzero roots of the
simple algebras \( I_j \) [4, Theorem 4.4]. Thus the Chevalley basis breaks into \( k \) corresponding disjoint pieces. Hence forming the direct sum of the Chevalley algebras generated by these pieces yields all of \( L_R \).

It is apparent that the Chevalley algebras generated by the \( I_j \) are still ideals in \( L_R \). But they need no longer be simple, as we see by considering the ideal generated in \( L_R \) by \( 2e_1 \) if \( L \) is simple. (We denote by \( e_i \) the Chevalley basis element chosen from the root space of the simple root \( r_i \).)

In view of 2.1 we shall henceforth consider only simple algebras \( L \). In this case it is known that \( L \) has at most two root lengths [7, p. V–9], so in the sequel we shall feel free to refer to “short roots” or “long roots” of \( L \).

The example just considered indicates that the invertibility of certain integers will be of great importance in studying the structure of \( L_R \), and indeed the integers we shall need to study most closely are \( N_{rs} \) and \( c(r, s) \). We take the opportunity now to list a series of elementary computational properties of \( N_{rs} \) in the various simple algebras, properties we shall need to make reference to later. Concrete realizations of the root systems of the simple Lie algebras may be found in [4, pp. 135–145] or in [7, pp. V–27 to V–30]. These realizations make possible the identifications given below between certain subsets of root systems of one type and entire systems of another type. In (2.2)–(2.10) we suppose that \( r \) and \( s \) are roots such that \( r \in S \).

2.2. If \( L \) is of one of the types \( A_n \), \( D_n \), \( E_6 \), \( E_7 \), or \( E_8 \), then since \( r \) and \( s \) are not orthogonal, they must be part of a system of type \( A_2 \). Since \( r + s \) is a root, \( s - r \) is not and so \( N_{rs} = 1 \).

2.3. If \( r \) and \( s \) are long and \( L \) is of type \( B_n \), \( C_n \), \( F_4 \), \( G_2 \), or \( G_8 \), then since the long roots constitute a system of type \( D_n \), \( C_n \), \( F_4 \), \( G_2 \) respectively, we have \( N_{rs} = 1 \).

2.4. If \( L \) is of type \( G_2 \) and \( r \) and \( s \) are long, then since the long roots form a system of type \( A_2 \), \( N_{rs} = 1 \).

2.5. If \( r, s, \) and \( r + s \) are all short and \( L \) is of type \( C_n \), \( n \geq 3 \), then as in 2.3, \( N_{rs} = 1 \).

2.6. If \( r \) and \( s \) are short, but \( r + s \) is long and \( L \) is of type \( C_n \), then imbedding \( r \) and \( s \) in a system of type \( B_2 \), we see that \( s - r \) is also a root, \( s - 2r \) is not, and so \( N_{rs} = 2 \).

2.7. If \( r, s, \) and \( r + s \) are all short and \( L \) is of type \( F_4 \), then they are seen to generate a system of type \( A_2 \), so that \( N_{rs} = 1 \).

2.8. If \( r \) and \( s \) are short and \( L \) is of type \( B_n \), \( n \geq 2 \), then imbedding \( r \) and \( s \) in a system of type \( B_2 \), we see that \( r + s \) must be long, \( s - r \) is a root, but \( s - 2r \) is not. Thus \( N_{rs} = 2 \). The same reasoning applies if \( r \) and \( s \) are short but \( r + s \) is long for \( L \) of type \( F_4 \).

2.9. If \( r \) and \( s \) are of unequal length, then consideration of types \( B_n \), \( C_n \), \( F_4 \), and \( G_2 \) separately shows that \( r + s \) must be short, \( s - r \) is not a root, and so \( N_{rs} = 1 \).

2.10. If \( r \) and \( s \) are short and \( L \) is of type \( G_2 \), then \( r + s \) also short implies that \( s - r \) is a root, \( s - 2r \) is not, and \( N_{rs} = 2 \). But \( r + s \) long implies that both \( s - r \) and \( s - 2r \) are roots, and hence \( N_{rs} = 3 \).

2.11. If \( S \) is a set of roots which is itself a system of roots of some simple algebra,
then \( \{ e_r | r \in S \} \) generates a simple subalgebra of \( L \) which will have a Chevalley basis consisting of \( \{ e_r | r \in S \} \) together with suitably chosen elements from \( H \).

3. Principal results. Throughout the sequel we assume that neither 2 nor 3 is a zero divisor in \( R \), where we consider 0 to be a zero divisor. We shall denote by \( C \) the Cartan matrix of \( L \) as well as the transformation on \( H \) represented by this matrix, relative to the basis \( \{ h_1, h_2, \ldots, h_n \} \) of \( H \) given in \( \S 2 \). We shall use \( s \) and \( t \) as generic symbols for long and short roots of \( L \) respectively. We define the integer \( m = (t, t)/(s, s) \). We denote by \( E_R \) the \( R \)-module generated by the Chevalley basis elements \( e_r, r \in \Sigma \), and by \( H_R \) the \( R \)-module generated by \( \{ h_1, \ldots, h_n \} \). The symbol \( E_L \) (respectively \( H_L \)) will denote the \( R \)-module generated by \( \{ e_t | t \text{ a long root in } \Sigma \} \) (respectively \( \{ e_t | t \text{ a long root in } \Sigma \} \)), and \( E_s \) (respectively \( H_s \)) will denote the corresponding \( R \)-modules with short roots \( s \) replacing \( t \).

3.1. For every ideal \( I \) of \( L_R \) such that \( I \not\subseteq H_R \), there is a nonzero ideal \( J \) in \( R \) for which \( I \not\subseteq L_J \).

3.2. If \( L \) is not of type \( A_1 \) or \( C_n, n \geq 2 \), and \( I \) is an ideal in \( L_R \), then \( I = (I \cap E_R) \oplus (I \cap H_R) \).

3.3. Main theorem. A necessary and sufficient condition for every ideal in \( L_R \) to have the form \( L_J \) for some ideal \( J \) in \( R \) is that \( m \) and \( \det C \) be invertible in \( R \).

3.4. Suppose that \( L \) is of one of the types \( A_n, n \geq 2 \), \( D_n, n \geq 3 \), \( E_6, E_7, \) or \( E_8 \). Let \( I \) be an ideal in \( L_R \). Then \( I \cap E_R = E_J \) where \( J \) is the ideal in \( R \) generated by the scalar coefficients of the elements in \( I \cap E_R \). Also \( H_J \subseteq I \cap H_R \subseteq C^{-1}(H_J) \). Conversely, if \( \tilde{H} \) is any \( R \)-module such that \( H_J \subseteq \tilde{H} \subseteq C^{-1}(H_J) \) for some ideal \( J \) in \( R \), then \( \tilde{I} = \tilde{H} \oplus E_J \) is an ideal in \( L_R \).

3.5. Suppose that \( L \) is of one of the types \( B_n, n \geq 3 \), \( F_4, \) or \( G_2 \). Let \( I \) be an ideal in \( L_R \). Let \( J \) be the ideal in \( R \) generated by the scalar coefficients of elements in \( I \cap E_R \). Then \( I \cap E_R = J_E \oplus J_E \) where \( J' \) is an ideal of \( R \) such that \( J \subseteq J' \subseteq m^{-1}J \); also, \( J'H_J + JH_L \subseteq I \cap H_R \subseteq C^{-1}(J'H_J + JH_L) \). Conversely, given any \( E = J_E_L + J_E_S \) with \( J \) and \( J' \) ideals of \( R \) such that \( J \subseteq J' \subseteq m^{-1}J \), then \( E = J_E_L \oplus J'E_S \) is an ideal in \( L_R \); also, given any \( R \)-module \( \tilde{H} \) such that \( J'H_J + JH_L \subseteq \tilde{H} \subseteq C^{-1}(J'H_J + JH_L) \), then \( \tilde{I} = \tilde{H} \oplus J_E \oplus J'_E_S \) is an ideal in \( L_R \).

3.6. Suppose \( L \) is of type \( C_n, n \geq 2 \). Let \( I \) be an ideal in \( L_R \). Then \( I \cap E_S = J_E \) where \( J \) is the ideal in \( R \) generated by the scalar coefficients of elements in \( I \cap E_S \); also,

\[
2JE_L + 2JH_L + JH_S \subseteq I \cap (E_L \oplus H_R) \subseteq JE_L \oplus C^{-1}(JH_R).
\]

Conversely, given any \( R \)-module \( N \) of the form \( N = J'E_L \oplus \tilde{H} \) where \( J' \) is an ideal of \( R \) such that \( 2J \subseteq J' \subseteq J \) and where \( J'H_L + JH_S \subseteq \tilde{H} \subseteq C^{-1}(J'H_L + JH_S) \), then \( N \oplus J_E \) is an ideal in \( L_R \).

3.7. Let \( L \) be nonsymplectic of rank at least 2 and let \( I \) be an ideal in \( L_R \). Let \( J \) be the ideal in \( R \) defined in 3.4 and 3.5, and suppose \( I \cap E_S = E_J \). Let \( g \) be the minimal
number of generators of the R-module \((I \cap H_R)/H\). Suppose every ideal in R is generated by no more than \(yz\) elements where \(y = \min \{g_i | I \cap E_n = E_j\}\) and \(z\) is the number of roots of \(L\) (respectively, long roots of \(L\)) if \(L\) has a single (respectively two) root length(s). Then \(g_i\) is the minimum number of generators of the ideal \(I\).

4. **Lower bounds for ideals.** In writing elements of \(L_R\) in terms of the Chevalley basis, we shall use the following lexicographical ordering. If \(r = \sum_{i=1}^{n} k_i r_i\), then \(ht r = \sum_{i=1}^{n} k_i\). We shall write \(e_r\) before \(e_u\) if \(ht r < ht u\) or \(ht r = ht u\) and the first simple root whose coefficient is different for \(r\) and \(u\) has smaller integral coefficient in the expression for \(r\). The \(h_i\)'s will occur before the \(e_i\)'s corresponding to positive roots, and after those corresponding to negative roots, and will be arranged in increasing order of subscripts. The root \(d\) attached to the largest \(e_r\) in this ordering will be called the highest root.

4.1. If \(r \neq u\) are roots of \(L\), then there is a sequence of roots \(t_0 = r, t_1, t_2, \ldots, t_k = u\) such that \(t_{i+1} - t_i\) is a root for \(0 \leq i \leq k - 1\). (Note that some \(t_i\) may be zero. See 5.1 below where the assumption that rank \(L \geq 2\) is needed to avoid this possibility.)

**Proof.** Since \(L\) is simple, \(e_r\) will generate all of \(L\) as an ideal, and so multiplications by a suitable sequence of root elements and field elements will bring us to \(e_u\). The subscripts of the intermediate root elements furnish us with the required sequence.

4.2. **Proof of 3.1.** Let \(I\) be an ideal in \(L_R\) with \(I \neq H_R\). Then \(I\) contains an element \(x\) whose expression in terms of basis elements involves at least one \(e_r\) with a nonzero coefficient \(c_r\). Now calling upon the sequence of roots in 4.1 which begins at \(r\) and ends at \(d\), we multiply \(x\) successively by root elements attached to \(t_{i+1} - t_i\) until we obtain the element \(x' \in I\) with final component \(c_r e_d\). Here in view of 2.2 through 2.10 and our assumptions on \(R, c_r \neq 0\). Next we form a sequence of roots beginning at \(d\) and ending at \(-d\) and proceed in the same way to obtain \(x'' = c_r e_{-d} \in I\). Then given any root \(u\) we can obtain \(n_u c_r e_u \in I\) and given any \(i\) we can also obtain \(n_i c_r h_i \in I\), where the \(n_i\)'s are products of powers of 2 or 3. If \(n\) is the least common multiple of all the \(n_u\) and \(n_i\), then \(n c_r \neq 0\), and letting \(J\) be the principal ideal generated by \(n c_r\), we have \(I \supseteq J L_R = L_J\).

Theorem 3.1 shows that most ideals in \(L_R\) are large in that they contain a complete Chevalley algebra of \(L\) over a smaller ring. The reason for requiring \(I \neq H_R\) is shown by the following example.

Consider the algebra \(L\) of type \(A_1\) over the ring \(Z_5\) of integers modulo 5. Then the Cartan matrix of \(L\) has rank 3 since it diagonalizes to diag (1, 1, 1, 0). So the transformation \(C\) on \(H_R\) defined by this matrix has a 1-dimensional kernel. We claim that \(I = \text{Ker} C\) is an ideal in \(L_R\). Clearly \(H_R I = 0\). If \(h = \sum_{i=1}^{n} n_i h_i \in I\) and \(r = \sum_{j=1}^{n} k_j r_j\), then

\[
he_r = \sum_{j=1}^{n} k_j \left[ \sum_{i=1}^{n} n_i c(r_j, r_i) \right] e_r = 0
\]
5. **Decomposition lemmas and the main theorem.** The main result we use in studying ideals in nonsymplectic $L_R$ is 3.2.

5.1. **Proof of 3.2.** We may suppose that $I \neq 0$. It is clear that $I \supseteq (I \cap E_R) \oplus (I \cap H_R)$. To establish the converse inclusion we distinguish the cases in which $L$ has one root length and in which $L$ has two root lengths. In the first case let $x = \sum_{r \neq 0} c_r e_r + \sum_i n_i h_i$ be a nonzero element of $I$. We need to show that each partial sum is also in $I$. First suppose $c_r e_r$ occurs in this expression for $x$. Then we proceed as in 4.2. Owing to the fact that rank $L \geq 2$, we see from 2.2 that we can so proceed as to obtain $x' \in I$ with last component $c_r e_d$. In the same manner we can finally obtain $c_r e_u$ in $I$. Hence $\sum_{r \neq 0} c_r e_r$ belongs to $I$, as does $x - \sum_{r \neq 0} c_r e_r = \sum_{i=1}^n n_i h_i$.

The second case is more complicated in that the sequences of roots of 4.2 must be more carefully chosen. These choices are dictated by the following two lemmas.

5.2. Suppose $L$ is of type $B_n$, $n \geq 3$, or $F_4$. If $k e_r \in I$, then $k a_r \in I$ for every root $r$. If $k e_s \in I$, then $2 k e_t \in I$ for every long root $t$, and $k$ times every short root element also belongs to $I$.

**Proof.** Let $k e_r \in I$. Consider $t$ to be embedded in a system of type $D_n$, $n \geq 3$, or $D_4$, as the case may be. Then given a long root $t'$, we proceed to multiply $k e_r$ by long root elements (considered likewise to be in a system of type $D_n$ or $D_4$), which we choose judiciously as in 5.1. By 2.2, we can thus obtain $k e_r \in I$. Now also, if $s'$ is any short root, then from 2.9, $k e_r s' e_r = \pm k e_{t+u} \in I$, where $t+s'$ is short. So at least $I$ contains $k$ times one short root element whenever $k e_r \in I$. Presently we show that then $I$ contains $k$ times every short root element. Before turning to that however, let us suppose that $k e_s \in I$. Choose a short root $u$ such that $s+u$ is a root. One sees that this is always possible from a study of the previously cited lists of roots in [4] or [7]. Then $s+u$ is long (by 2.8), and $k e_s s' e_s = \pm 2 k e_{s+u} \in I$. But by the above, then $I$ contains $2k$ times every long root element. All that remains now is to prove that if $k e_s \in I$, then $k$ times every short root element is in $I$. So let $r$ be any short root element. Suppose first that $(r, s) \geq 0$. Then $r-s = u$ is a root, and $r = s + u$. Since $r$ and $s$ are short, it is easily seen that $s - u = 2s - r$ cannot be a root. Hence $k e_s e_u = \pm k e_r \in I$. Now if $r = -s$, we need only pick a short root $r'$ orthogonal to $s$, and we can conclude from the preceding that $k e_r \in I$ since $(r', s) = 0$. Since $(r, r') = 0$, $k e_r \in I$ also. Finally, if $(r, s) < 0$ and $r$ and $s$ are linearly independent, then we can obtain $k e_{r-s} \in I$ as we just showed, and then $(r, -s) > 0$ allows us to conclude that $k e_r \in I$, completing the proof.

The following simple example may clarify the implementation in practice of the procedures in 5.2. Using the notation of [4], the roots of $B_n$ consist of $\pm \omega_i$, $\pm \omega_i + \omega_j$, $\pm \omega_j$, $i \neq j$, where $\{\omega_1, \ldots, \omega_n\}$ is the canonical basis for $\mathbb{R}^n$. If the short root element

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
ke_{\alpha i} \in I, then ke_{\alpha i} \cdot e_{\alpha j} = \pm 2ke_{\alpha i + \alpha j} \in I since \omega_i - \omega_j is also a root. Once we have 2ke_{\alpha i + \alpha j} \in I, we can obtain 2ke_{\alpha k + \alpha m} \in I as well, for any long root \omega_k + \omega_m. For,

\begin{align*}
2ke_{\omega_k + \omega_j} \cdot e_{\omega k - \omega_j} &= \pm 2ke_{\omega k + \omega_i} \in I
\end{align*}

since \omega_1 + \omega_j - (\omega_k - \omega_i) is not a root. Then

\begin{align*}
2ke_{\omega k + \omega m} \cdot e_{\omega m - \omega_i} &= \pm 2ke_{\omega k + \omega m} \in I,
\end{align*}

since \omega_k + \omega_i - (\omega_m - \omega_i) is not a root.

5.3. Suppose \(L\) is of type \(G_2\). If \(ke_i \in I, then ke_i \in I for every root \(r\). If \(ke_i \in I, then 3ke_i \in I for every long root \(t\), and \(k\) times every short root element belongs to \(I\).

Proof. If \(ke_i \in I, then we consider \(t\) to be imbedded in a system of type \(A_2\) (see 2.4). Then the reasoning of 5.2 shows that \(k\) times all long root elements are in \(I\), as well as \(k\) times at least one short root element, using again 2.9. Also, if \(ke_s \in I, then 2.10\) and 2.4 show that \(3ke_s \in I\) for all long roots \(r\). So as in 5.2, we are reduced to showing that \(ke_s \in I\) implies \(ke_s \in I\) for all short roots \(r\). We know that \(r\) and \(s\) are not orthogonal. If \((r, s) > 0, then r - s\) is a root \(u\), and \(r = s + u\). If \(s - u\) is not a root, then \(ke_t \cdot e_u = \pm ke_t \in I\). If \(s - u\) is a root, then \(s - 2u = 3s - 2r\) is not, so \(ke_s \cdot e_u = \pm 2ke_t \in I\). But we can also obtain \(3ke_t\) and then \(3ke_s \in I\) by the preceding, thus enabling us to obtain \(ke_s \in I\). If \((r, s) < 0, then r + s\) is a root. We use \(e_t\) and \(e_s\) as multipliers and unwanted coefficients of 2 once more are eliminated using a suitable \(3ke_i \in I\), giving us \(ke_i \in I\).

Now, continuing with 5.1, let \(x = \sum c_t e_t + \sum c_t e_t + \sum_{n=1}^{\infty} n_i h_i\) be a nonzero element in \(I\). If \(r\) is any long root whose coefficient \(c_r\) is nonzero, then the above proofs show us how to obtain \(c_r e_r \in I\), confining our operations to the set of long root elements. If \(u\) is a short root of \(L\) and \(c_u \neq 0\) in the expression for \(x\), then the preceding proofs again show that we can find an element \(x'\) in \(I\) with last component \(c_u e_v\) where \(v\) is the largest short root in our lexicographical ordering. Similarly, we can obtain \(x'' \in I\) with last component \(c_u e_v\). Any earlier components will be of the form \(c_u e_t\) and so may be subtracted off leaving \(c_u e_v \in I\). Then it is a simple matter to finally get \(c_u e_v \in I\). Hence \(\sum c_t e_t + \sum c_t e_t\) belongs to \(I\), as therefore does \(\sum_{n=1}^{\infty} n_i h_i\), completing the proof. We note that in the second case our proof actually shows that \(I = (I \cap E_3) \oplus (I \cap E_3) \oplus (I \cap H_R)\), a fact we shall find useful in §7.

We have need of a lemma similar to 3.2 for symplectic algebras.

5.4. Suppose \(L\) is of type \(C_n, n \geq 2\). If \(I\) is an ideal in \(L_R\), then

\begin{align*}
I = (I \cap E_3) \oplus (I \cap [E_L \oplus H_R]).
\end{align*}

Proof. Let \(x = \sum c_t e_t + \sum c_t e_t + \sum_{n=1}^{\infty} n_i h_i\) be a nonzero element of \(I\). Suppose \(u\) is short and \(c_u \neq 0\). As in 4.2 we can, in view of 2.9, obtain an element \(x' \in I\) with component \(c_u e_v\), where \(v\) is as in 5.1, using multipliers that are long root elements. Later components will all have been eliminated in the process since the sum of two
linearly independent long roots is never a root. Similarly, we can obtain \( c_u e_{-v} \) in \( I \)
and hence \( c_u e_v \in I \). Then \( \sum c_v e_v \in I \) and the lemma follows immediately.

We are now in a position to prove the sufficiency part of the main theorem.

5.5. **If \( m \) and \( \det C \) are invertible in \( R \), then every ideal in \( L_R \) is of the form \( L_J \) for some ideal \( J \) in \( R \).**

**Proof.** We may suppose that \( I \neq 0 \). Let \( J \) be the ideal generated in \( R \) by all coefficients of elements in \( I \cap E_R \). Since \( C \) is invertible, the reasoning of §4 shows that \( I \) cannot be confined to \( H_R \). For if \( h = \sum_{i=1}^{n} n_i h_i \in I \), then

\[
  h e_j = \sum_{i=1}^{n} n_i c(r_j, r_i) e_j.
\]

If \( h \neq 0 \), then \( C(h) \neq 0 \) and so for some \( j \), \( \sum_{i=1}^{n} n_i c(r_j, r_i) \neq 0 \), whence \( h e_j \neq 0 \). So we may suppose \( x = \sum c_v e_v \in I \), with the \( r \)'s short in case \( C_n \), arbitrary otherwise. Now in attempting to isolate a given \( c_u e_u \in I \), the multiplications used in 4.2 will introduce at worst some powers of 2 or 3. But the invertibility of \( m \) will permit us to obtain \( c_u e_u \in I \) in every case except \( A_1 \), where the invertibility of \( \det C = 2 \) can be used. [Note that judicious procedure in case \( G_2 \) will avoid introducing any powers of 2, in view of Lemma 5.3.] Then \( c_u e_u \in I \) quickly gives \( c_u e_r \in I \) for all roots \( r \). Thus \( I \cap E_R = J E_R \). Hence \( I \cap H_R \supseteq J H_R \). Furthermore, if \( h = \sum_{i=1}^{n} n_i h_i \in I \cap H_R \), then \( h e_j = \sum_{i=1}^{n} n_i c(r_j, r_i) e_j \in I \cap E_R \) showing that \( C(h) \in J H_R \). Otherwise put, \( I \cap H_R \subseteq C^{-1}(J H_R) = J H_R \). Thus \( I = J L_R = L_J \).

5.6. **Corollary.** If \( L \) is of type \( E_n \), then every ideal in \( L_R \) is of the form \( L_J \) for some ideal \( J \) in \( R \).

5.7. **Corollary.** If \( m \) and \( \det C \) are invertible in \( R \), then the maximal ideals in \( L_R \) consist precisely of the ideals \( L_M \), where \( M \) is maximal in \( R \).

In studying \( L_R \) more closely, we shall have occasion to use bases for \( H_R \) other than the given Chevalley basis. Over the original ground field of \( L \) we can find a basis \( \{h_i\} \) of \( H \) which is dual to the system \( \Pi \) of simple roots. We denote by \( H'_R \) the \( R \)-module generated by \( \{h_i\} \). Over the original field we can express the elements of \( \{h_i\} \) in terms of the elements of \( \{h_i\} \), \( h_i = \sum_{j=1}^{n} n_j h_j \). Evaluation of each side at the simple root \( r_k \) shows that \( n_{rk} = c(r_k, r_i) \), and so \( h_i = \sum_{j=1}^{n} c(r_j, r_i) h_j \) is an expression valid over \( R \) also, and it shows that \( H_R \subseteq H'_R \).

We can diagonalize \( C \) using elementary matrices and so can view the diagonalization as taking place over \( R \). This process gives us new bases \( \{\tilde{h}_i\} \) and \( \{\tilde{h}_i\} \) of \( H_R \) and \( H'_R \) respectively, with \( \tilde{h}_i = d_i h_i \), where the \( d_i \) are the elementary divisors of \( C \). For reference we list here the values of the \( d_i \) for each of the simple algebras.

\[
A_n: 1, 1, \ldots, 1, n+1; \quad B_n: A_n; \quad C_n: 1, 1, \ldots, 1, 2; \quad D_n, n \text{ even}: 1, 1, \ldots, 1, 2; \quad D_n, n \text{ odd}: 1, 1, \ldots, 1, 4; \quad E_6: 1, 1, \ldots, 1, 3; \quad E_7: 1, 1, \ldots, 1, 2; \quad E_8: 1, 1, \ldots, 1; \quad F_4: 1, 1, 1, 1; \quad G_2: 1, 1.
\]

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
6. Algebras of one root length. In this section we complete our main theorem and also the characterization of ideals in $L_R$ when $L$ has a single root length.

6.1. Proof of 3.4. The first two assertions follow directly from the proof of 5.5. Conversely, suppose $\mathcal{H}$ is any $R$-module such that $JH_R \subseteq \mathcal{H} \subseteq C^{-1}(JH_R)$ where $J$ is an ideal in $R$. Then relations (1), (3), and (4) of §2 show at once that $L_R \otimes E_f \subseteq \mathcal{H} \oplus E_f$. It is equally clear from (2) that $H_R \cdot \mathcal{H} = 0$, so we are reduced to considering $E_R \otimes \mathcal{H}$. If $h \in \mathcal{H}$ and $r$ are as in the discussion following 4.2, then $hr, e \in \mathcal{H}$, since $\sum_{i=1}^n n_i c(r_i, r_i) e^j$ follows from $\mathcal{H} \subseteq C^{-1}(H_R)$. Hence $\mathcal{H} \oplus JE_R$ is an ideal in $L_R$.

6.2. If $I$ is an ideal in $L_R$ with $I \cap E_R = E_f$ for some ideal $J$ in $R$, then the possibilities for $I \cap H_R$ in 3.4 are as follows. If $L$ is of type $A_n$, $n \geq 2$, $D_n$, $n$ odd $\geq 3$, $E_6$, or $E_7$, and $k$ is respectively $n+1$, 4, 3, or 2, then

$$I \cap H_R = Jh_1 \oplus \cdots \oplus Jh_{n-1} \oplus J'h_n$$

where $J'$ is an ideal in $R$ such that $J \subseteq J' \subseteq e^{-1}J$. If $L$ is of type $D_n$, $n$ even $> 3$, then

$$I \cap H_R = Jh_1 \oplus \cdots \oplus Jh_{n-2} \oplus J'h_{n-1} \oplus J''h_n$$

where $J'$ and $J''$ are ideals of $R$ lying between $J$ and $eJ$.

Proof. Let $h \in I \cap H_R, h = \sum n_i h_i = \sum c_i h_i = \sum \bar{c}_i h_i$. Since $he_i = c_i e_i$, we see that every $c_i \in J$. Thus all the $\bar{c}_i \in J$ also. We have further $(\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_n) = D(\bar{n}_1, \bar{n}_2, \ldots, \bar{n}_n)$ where $D$ is the diagonal matrix of elementary divisors of $C$. From the lists of elementary divisors, we read off the asserted possibilities for the $\bar{n}_i$.

6.3. Corollary. If $L$ is of type $A_n$, $n \geq 2$, $D_n$, $n$ odd $\geq 3$, $E_6$, or $E_7$, then the maximal ideals of $L_R$ have the form

$$ME_R \oplus Mh_1 \oplus \cdots \oplus Mh_{n-1} \oplus Rh_n$$

where $M$ is maximal in $R$. If $L$ is of type $D_n$, $n$ even $> 3$, then the maximal ideals of $L_R$ have the form

$$ME_R \oplus Mh_1 \oplus \cdots \oplus Mh_{n-2} \oplus Rh_{n-1} \oplus Rh_n$$

6.4. Corollary. (Necessity part of 3.3 for algebras of rank at least two and one root length.) If $L$ has one root length and rank at least 2, then every ideal in $L_R$ is of the form $L_J$ for some ideal $J$ in $R$ only if det $C$ is invertible in $R$.

Proof. In case det $C$ is not invertible, then the maximal ideals of Corollary 6.3 provide us with examples of ideals in $L_R$ which do not have the form $L_J$ for any ideal $J$ in $R$.

6.5. The preceding corollary holds also for $L$ of type $A_1$.

Proof. Here det $C = 2$. The ideal generated by $e$ is maximal, but is not $L_J$ for any $J$ if 2 is not invertible, since it consists precisely of the $R$-module generated by $e$, $h$, and $2f$. 

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
7. Nonsymplectic algebras with two root lengths. Our attention is now turned to
the simple algebras of type $B_n$, $n \geq 3$, $F_4$, and $G_2$.

7.1. Proof of 3.5. The reasoning of §5 shows that $I \cap E_L = J E_L$ and thus $I \cap E_R \supseteq J E_R$. If the inclusion is strict, then there is some short root element $e_s$ occurring
in an expression for an element of $I$ whose coefficient $c_s$ is not in $J$. From §5 again
we see that $c = mc_s \in J$, so that $c_s \in m^{-1}J$. Then also $c_s e_u \in I$ for each short root $u$ and
$m c e_u \in I$ for each long root $t$. Let $J'$ denote the collection of all coefficients $c_s$
such that $c_s e_s$ is a component of an element of $I$ and $s$ is short. Then $J'$ is an ideal
in $R$, and $J' \subseteq m^{-1}J$, and thus $I \cap E_R = J E_L \oplus J' E_S$. Again $I \cap H_R \supseteq J H_S + J H_L$ is immediate and $h e_j \in I \cap H_R$ for every $h \in I \cap H_R$ shows that $C(h) \in J H_S + J H_L$, completing the first half of 3.5.

Now let $E = J E_R \oplus J' E_S$ be of the hypothesized sort. Since $c(t, s) = m$ if $t$ is long
and $s$ is short, we have $L_R \cdot J' H_S \subseteq J' E_S + J E_L$. Also

$$L_R E = (E_S \oplus E_L \oplus H_S) \cdot (J E_L \oplus J' E_S) \subseteq J E_R + J H_L + J E_R + J E_L + J' E_S + J' H_S$$

as follows from 2.3, 2.4, 2.7, 2.8, 2.9, and 2.10. Thus $E \oplus J' E_S$ is an ideal in $L_R$.

Finally let $\tilde{H}$ satisfy the hypotheses of the last statement. Then we assert that
$I = \tilde{H} \oplus J E_L \oplus J' E_S$ is an ideal in $L_R$. Clearly $H_R \cdot I \subseteq I$. From the above,

$$E_R \cdot (J E_L \oplus J' E_S) \subseteq I.$$ 

If $h = \sum n_i h_i \in \tilde{H}$, then $h e_j = \sum n_i c(r_j, r_i) e_j$ and $n_i c(r_j, r_i)$ belongs to $J'$ if $r_j$ is short, to $J$ if $r_j$ is long. Thus $h e_j \in J' E_S \oplus J E_L \subseteq I$. If $r = \sum_{j=1}^n k_j r_j$, then

$$h e_r = \sum_{i=1}^n k_i \left( \sum_{j=1}^n n_i c(r_j, r_i) \right) e_r.$$ 

If $r$ is short, then clearly $h e_r \in J' E_S \supseteq J E_S$. If $r$ is long, then consideration of separate
cases shows that all coefficients $k_i$ of short roots in the expression for $r$ are multiples
of $m$. Specifically, in type $B_n$ we take for a simple system $r_1 = \omega_1$, $r_i = \omega_{i+1} - \omega_i$ for
$1 < i < n$, where $\{\omega_1, \omega_2, \ldots, \omega_n\}$ is the standard basis for $R^n$. Thus there is only one
short simple root, and in this case the preceding assertion follows from 2.3 and 2.9.

In type $F_4$, we take for a simple system $r_1 = \frac{1}{2}(\omega_1 - \omega_2 - \omega_3 - \omega_4)$, $r_2 = \omega_4$, $r_3 = \omega_3 - \omega_4$, and $r_4 = \omega_2 - \omega_3$. If one lists the 24 long roots in the manner of [8, p. 500], the assertion
is readily seen to be true. Finally, in type $G_2$, we take $r_1$ long and $r_2$ short, so
the long roots consist of $r_1$, $r_1 + 3r_2$, $2r_1 + 3r_2$, and their negatives, so the assertion
is verified immediately. Since $\sum n_i c(r_j, r_i)$ belongs to $J$ if $r_j$ is long and to $m^{-1}J$ if $r_j$
is short, we see in either case that $h e_r \in J E_L$, completing the proof.

7.2. Corollary. If $L$ is of type $F_4$ or $G_2$, then in 3.5 the only possibility for
$I \cap H_R$ is $J' H_S + J H_L$ if $I \cap E_R = J E_S \oplus J E_L$.

In order to give a parallel to 6.2, more needs to be said about the diagonalization of $C$. To do this we divide the algebras of type $B_n$, $F_4$, and $G_2$ into separate cases.

For $B_n$, $n \geq 3$, we take $\{r_1\}$ to be the above standard simple system, where $r_1$ is the
only short simple root. One can diagonalize $C$ in this case by adding its second
column to its first, then the first column of the new matrix to its second, and then repeating these steps on the \((n-1)\) by \((n-1)\) submatrix of \(B_{n-1}\) obtained by adding the first row of this matrix to its third row. What is of interest is that the first row and column affect and are in turn affected by only the two following rows and columns. Now if \(I\) is an ideal in \(L_R\) with \(I \cap E_R = J E_L \oplus J' E_S\), then we see that, if \(h = \sum c_i h_i \in I \cap H_R\), we have \(c_1 \in J'\) and \(c_i \in J\) for \(j=2, \ldots, n\). In view of the above, we can thus restrict attention to \(B_n\) in studying the nature of the \(h_i\). We have
\[
\text{diag}(1, 1, 2) = Q^{-1}CP
\]
where
\[
Q^{-1} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
1 & 0 & 1
\end{bmatrix}
\quad \text{and} \quad
P = \begin{bmatrix}
1 & 1 & 1 \\
1 & 2 & 2 \\
0 & 1 & 2
\end{bmatrix}.
\]
Thus \(c_1 = c_1, c_2 = c_2,\) and \(c_3 = c_3 + c_1\). In general, then \(c_1\) and \(c_3\) belong to \(\frac{1}{2}J\), while the other \(c_i\) belong to \(J\). Then \((\bar{c}_1, \ldots, \bar{c}_n) = D(\bar{n}_1, \ldots, \bar{n}_n)\) shows that in general \(\bar{n}_1, \bar{n}_2,\) and \(\bar{n}_n\) belong to \(\frac{1}{2}J\), while the other \(\bar{n}_i\) belong to \(J\). Similar reasoning in case \(I \cap E_R = J E_R\) provides the rest of the following result.

7.3. If \(L\) is of type \(B_n, n \geq 3\), \(I\) is an ideal in \(L_R\), and the basis \(\{h_i\}\) is obtained as above, then the possibilities for \(I \cap H_R\) in 3.5 are as follows.

(a) If \(I \cap E_R = J E_R\), then
\[
I \cap H_R = J h_1 \oplus \cdots \oplus J h_{n-1} \oplus J' h_n
\]
with \(J'\) an ideal of \(R\) satisfying \(J \subseteq J' \subseteq \frac{1}{2}J\).

(b) If rank \(L > 3\) and \(I \cap E_R = J E_L \oplus J' E_S\) with \(J\) and \(J'\) as in 3.6, then
\[
I \cap H_R = J^* h_1 \oplus J h_2 \oplus J^{(ui)} h_3 \oplus J h_4 \oplus \cdots \oplus J h_{n-1} \oplus J^{(iv)} h_n,
\]
where \(J^*, J^{(ui)},\) and \(J^{(iv)}\) are ideals of \(R\) in the same range as \(J'\).

(c) If rank \(L = 3\) and \(I \cap E_R\) is as in (b), then
\[
I \cap H_R = J^* h_1 \oplus J h_2 \oplus J^{(ui)} h_3
\]
where \(J^{(ui)}\) is an ideal of \(R\) lying between \(J\) and \(\frac{1}{2}J\), and \(J^*\) as is in (b).

7.4. Corollary. If \(L\) is of type \(B_n, n \geq 3\), then the maximal ideals of \(L\) are of the form
\[
M E_L \oplus R E_S \oplus R h_1 \oplus M h_2 \oplus R h_3 \oplus M h_4 \oplus \cdots \oplus M h_{n-1} \oplus R h_n,
\]
where \(M\) is a maximal ideal in \(R\), and \(\{h_i\}\) is the basis of 7.3.

For the algebra of type \(F_4\) we again take the above standard simple system with \(r_1\) and \(r_2\) short. If \(I \cap E_R = J E_L \oplus J' E_S\) as before and \(h \in I \cap H_R\) is written as before, then we see that \(c_1\) and \(c_2\) are in \(\frac{1}{2}J\), with \(c_3\) and \(c_4\) in \(J\). Here we have \((\bar{n}_1, \ldots, \bar{n}_n) = (\bar{c}_1, \ldots, \bar{c}_4)\), and \(I_4 = I_4 CP\) where \(P\) is a suitable 4 by 4 matrix. Thus each \(\bar{c}_1 = c_1\) and hence \(\bar{n}_1\) and \(\bar{n}_2\) belong to \(\frac{1}{2}J\), while \(\bar{n}_3\) and \(\bar{n}_4\) belong to \(J\). Similar reasoning shows that all \(\bar{n}_i \in J\) if \(I \cap E_R = J E_R\). We have the following results then.
7.5. If $L$ is of type $F_4$, $I$ is an ideal in $L_R$, and the basis $\{\tilde{h}_1\}$ is that obtained above, then
\begin{enumerate}[(a)]  
    
\item $I \cap E_R = J E_R$ implies $I \cap H_R = J H_R$, and $I = L_I$;
\item if $I \cap E_R = J E_L \oplus J' E_S$ as in 3.5, then
\[ I \cap H_R = J'\tilde{h}_1 \oplus J^{(iii)}\tilde{h}_2 \oplus J\tilde{h}_3 \oplus J\tilde{h}_4, \]
\end{enumerate}
where $J'$ and $J^{(iii)}$ are ideals in $R$ lying between $J$ and $\frac{1}{2}J$.

7.6. **Corollary.** If $L$ is of type $F_4$, then the maximal ideals of $L_R$ are of the form
\[ ME_L \oplus RE_S \oplus R\tilde{h}_1 \oplus R\tilde{h}_2 \oplus M\tilde{h}_3 \oplus M\tilde{h}_4 \]
where $M$ is a maximal ideal in $R$, and $\{\tilde{h}_i\}$ is the elementary divisor basis for $H_R$ obtained above.

If $L$ is of type $G_2$, then we take $r_1$ long and $r_2$ short. If $I \cap E_R$ is as in 3.5 and $h \in I \cap H_R$ is written as above, then $c_1 \in J$ and $c_2 \in \frac{1}{2}J$. Also $\tilde{n}_i = \tilde{c}_i$, $i = 1, 2$, and
\[ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} = Q^{-1}CP. \]
Thus $\tilde{c}_1 = c_1$ and $\tilde{c}_2 = c_2 \in \frac{1}{2}J$.

7.7. If $L$ is of type $G_2$, $I$ is an ideal of $L_R$, and $\{\tilde{h}_i\}$ is the basis obtained above, then
\begin{enumerate}[(a)]  
\item $I \cap E_R = J E_R$ implies $I \cap H_R = J H_R$, and $I = L_I$;
\item if $I \cap E_R = J e_1 \oplus J' e_2$ as in 3.5, then $I \cap H_R = J\tilde{h}_1 \oplus J^*\tilde{h}_2$, where $J^*$ is an ideal of $R$ lying between $J$ and $\frac{1}{2}J$.
\end{enumerate}

7.8. **Corollary.** If $L$ is of type $G_2$, then the maximal ideals in $L_R$ are of the form
\[ ME_1 \oplus RE_2 \oplus M\tilde{h}_1 \oplus R\tilde{h}_2, \]
where $M$ is a maximal ideal of $R$ and $\tilde{h}_1$ and $\tilde{h}_2$ are as in 7.7.

Combining 7.4, 7.6, and 7.8, we obtain the necessity part of the main theorem.

7.9. **Corollary.** If $L$ is nonsymplectic and has two root lengths, then every ideal in $L_R$ is of the form $L_I$ for some ideal $I$ in $R$ only if $m$ and $\det C$ are invertible in $R$.

8. **The symplectic algebra.** In this section the proof of the main theorem is completed, with the aid of 3.6 whose proof we now give.

8.1. **Proof of 3.6.** That $I \cap E_S = J E_S$ follows directly from the proof of 5.4. If $x = \sum c_i e_i + \sum n_i h_i \in I$, then multiplication of $x$ by suitable short root elements in the manner of the proof of 5.4 yields $c_i e_s \in I$ for every short root $s$. Thus each $c_i \in J$, and $2c_i e_t \in I$ for every long root $t$. If $r_1$ is long, then $2n_i h_1$ can be isolated in $I$ similarly, and if $r_1$ is short, then $n_i h_1$ can be isolated in $I$ by confining work to the system of short roots since rank $L \geq 2$, and $n_i \in J$ in either case. Thus
\[ I \cap (E_L \oplus H_R) \supseteq 2JE_L \oplus 2JH_L \oplus JH_S. \]
To get the second inclusion of 3.6, note that multiplication of $x$ by $e_i$ maps
\[ \sum_{i=1}^{n} n_i h_i \] to \[ \sum_{i=1}^{n} n_i c(r_i, r_i) e_i \in JE_S. \]

Again the coefficient is the $j$th component of $C(\sum n_i h_i)$. Also, each coefficient belongs to $J$ since if $r_j$ is long, it is a $c_t$ and if $r_j$ is short, it is a $c_s$. Hence \( n_i h_i \in C^{-1}(JH_R) \), so an upper bound for $I \cap (E_L \oplus H_R)$ is $JE_L \oplus C^{-1}(JH_R)$. Next, let $N$ be as in 3.6. Then $L_R N \subseteq N \oplus JE_S$. To see this, note first that $L_R \cdot JE_S \subseteq JE_S + 2JE_L + JH_S + 2JH_L$ in view of 2.5, 2.6, and 2.9, so we need only check $L_R \cdot \tilde{H}$. Let $h = \sum n_i h_i \in \tilde{H}$. Then $he_j = \sum n_i c(r_i, r_i) e_j$ and the coefficient is the $j$th component of $C(h)$, so belongs to $J'$ if $r_j$ is long, to $J$ if $r_j$ is short. Thus $he_j \in J'E_L$ if $r_j$ is long, and $he_j \in JE_S$ if $r_j$ is short. These conclusions remain true if we replace $r_j$ by an arbitrary root $r_i$ as in 7.1. Thus $L_R H' \subseteq JE_L \oplus JE_S$ and hence $N \oplus JE_S$ is an ideal in $L_R$.

8.2. Corollary. If $L$ is of type $C_n$, then every ideal in $L_R$ is of the form $L_J$ for some ideal $J$ in $R$ only if $m = 2 = \det C$ is invertible in $R$.

**Proof.** If $2$ is not invertible in $R$, then 3.7 tells us that $(H_R \oplus 2E_L) \oplus E_S$ is an ideal of $L_R$.

If we specialize 5.5 to $C_n$ and combine it with 8.2, we have a recent result of W. Jehne [5, Theorem 4.1], which was obtained using the explicit classical model of the symplectic algebra as a Lie algebra of matrices.

9. Generators. Our previous results enable us to make some statements regarding generating sets for ideals.

9.1. **Proof of 3.7.** Suppose the cosets $H_1$, $H_2$, ..., $H_g$ constitute a minimal generating set for $(I \cap H_R)/H_J$. Then we claim that
\[ G = \left\{ h^{(1)} + \sum_{j=1}^{g} n_{1j} e_{r_j}, h^{(2)} + \sum_{j=1}^{g} n_{2j} e_{r_j}, \ldots, h^{(g)} + \sum_{j=1}^{g} n_{gj} e_{r_j} \right\} \]
is a minimal generating set for $I$, where the $n_{ij}$ are chosen from a minimal generating set $\{n_1, \ldots, n_k\}$ of $J$ and are distinct until all $k$ generators of $J$ have appeared, arbitrary thereafter. Also $h^{(i)} \in H_i$, and $e_{r_j}$ is a Chevalley root element. In either the single or the two root length case, we can break off each $n_{ij} e_{r_j}$ by proceeding as in 5.1, 5.2, and 5.3. Hence $G$ generates all of $I \cap E_R$. It also generates all of $I \cap H_R$ since $JH_R$ comes from $JE_R$, and the remainder of $I \cap H_R$ can be generated by the elements $h^{(i)}$ and those in $JH_R$ since each $h^{(i)}$ is a representative of a generating coset for $(I \cap H_R)/H_R$. So $G$ generates $I$. Furthermore, none of the $h^{(i)}$ can be obtained from any expression involving the others by multiplications or scalar operations since they are all representatives of a minimal generating set for $(I \cap H_R)/H_J$. Hence any generating set for $I$ must contain at least $g$ elements in order to yield all the $h^{(i)}$. Hence $G$ is a minimal generating set.

The exclusion of the symplectic algebra results from the chaotic relationship
between long and short root elements in ideals in symplectic $L_R$, together with the
fact that $I \cap H_R$ is not in general a direct summand of $I$.

9.2. Corollary. If $R$ is a principal ideal ring, $L$ is nonsymplectic, $I$ is an ideal in $L_R$, $J$ is the ideal of $R$ defined in 3.4 and 3.5 with $I \cap E_R = E_I$, and $g$ is the minimum number of generators of the $R$-module $I \cap H_R/H_I$, then $g$ is the minimum number of generators for $I$ also.

9.3. Corollary. If in 9.2 we also have $m$ and $\det C$ invertible in $R$, then every ideal in $L_R$ is principal.

Bibliography

3. J. Dieudonné, Les algèbres de Lie simples associées aux groupes simples algébriques sur un corps de caractéristique $p > 0$, Rend. Circ. Mat. Palermo (2) 6 (1957), 198–204.

University of California, Riverside, California