ORTHOGONAL REPRESENTATIONS OF
ALGEBRAIC GROUPS

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Introduction. Let $G_1$ and $G$ be connected semisimple algebraic groups defined over a field $K$ of characteristic zero and assume that there is an isomorphism $f$ of $G_1$ onto $G$ which is defined over $\bar{K}$, the algebraic closure of $K$. If $\rho: G \to GL(V)$ is an absolutely irreducible (finite-dimensional) representation of $G$ defined over $K$, then $\rho \circ f$ is an absolutely irreducible representation of $G_1$ defined over $\bar{K}$.

Satake [7, p. 230] has shown that there is a field $K_1$ which is a finite extension of $K$, a (unique) central simple division algebra $K_1^{\#}$ defined over $K_1$, a finite-dimensional right vector space $V_1$ over $K_1^{\#}$, and a $K_1$-homomorphism $\rho_1: G_1 \to GL(V_1/K_1^{\#})$ (the group of all nonsingular $K_1^{\#}$-linear endomorphisms of $V_1$) such that $(\rho \circ f)(g) = \theta_f(\rho_1(g))$ for all $g \in G_1$ where $\theta_f$ is a unique absolutely irreducible representation of $\text{End}(V_1/K_1^{\#})$ (the algebra of all $K_1^{\#}$-linear endomorphisms of $V_1$) onto $\text{End}(V)$.

In this paper we are interested in the case where $K = K_1$ and where there are invariant forms on $V$ and $V_1$. More precisely, we state the following two problems.

Problem 1. Assume that $K_1^{\#} = K$ and that there are invariant bilinear forms $B$ on $V$ and $B_1$ on $V_1$ which are defined over $K$. What is the relationship between these two forms over $F$? Of course, if $F$ is alternating, so is $B_1$ and both are determined by $\dim V = \dim V_1$. Hence, we shall always take $B$ and $B_1$ to be symmetric.

Problem 2. Assume that $K_1^{\#}$ is a nontrivial division algebra over $K$ (i.e., $K_1^{\#} \neq K$) and that there is an invariant bilinear form $F$ on $V$ and an invariant $\varepsilon$-hermitian form $F_1 (\varepsilon = +1$ or $-1)$ on $V_1$ both of which are defined over $K$. What is the relationship between these two forms over $K$?

We are especially interested in the case $K = Q_p$, a $p$-adic field. (In a future paper, we shall discuss the case $K = \mathbb{R}$.) Here, some simplifications are immediately available. In Problem 2, it can be shown [7, p. 232] that $K_1^{\#}$ has an involution of the first kind; but over $Q_p$, it is known that the only such division algebra is the quaternion division algebra. Furthermore, it is known that a hermitian form on a finite-dimensional vector space over a quaternion division algebra defined over $Q_p$ is determined only by the dimension of the vector space. Therefore, in Problem 2 we shall always take $F$ to be skew-hermitian; in the case where $K_1^{\#}$ is a quaternion division algebra, this means that the form $B$ is symmetric [7, p. 233].

If $W$ is a vector space defined over $K$ and if $S$ is a symmetric form on $W$ which is also defined over $K$, then three invariants can be associated with the pair $(W, S)$,
namely, (1) the dimension of $W$, $\dim W$, (2) the discriminant of $S$, $\Delta(S)$, and (3) the Hasse invariant, $c(S)$. In answering Problem 1, we describe these three invariants of $B_1$ in terms of those of $B$. Over $\mathbb{Q}_p$, these invariants completely describe a symmetric form.

Similarly, in Problem 2 we deal with two invariants of the space $(V_1, F)$, namely, (1) the dimension of $V_1$ (over $\mathbb{K}_f$), $\dim V_1$, and (2) the discriminant of $F$, $\delta(F)$. We describe these invariants in terms of the invariants of $B$. Over $\mathbb{Q}_p$, the two invariants above completely describe a skew-hermitian form.

The answers to the questions above fall into two main parts. In Part I, we assume that the isomorphism $f: G_1 \to G$ is of inner type, i.e., for each $\sigma \in \Gamma$ (the Galois group of $\mathbb{K}$ over $\mathbb{K}$), $f^{-\sigma} \circ f = I_{a_\sigma}$ where $g_\sigma \in G_1$ and $I_{a_\sigma}(g) = g_\sigma g g_\sigma^{-1}$ for all $g \in G_1$. (By $f^{-\sigma}$, we shall always mean $(f^{-1})^\sigma$.)

For absolutely simple groups $G_1$, it is well known that there is a Chevalley group $G$ defined over $K$ and an isomorphism $f: G_1 \to G$ defined over $\overline{K}$ of inner type, except possibly when $G_1$ is of type $A_n$, $D_n$, or $E_6$. These last three cases are discussed in Part II.

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**PART I**

1.1. **The standard situation.** Throughout this part, we shall assume that $f$ is of inner type, i.e., $f^{-\sigma} \circ f = I_{a_\sigma}$ for each $\sigma \in \Gamma$. The elements $g_\sigma$ in $G_1$ are determined modulo the center of $G_1$, $Z(G_1)$, and so for $\sigma, \tau \in \Gamma$, the element $c_{\sigma, \tau} = g_\sigma g_\tau g_{\sigma \tau}^{-1}$ are in $Z(G_1)$. It follows that the cohomology class $(c_{\sigma, \tau})$ of the 2-cocycle $c_{\sigma, \tau}$ of $\Gamma$ in $Z(G_1)$ is independent of the choice of elements $g_\sigma$. This 2-cocycle will play an important role in what follows.

Let $\rho: G \to SO(V, B)$ be an absolutely irreducible orthogonal representation defined over $K$ and assume that $B$ is also defined over $K$. In general, such a representation will be denoted by the triple $(V, \rho, B)$ and will be called an orthogonal representation of $G$ defined over $K$. Then $\rho \circ f$ is an orthogonal representation of $G_1$ defined over $\overline{K}$ and, setting $A_\sigma = (\rho \circ f)(g_\sigma^{-1})$, we have that for each $\sigma \in \Gamma$

\[ (\rho \circ f)^\sigma(g) = A_\sigma (\rho \circ f)(g) A_\sigma^{-1} \]

for all $g \in G_1$. Also, by definition of $A_\sigma$ and (1), it follows that

\[ A_\sigma A_\tau = (\rho \circ f)(c_{\sigma, \tau}) A_{\sigma \tau} \]

for all $\sigma, \tau \in \Gamma$. The continuous 2-cocycle $(\rho \circ f)(c_{\sigma, \tau})$ defines $\mathbb{K}_f$ as a normal division algebra if we require that $c(\mathbb{K}_f) \sim ((\rho \circ f)(c_{\sigma, \tau}))$ [7, p. 227].

1.2. **Problem 1.** Our concern in this section is the case where $((\rho \circ f)(c_{\sigma, \tau})) \sim 1$. As we shall see, this is the case of Problem 1. However, before proving the theorem describing completely this situation, we need two lemmas.
Lemma 1.1. Assume that \((\rho \circ f)(c_{\sigma,\tau})\sim 1\). Then there exist elements \(h_{\sigma}\) in \(G_1\) such that \(h_{\sigma} = g_{\sigma} \mod Z(G_1)\) and \((\rho \circ f)(h_{\sigma}^{-1}h_{\tau}) = 1\) for all \(\sigma, \tau \in \Gamma\).

**Proof.** We set \(d_{\sigma,\tau} = (\rho \circ f)(c_{\sigma,\tau})\) for all \(\sigma, \tau \in \Gamma\). Then, as is well known, since \(d_{\sigma,\tau}\) is a 2-cocycle of \(\Gamma\) in \(\{+1, -1\}\) which is equivalent to 1, there exist elements \(a_{\sigma}\) in \(\{+1, -1\}\) for each \(\sigma \in \Gamma\) such that \(d_{\sigma,\tau} = a_{\sigma}a_{\tau}^{-1}\).

If \(\dim V = 1 (2)\), it is immediate that the elements \(d_{\sigma,\tau}\) are always 1 as can be seen by taking determinants of both sides of (2). The case where \(\dim V = 0 (2)\) is harder; however, if \(d_{\sigma,\tau}\) is always 1 then there is nothing to prove. Therefore, we may assume that there is an element \(z \in Z(G_1)\) such that \((\rho \circ f)(z) = -1\). In particular, for each \(\sigma \in \Gamma\), there is an element \(z_{\sigma} \in Z(G_1)\) such that \((\rho \circ f)(z_{\sigma}) = a_{\sigma}\). Using these \(z_{\sigma}\), we define \(h_{\sigma}\) to be \(g_{\sigma}z_{\sigma}\). It is easy to see that these \(h_{\sigma}\) satisfy the conditions above and so this lemma is proved.

From now on, we shall assume that the \(g_{\sigma}\) are chosen so that \((\rho \circ f)(c_{\sigma,\tau}) = 1\) for all \(\sigma, \tau \in \Gamma\). Actually, in practice this choice is frequently trivial, for in many cases \((\rho \circ f)(Z(G_1)) = \{1\}\). Also, we shall assume that \(G_1\) is simply connected. This assumption will be removed following the proof of Theorem 1.1.

Denote the “spin group” of \(B\) by \(\text{Spin}(B)\) and let \(\pi\) be the canonical mapping from \(\text{Spin}(B)\) onto \(SO(V, B)\). It is known that \(\pi\) is defined over \(K\) and that its kernel is \(\{+1, -1\}\). Since \(G_1\) is simply connected, there is a (polynomial) map \(\rho_x : G_1 \to \text{Spin}(B)\) such that \(\pi \circ \rho_x = \rho \circ f\). We define elements \(\overline{A}_{\sigma} \in \text{Spin}(B)\) by \(\overline{A}_{\sigma} = \rho_x(g_{\sigma}^{-1})\). Then \(\pi(\overline{A}_{\sigma}) = A_{\sigma}\) and the system \(\{\overline{A}_{\sigma}\}\) satisfies the relation \(\overline{A}_{\sigma}\overline{A}_{\tau} = e_{\sigma,\tau}\overline{A}_{\sigma}\overline{A}_{\tau}\) where each \(e_{\sigma,\tau}\) is \(+1\) or \(-1\).

Lemma 1.2. Let \(\rho_x : G_1 \to \text{Spin}(B)\) be such that \(\pi \circ \rho_x = \rho \circ f\) and assume that each \((\rho \circ f)(c_{\sigma,\tau}) = 1\). Then the \(e_{\sigma,\tau}\) above are given as follows: \(e_{\sigma,\tau} = \rho_x(c_{\sigma,\tau})\).

**Proof.** For each \(\sigma \in \Gamma\), we have \(\pi \circ \rho_x^\sigma = (\rho \circ f)^\sigma = A_{\sigma}(\rho \circ f)A_{\sigma}^{-1} = \pi(\overline{A}_{\sigma}\rho_x(\overline{A}_{\sigma}^{-1}))\).

So \(\rho_x^\sigma(g) = e(g)\overline{A}_{\sigma}\rho_x(g)\overline{A}_{\sigma}^{-1}\) where \(e(g) = +1\) or \(-1\). But, since \(G_1\) is connected, \(e(g)\) is always 1 and so \(\rho_x^\sigma(g) = \overline{A}_{\sigma}\rho_x(g)\overline{A}_{\sigma}^{-1}\) for all \(g \in G_1\). Using this fact, the lemma follows immediately.

Before stating Theorem 1.1, we recall a few definitions about quadratic spaces \((W, S)\) defined over \(K\). Assume that \(n = \dim W\) and that in diagonal form \(S\) is \(\text{diag}(a_1, \ldots, a_n)\) (the multiplicative group of nonzero elements in \(K\)). Then one puts \(\Delta(S) = (-1)^{(n-1)/2}a_1 \cdot \cdots \cdot a_n \mod (K^*)^2\). The invariant \(c(S)\) is the cohomology class of a certain 2-cocycle of \(\Gamma\) in \(K^*\) and is defined in the proof of Theorem 1.1. It can be shown \([4]\) that the invariants \(\dim, \Delta,\) and \(c\) are enough to determine \(S\) if \(K\) is a nonarchimedean local field.

**Theorem 1.1.** Let \(G_1\) and \(G\) be simply connected algebraic groups defined over \(K\) (\(\text{char}\, K = 0\)) and assume that there is a \(K\)-isomorphism \(f : G_1 \to G\) such that \(f^{-\sigma} \circ f = I_{g_{\sigma}}\) for each \(\sigma \in \Gamma\). Define elements \(c_{\sigma,\tau} \in Z(G_1)\) by setting \(c_{\sigma,\tau} = g_{\sigma}^{-1}g_{\tau}g_{\sigma}\). Let \((V, \rho, B)\) be an orthogonal representation of \(G\) defined over \(K\) and assume that
each \((\rho \circ f)(c_{\sigma,t})\) is 1. Then there is an orthogonal representation \((V_1, \rho_1, B_1)\) of \(G_1\) defined over \(K\) such that \(\rho_1 \sim \rho \circ f\) and \(B_1\) is related to \(B\) as follows: \(\dim V_1 = \dim V\), \(\Delta(B_1) = \Delta(B)\), and \(c(B_1) = c(B)(\rho_1(c_{\sigma,t}))\) where \(\rho_1: G_1 \to \text{Spin}(B)\) and \(\pi \circ \rho_2 = \rho \circ f\).

**Proof.** As before, we set \(A_0 = (\rho \circ f)(g^{-1})\) and \(\overline{A}_0 = \rho_0(g^{-1})\). Since \(A_0^T A_1 = A_0\), there is an element \(X \in GL(V)\) such that \(A_0 = X^{-1} A X\). Using \(X\), we set \(\rho_1 = X(\rho \circ f)X^{-1}\) and \(B_1 = X^{-1} BX^{-1}\). It is easy to check that \(\rho_1\) is defined over \(K\) and that the image of \(G_1\) under \(\rho_1\) preserves \(B_1\) which is also defined over \(K\). Also, since \(A_0 \in SO(V, B)\), \((\det X)^{(\det X)^{\sigma}} = 1\) for all \(\sigma \in \Gamma\) and so \((\det X) \in K^*\). Hence, \(\Delta(B_1) = \Delta(B)\).

Finally, it is necessary to compute \(c(B_1)\). To do this, we look at the Clifford algebra \(C(B)\) of \(B\). (If \(\dim V = 1 (2)\), we really need \(C^+(B)\), the set of even elements of \(C(B)\), but we write \(C(B)\) to avoid some notational clumsiness.) Let \(h: C(B) \to M(t, K)\) be an isomorphism of \(C(B)\) onto a total matrix algebra. For each \(x \in C(B)\), there is \(Y_x \in \text{GL}(t, \overline{K})\) such that \(h(x) = Y_x h(x) Y_x^{-1}\) for all \(x \in C(B)\). The system \(\{Y_x\}\) satisfies the relation \(Y_x Y_y = b_{\sigma,x} Y_{x \sigma,y}\) with \(b_{\sigma,x} \in \overline{K^*}\) and, by definition, the cohomology class of the 2-cocycle \(b_{\sigma,x}\) is \(c(B)\).

The map \(X^{-1}: (V_1, B_1) \to (V, B)\) is a quadratic space isomorphism and induces a mapping \(X^{-1}: C(B_1) \to C(B)\). (In the following when we write \(X^{-1}\), we shall always mean the mapping of the Clifford algebras.) The composite map \(H = h \circ X^{-1}\) gives an isomorphism of \(C(B_1)\) with a total matrix algebra. We now determine the corresponding 2-cocycle. For each \(\sigma \in \Gamma\), \(H^\sigma \circ H^{-1} = I_{h_\sigma}\) where \(N_x = Y_x h(\overline{A}_0)\). From this it follows that \(N_x N_y = b_{\sigma,x} \rho_0(c_{\sigma,y}) N_{x \sigma,y}\) and our theorem is proved.

It is not difficult to reduce the general case where \(G_1\) is not simply connected to the case above. For it is known that there are simply connected covering groups \((\overline{G}, \rho_1)\) and \((G, \rho)\) of \(G_1\) and \(G\) respectively which are defined over \(K\). Then, it also can be shown that there is a \(\overline{K}\)-isomorphism \(\overline{f}: \overline{G_1} \to \overline{G}\) such that for each \(\sigma \in \Gamma\), \(\overline{f}^{-\sigma} \circ \overline{f} = I_{h_\sigma}\); here, \(h_\sigma\) is an element in \(\overline{G}_1\) such that \(p_0(h_\sigma) = g_{\sigma}\). In the statement of Theorem 1.1, \(G\) is replaced by \(\overline{G}, \rho\) by \(\rho \circ \rho, g_{\sigma}\) by \(h_{\sigma}\), and so on.

### 1.3. Problem 2

In this section, we consider the case where \(K\#\) is a quaternion division algebra \((\beta, \gamma)\) and we begin by summarizing some results which can be found in [7, p. 235]. The algebra \(K\#\) has a basis \((1, x_1, x_2, x_1 x_2)\) over \(K\) such that \(x_1^2 = \beta, x_2^2 = \gamma,\) and \(x_1 x_2 = -x_2 x_1\). The elements \(\beta\) and \(\gamma\) are in \(K^*\) and we assume that the equation \(\beta X^2 + \gamma Y^2 = 1\) has no solution \((X, Y)\) in \(K\). An isomorphism \(M: K\# \to M(2, \overline{K})\) is given by

\[
M(Y_0 + Y_1 x_1 + Y_2 x_2 + Y_3 x_1 x_2) = \begin{pmatrix}
Y_0 + Y_1 \beta^{1/2} & \gamma (Y_2 + Y_3 \beta^{1/2}) \\
Y_2 - Y_3 \beta^{1/2} & Y_0 - Y_1 \beta^{1/2}
\end{pmatrix}.
\]

\(M\) is defined over \(L = K(\beta^{1/2})\) and if we set \(\text{Gal}(L/K) = \{1, \sigma\}\), then \(M^\sigma(x) = M(n^{-1}_x x_n)\) for all \(x \in K\#\) where \(n_x = x_2\). There is a canonical involution \(x \to \overline{x}\)
of the first kind on $K^\#$, namely, if $x = Y_0 + Y_1 x_1 + Y_2 x_2 + Y_3 x_1 x_2$, then $\bar{x} = Y_0 - Y_1 x_1 - Y_2 x_2 - Y_3 x_1 x_2$. Setting

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

we see that $M(\bar{x}) = J^{-1} M(x) J$ for all $x \in K$. Furthermore, $J = -J$.

Now we return to the situation in Problem 2 and assume that $K^\#$ is a quaternion division algebra. If $e_{ij}$ are matrix units in $K^\#$, then, considering $V_1 e_{11}$ as a vector space over $K$, there is a $K$-isomorphism $f_1 : V \to V_1 e_{11}$ defined over $L$ such that

$$(3) \quad R_{n_\alpha} = a_\alpha f_1 \circ A_{\sigma}^{-1} \circ f_1^{-\sigma}$$

where $R_{n_\alpha} : V_1 e_{22} \to V_1 e_{11}$ is given by $R_{n_\alpha}(v) = v n_\alpha$ for all $v \in V_1 e_{22}$. The element $a_\alpha$ is in $K^*$ [7, p. 229].

Define $B_{11}$ on $V_1 e_{11}$ so that $f_1$ is a quadratic space isomorphism and set

$$(4) \quad J M(F(v, w)) = (B_{11}(v, w)).$$

$F$ is skew-hermitian if $B$ is orthogonal.

**Lemma 1.3.** In formula (3), $a_\alpha^2 = -\gamma$.

**Proof.** First we show that $-\gamma B_{11}(v n_\alpha^{-1}, v n_\alpha^{-1}) = B_{11}(v, w)$ for all $v, w \in V_1$. This is done by applying $\sigma$ to (4) and remembering that $F$ is defined over $K$, $M^\sigma \circ I_{n_\alpha} = M$, and $n_\alpha = \gamma e_{12} + e_{21}$.

Using this result we are able to prove the lemma. Again we use (3) and the fact that $A_\alpha \in SO(V, B)$. For choosing $v$ to be $K$-rational in $V_1$, such that $B_{22}(v) = B_{22}(v, v) \neq 0$, we have: $B_{11}(R_{n_\alpha}(v e_{22})) = B_{11}(v e_{22}) = B_{22}(v)$. But also $a_\alpha^{-2} B_{11}(R_{n_\alpha}(v e_{22})) = B_{11}(f_1 \circ A_{\sigma}^{-1} \circ f_1^{-\sigma})(v e_{22})) = (B(f_1^{-1}(v e_{11})))^\sigma = (B_{11}(v e_{11}))^\sigma = B_{11}(v n_\alpha e_{11}, n_\alpha^{-1})$ which by the first part of this lemma is just $(-\gamma)^{-1} B_{11}(v n_\alpha e_{11}) = (-\gamma)^{-1} B_{11}(v e_{22}) = (-\gamma)^{-1} B_{22}(v)$ and the lemma is complete.

Before stating Theorem 1.2, we again review some fundamental definitions. For a skew-hermitian form $F$ on a space $V_1$ over $K^\#$, Tsukamoto [8] has determined a complete set of invariants when $F$ is a nonarchimedean local field such that $[K^* : (K^*)^2] > 2$. The invariants are dim $V_1$ and $\delta(F)$. This last invariant is defined in the following way: let $\{v_1, \ldots, v_m\}$ be an orthogonal basis defined over $K$ of $V_1$ over $K^\#$. Since $F$ is skew-hermitian, $F(v_i, v_i) = x_i = -\bar{x}_i$ for some $x_i \in K^\#$. But $\delta^2 = a_i \in K^*$ and we set $\delta(F) = a_1 \cdots a_m \mod (K^*)^2$.

**Theorem 1.2.** Let $G_1$ and $G$ be semisimple algebraic groups defined over $K$ (char $K = 0$) and assume that there is a $K$-isomorphism $f_1 : G_1 \to G$ such that $f^{-\sigma} \circ f = I_{g_\alpha}$ for each $\alpha \in \Gamma$. Let $(V, \rho, B)$ be an orthogonal representation of $G$ defined over $K$ and let $(V_1/K^\#, \rho_1, F)$ be a skew-hermitian representation of $G$ defined over $K$ where $K^\#$ is a quaternion division algebra over $K$. Assume also that there is an
absolutely irreducible representation \( \theta_i : \text{End} (V_i/K\#) \to \text{End} (V) \) defined over \( \bar{K} \) such that \( \theta_i(g) = (\rho \circ f)(g) \) for each \( g \in G_1 \). Then the invariants of \( F \) are as follows: \( \dim V_i = \frac{1}{2} \dim V \) and \( \delta(F) = \Delta(B) \).

**Proof.** The dimension formula follows from the existence of \( f \) in (3). To prove the relation on discriminants, let \( \{v_1, \ldots, v_m\} \) be an orthogonal basis of \( F \) defined over \( K \). Then \( E = \{v_1e_{11}, \ldots, v_me_{11}, v_1e_{21}, \ldots, v_me_{21}\} \) is a basis for \( V_1e_{11} \) and \( \delta(F) = (-1)^m \det (B_{11}, E) \). By this last term we mean the determinant of \( B_{11} \) in the basis \( E \).

Let \( \{x_1, \ldots, x_{2m}\} \) be a basis of \( V \) defined over \( K \) and let \( P \) be the matrix of \( f^{-1}(E) \) with respect to \( \{x_i\} \). Then \( \delta(F) = (-1)^m \det (B, \{x_i\}) \cdot (\det P)^2 \). Hence, \( (\det P)^2 \in K^* \). If we can show that \( \det P \in K^* \), we are done. Stated differently, it remains to be proved that \( (\det P)^a(\det P)^{-1} = 1 \) where \( \text{Gal} (L/K) = \{1, \sigma\} \).

To prove this statement, we compute determinants of both sides of (3). The matrix of \( R_{n-1}(E) \) in the basis \( E^a = \{v_1e_{22}, \ldots, v_me_{22}, \gamma v_1e_{12}, \ldots, \gamma v_me_{12}\} \) is

\[
\begin{pmatrix}
0 & 1_m \\
\gamma^{-1} & 0
\end{pmatrix}
\]

and has determinant \( (-\gamma)^{-m} \). So, by (3), it follows that \( (\det P)^a(\det P)^{-1} = (-\gamma)^{-m}a_2m = (-\gamma)^{-m}(-\gamma)^m \), by Lemma I.3, and we have proved the theorem.

1.4. Steinberg groups. In this brief section, we look at the results in this part from a slightly different viewpoint, namely that of Steinberg groups. A group \( G \) defined over \( K \) is called Steinberg if there is a Borel subgroup of \( G \) which is also defined over \( K \). It is known that if \( G_1 \) is a connected semisimple group defined over \( K \), then there is a Steinberg group \( G \) defined over \( K \) and a \( \bar{K} \)-isomorphism \( f: G_1 \to G \) of inner type. In this case, the cohomology class of \( e_{a_1} \) is independent of \( f \) and is denoted by \( \gamma_K(G_1) \). This last invariant has been studied by Satake [6], [7].

The division algebra associated with an irreducible representation of a Steinberg group is always trivial, i.e., is the underlying field [7, p. 241]. Hence, in terms of Steinberg groups, Theorems I.1 and I.2 say that to determine the form on a representation of \( G_1 \) it is enough to know the form on the corresponding representation of the Steinberg group \( G \) associated with \( G_1 \). Of course, for absolutely simple groups \( G_1 \), the associated Steinberg group \( G \) will always be the corresponding Chevalley group except possibly when \( G_1 \) is of type \( A_n, D_n, \) or \( E_6 \). In Part II, we shall study these three cases and show how orthogonal representations of Steinberg and Chevalley groups are related.

**Part II**

2.1. The group \( G^* \). Throughout this section, let \( G \) be a semisimple Chevalley group defined over \( K \) (char \( K = 0 \)) and let \( T \) be a maximal split torus in \( G \) defined over \( K \). Denote by \( \Delta = \{a_1, \ldots, a_n\} \) the corresponding fundamental root system.
The automorphism group of $G$ is the semidirect product of a finite group $\Theta$ and the inner automorphisms of $G$. We choose $\Theta$ in such a way that for each $\theta \in \Theta$, $\theta$ is defined over $K$, $\theta(T) = T$, and $\theta(\Delta) = \Delta$. We define an algebraic group $G^*$ to be $G \cdot \Theta$, the semidirect product of $G$ and $\Theta$ where group multiplication is given in the following way: $(g_1, \theta_1)(g_2, \theta_2) = (g_1g_2, \theta_1\theta_2)$. In what follows, we consider $G$ as a subgroup of $G^*$. By our choice of $\Theta$, both are algebraic groups defined over $K$.

**Lemma II.1.** Let $\rho: G \to GL(V)$ be an absolutely irreducible representation of $G$ defined over $K$. Then there exists a representation $\rho^*: G^* \to GL(V)$ defined over $K$ such that $\rho^*|_G = \rho$ if and only if there is a homomorphism $\theta \to A_\theta$ of $\Theta$ to $GL(V)$ such that $\rho(\theta(g)) = A_\theta \rho(g) A_\theta^{-1}$ for all $g \in G$.

**Proof.** If $\rho^*$ exists, set $\rho^*(1, \theta) = A_\theta$. Then $\rho^*[\theta((1, \theta)(g, 1)(1, \theta^{-1}))] = A_\theta \rho(g) A_\theta^{-1}$ and is also $\rho^*((\theta(g), 1)) = \rho(\theta(g))$.

Conversely, if such $A_\theta$ exist, define $\rho^*(g, \theta) = \rho(g) A_\theta$. It is easy to check that $\rho^*$ becomes a homomorphism and so the lemma is proved.

**Corollary.** Assume that $\Theta$ is a cyclic group generated by $\theta$. Then $\rho^*$ exists if and only if $\rho \circ \theta \sim \rho$.

**Proof.** Assume that $\theta^2 = 1$ and $\rho \circ \theta = A_\theta \rho A_\theta^{-1}$. It is easy to see that $A_\theta = aI$ for some $a \in K^*$ and modifying $A_\theta$ we can assume $A_\theta = 1$. This completes the proof.

2.2. The groups $A_n$, $D_n$, and $E_6$. In this section, we shall take a closer look at the group $G^*$ when $G$ is a Chevalley group of type $A_n$, $D_n$, or $E_6$. In particular, let $(V, \rho, B)$ be an orthogonal representation of $G$ defined over $K$ with highest weight $\lambda$. We shall give conditions on $\lambda$ in order that $\rho^*: G^* \to GL(V)$ exists; furthermore, in each case we shall show that $\rho^*$ can be chosen to be defined over $K$ and $\rho^*: G^* \to O(V, B)$.

**Lemma II.2.** Let $G$ be a Chevalley group of type $A_n$ defined over $K$ (char $K = 0$) and let $(V, \rho, B)$ be an orthogonal representation of $G$ defined over $K$. Then $\rho^*: G^* \to O(V, B)$ exists and is defined over $K$. Furthermore, if $\dim V = 1$ (2), $\rho^*$ can be chosen so that $\rho^*: G^* \to SO(V, B)$.

**Proof.** For easy reference, the proof is divided into small sections.

(i) The group $\Theta$ is of order 2 and is generated by $\theta$ where $\theta(a_r) = a_{n-r+1}$. If $\lambda = \sum_{r=1}^n m_r a_r$ with $\lambda \in Q$, $m_r \geq 0$, then $\rho \circ \theta \sim \rho$ if and only if $m_r = m_{n-r+1}$. But it is known [3, p. 196] that all orthogonal representations of $A_n$ have this property and also that each $m_r \in Z$. Since $\rho$ and $\rho \circ \theta$ are both defined over $K$, there is an $A \in GL(V, K)$ such that $A \rho(g) = \rho(\theta(g)) A$. Let $x$ be a $K$-rational highest weight vector in $V$. Since $\theta \lambda = \lambda$, it is easy to see that $A x$ is also a $K$-rational highest weight vector. Hence, $A x = ax$ for some $a \in K^*$ and $A^2 = a^2$. Set $A_\theta = a^{-1} A$; then $A_\theta \in GL(V, K)$, $A_\theta \rho(g) = \rho(\theta(g)) A_\theta$ for all $g \in G$, and $A_\theta^2 = 1$. If $\dim V = 1$ (2), we may assume that $\det A_\theta = 1$, multiplying $A_\theta$ by $-1$ if necessary. We also note that $A_\theta x = ex$ where $e^2 = 1$. Next, we shall show that $A_\theta$ is in $O(V, B)$. 

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Let $W = N(T)/T$ be the Weyl group of $G$. It is known that there is an element $w$ in $W$ such that $w(\Delta) = -\Delta$, i.e. $w(\alpha_i) = -\alpha_{n-r+1}$. Choose a representative $g$ in $N(T)$ for $w$, i.e. $w = gT$. The element $\theta(g)$ is also in $N(T)$ and it is easy to see that $\lambda(g) = \theta \circ I_g \circ \theta$. Hence, there is a $t$ in $T$ such that $\theta(g) = gt$. Applying $\theta$ again to this equation we get

$$t \theta(t) = 1.$$  

Next, we show that $B(x, \theta(g)x) \neq 0$. If $x_1$ and $x_2$ are weight vectors in $V$ corresponding to weights $\lambda_1$ and $\lambda_2$, respectively, then for $t$ in $T$, $B(x_1, x_2) = B(\rho(t)x_1, \rho(t)x_2) = \lambda_1(t)\lambda_2(t)B(x_1, x_2)$. So $B(x_1, x_2) = 0$ except possibly when the character $\lambda_1 + \lambda_2$ is zero. (We use additive notation on the character module of $T$.)

In the case above, the highest weight space has dimension 1 and so if $\rho(g)x$ has weight $-\lambda$, then we are done with (iii). But this follows from the facts that $g \in N(T)$ and $I_g(\lambda) = -\lambda$.

Since $A_0B_A$ is also invariant under $\rho(G)$, there is $a_0 \in K^*$ such that $A_0B_A = a_0B$. In particular $0 \neq a_0B(x, \rho(g)x) = B(A_0, A_0\rho(g)x) = B(A_0, \rho(\theta(g))A_0x) = B(x, \rho(g)x) = -\lambda(t)B(x, \rho(g)x)$. Hence, $a_0 = \lambda(t)$. The map $\theta \rightarrow a_0$ is a homomorphism and so $a_0 = 1$, i.e. $\lambda(t)^2 = 1$, a result which can also be seen by applying $\lambda$ to (5).

Finally, we show that $\lambda(t) = 1$. If $n \equiv 0 \pmod{2}$, this follows immediately. For by (5), $\lambda_r(t) = 1$; but $t$ is an integral combination of such terms. If $n \equiv 1 \pmod{2}$, then it is enough to show that $\lambda_r(t) = 1$ where $r = \frac{1}{2}(n+1)$. We saw that $A_0B_A = \lambda(t)B$. In particular, if $\dim V \equiv 1 \pmod{2}$, then $\lambda(t) = 1$ (as can be seen by taking determinants). But for $n \equiv 1 \pmod{2}$, the representation with highest weight $\lambda = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ is orthogonal and has dimension $n(n+2)$ which is odd. Hence, $\lambda(t) = \alpha_r(t) = 1$ and the lemma is proved.

We have proved this lemma in such generality so that the proof will apply in the cases $D_n$ and $E_6$. We indicate below the way in which this happens.

**Lemma 11.3.** Let $G$ be a Chevalley group of type $D_n$ $(n \neq 4)$ defined over $K$ (char $K = 0$) and let $(V, \rho, B)$ be an orthogonal representation of $G$ defined over $K$ with highest weight $\lambda = \sum_{j=1}^{n} m_j \alpha_j$. Then $\rho^*: G^* \rightarrow O(V, B)$ exists and is defined over $K$ if and only if $m_n = m_{n-1}$. Furthermore, if $\dim V \equiv 1 \pmod{2}$, $\rho^*$ can be chosen so that $\rho^*: G^* \rightarrow SO(V, B)$.

**Proof.** We take $G = SO(2n)$, the special orthogonal group on a $2n$-dimensional vector space $W$ defined over $K$. Let $\{e_1, \ldots, e_{2n}\}$ be a $K$-rational basis of weight vectors where $e_i$ has weight $\lambda_i$ and $e_{n+i}$ has weight $-\lambda_i$ for $i = 1, \ldots, n$. A fundamental root system $\{\alpha_1, \ldots, \alpha_n\}$ is given by $\alpha_1 = \lambda_1 - \lambda_2$, $\ldots$, $\alpha_{n-1} = \lambda_{n-1} - \lambda_n$, and $\alpha_n = \lambda_{n-1} + \lambda_n$. Define a linear transformation $J \in O(2n)$ by $J e_i = e_r$, $r \neq n$, $2n$, $J e_n = e_{2n}$, and $J e_{2n} = e_n$. Then $\det(J) = -1$.

(i) The group $\Theta$ is of order 2 and is generated by $\theta$ where $\theta(\alpha_{n-1}) = -\alpha_n$. If $\lambda = \sum_{j=1}^{n} m_j \alpha_j$, with $m_r \in Q$, $m_{n-1} \geq 0$, then $\rho \circ \theta \sim \rho$ if and only if $m_n = m_{n-1}$. It is easy to see that $\theta = I_n$. Hence, $G^*$ may be identified with $O(2n)$.  

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(ii) The element \( w = gT \) is given in the following way: if \( n \equiv 1 \pmod{2} \), \( ge_t = e_{r+n} \) for \( r = 1, \ldots, n-1 \), \( ge_n = e_n \), \( ge_{2n} = e_{2n} \), and \( g^2 = 1 \). If \( n \equiv 0 \pmod{2} \), \( ge_t = e_{r+n} \) for \( r = 1, \ldots, n \) and \( g^2 = 1 \). In either case, \( \theta(g) = J_g J = g \) and so \( t = 1 \). The lemma now follows immediately.

The case \( D_4 \) is complicated by the fact that \( \Theta = S_3 \), the symmetric group on 3 elements. We postpone our study of it, looking first at \( E_6 \).

**Lemma II.4.** Let \( G \) be a Chevalley group of type \( E_6 \) defined over \( K \) (char \( K = 0 \)) and let \( (V, \rho, B) \) be an orthogonal representation of \( G \) defined over \( K \). Then \( \rho^*: G^* \to O(V, B) \) exists and is defined over \( K \). Furthermore, if \( \dim V \equiv 1 \pmod{2} \), \( \rho^* \) can be chosen so that \( \rho^*: G^* \to SO(V, B) \).

**Proof.** The group \( G \) has the following Dynkin diagram:

\[
\begin{array}{ccccccc}
0 & 
\alpha_1 & 
\alpha_2 & 
\alpha_3 & 
\alpha_4 & 
\alpha_5 & 
\alpha_6 \\
\end{array}
\]

(i) The group \( \Theta \) is of order 2 and is generated by \( \theta \) where \( \theta(\alpha_1) = \alpha_5 \), \( \theta(\alpha_2) = \alpha_4 \), \( \theta(\alpha_3) = \alpha_3 \), and \( \theta(\alpha_6) = \alpha_6 \). If \( \lambda = \sum_{\alpha_j} m_j \alpha_j \) with \( m_j \in \mathbb{Q}, m_j \geq 0 \), then \( \rho \circ \theta \sim \rho \) if and only if \( m_1 = m_5 \) and \( m_2 = m_4 \). But it is known [3, p. 202] that all orthogonal representations of \( E_6 \) have this property and also that each \( m_j \in \mathbb{Z} \).

(ii) The element \( w \) is given by: \( w(\alpha_1) = -\alpha_5 \), \( w(\alpha_2) = -\alpha_4 \), \( w(\alpha_3) = -\alpha_3 \), and \( w(\alpha_6) = -\alpha_6 \).

(iv) We know that \( \lambda \) is an integral combination of \( \alpha_1 + \alpha_5 \), \( \alpha_2 + \alpha_4 \), \( \alpha_3 \), and \( \alpha_6 \). From (5), it follows that \( (\alpha_1 + \alpha_5)(t) = 1 \), \( (\alpha_2 + \alpha_4)(t) = 1 \) and \( \alpha_3(t)^2 = \alpha_6(t)^2 = 1 \). Hence, it only remains to be shown that \( \alpha_3(t) = \alpha_6(t) = 1 \). The representation with highest weight \( \lambda = 2(\alpha_1 + \alpha_5) + 4(\alpha_2 + \alpha_4) + 6\alpha_3 + 3\alpha_6 \) is orthogonal and has odd dimension. But then \( \lambda(t) = \alpha_3(t) = 1 \). Similarly, the representation with highest weight \( \lambda = 5[(\alpha_1 + \alpha_5) + 2(\alpha_2 + \alpha_4) + 3\alpha_3 + 2\alpha_6] \) is orthogonal and has odd dimension. Hence, \( \alpha_3(t) = 1 \) and the proof of the lemma is completed.

**Lemma II.5.** Let \( G \) be a Chevalley group of type \( D_4 \) defined over \( K (K = \mathbb{Q}_p) \) and let \( (V, \rho, B) \) be an orthogonal representation of \( G \) defined over \( K \) with highest weight \( \lambda = m_1 \alpha_1 + m_2 \alpha_2 + m_3 \alpha_3 + m_4 \alpha_4 \). Then \( \rho^*: G^* \to O(V, B) \) exists and is defined over \( K \) if and only if \( m_1 = m_3 = m_4 \). Furthermore, if \( \dim V \equiv 1 \pmod{2} \), \( \rho^* \) can be chosen so that \( \rho^*: G^* \to SO(V, B) \).

**Proof.** The group \( G \) has the following Dynkin diagram:

\[
\begin{array}{cc}
0 & 0 \\
\alpha_1 & \alpha_2 \\
\alpha_3 & \alpha_4 \\
\end{array}
\]
(i) The group $\Theta$ is of order 6 and is the symmetric group on \{\(a_1, a_3, a_4\). We distinguish two elements $\theta$ and $\psi$ in $\Theta$. The element $\theta$ has order 2 and is defined by $\theta(a_3) = a_4$ and the element $\psi$, having order 3, is defined by $\psi(a_1) = a_2$, $\psi(a_3) = a_4$, and $\psi(a_4) = a_1$. If $\lambda = m_1a_1 + m_2a_2 + m_3a_3 + m_4a_4$, it follows that a necessary condition for $\rho^* : G^* \to GL(V)$ to exist is that $m_1 = m_2 = m_3$. We show now that these equalities are also sufficient. For let $x$ be a $K$-rational highest weight vector of $P$. Then, as in the proof for $A_n$, there are elements $A_\theta, A_\psi \in GL(V, K)$ such that $A_\theta^2 = A_\psi^3 = 1$, $A_\psi(g) = \rho(\theta(g))A_\theta$ and $A_\psi(g) = \rho(\psi(g))A_\psi$ for all $g \in G$, $A_\psi x = x$, and $A_\psi x = x$. The defining relations for $S_3$ are $\theta^2 = 1$ and $\psi \psi \theta = \psi^3$. Hence, we need to show that $A_\theta A_\psi A_\theta = A_\psi^3$. But since

$$
\rho(\psi^3(g)) = A_\psi^3 = (A_\theta A_\psi A_\theta)^{-1}
$$

it follows that there exists $a \in K^*$ such that $A_\theta \psi = aA_\theta A_\psi A_\theta$. Applying both sides to $x$, we see that $a = 1$ and this part of the lemma is proved. It should be noticed, also, that we can assume $\det A_\theta = 1$ if $\dim V = 1$ (2).

As in Lemma II.3, it can be shown that $A_\theta \in O(V, B)$. Therefore, if we can show that $A_\psi$ is in $O(V, B)$, the proof will be complete. As a matter of fact, since the mapping $\psi \to A_\psi$ gives a homomorphism of the group of order 3 generated by $\psi$, if $A_\psi \in O(V, B)$, then $A_\psi \in SO(V, B)$.

We know that $A_\psi B A_\psi = A_\psi B$ where $a_\psi \in K^*$ and $a_\psi^3 = 1$. But since $G$ is a Chevalley group, we may assume that $K = Q$ and then $a_\psi$ must be 1. This completes the proof of the lemma.

To conclude this section, we prove a result about the Clifford algebra $C(B)$ of $B$ which will be useful when we return to Problem 1. As above, the set of even elements in $C(B)$ will be denoted by $C^+(B)$.

**Lemma II.6.** Let $G$ be a Chevalley group of type $A_n$, $D_n$, or $E_6$ defined over $K$ (char $K = 0$) and let $\theta \in \Theta$ be an element of order 2. Let $(V, \rho, B)$ be an orthogonal representation of $G$ defined over $K$ and assume that $\rho^* : G^* \to O(V, B)$ exists and is defined over $K$. Then there is an element $\overline{A}_\theta$ in $C^+(B)$ if $\det A_\theta = 1$ or in $C(B)$ if $\det A_\theta = -1$ satisfying the following conditions:

(i) $\overline{A}_\theta x A_\theta^{-1} = A_\theta x$ for all $x \in V$.

(ii) $\overline{A}_\theta (\text{Spin } B) A_\theta^{-1} = \text{Spin } B$.

**Proof.** Since $A_\theta = \rho^*(\theta)$ is defined over $K$, $A_\theta^2 = 1$, and $A_\theta \in O(V, B)$, the spaces $V^+ = \{x \in A_\theta \mid x = x\}$ and $V^- = \{x \in V \mid A_\theta x = -x\}$ are defined over $K$, span $V$, and are perpendicular. Let $\{e_1, \ldots, e_r\}$ and $\{e_{r+1}, \ldots, e_n\}$ be orthogonal bases of $V^+$ and $V^-$, respectively, which are defined over $K$.

If $\det A_\theta = 1$ (i.e., $n - r = 0$ (2)), we set $\overline{A}_\theta = e_{r+1} \cdots e_n \in C^+(B)$. If $\det A_\theta = -1$ (i.e., $n = 0$ (2) and $n - r = 1$ (2)), we set $\overline{A}_\theta = e_1 \cdots e_r \in C(B)$. In both cases it is easy to see that $\overline{A}_\theta$ has the desired properties and so the lemma is proved.

**Corollary 1.** Let $\rho_\pi : G \to \text{Spin } B$ be such that $\pi \circ \rho_\pi = \rho$ where $\pi$ is the natural mapping from Spin (B) onto SO(V, B). Then $\rho_\pi(\theta(g)) = \overline{A}_\theta \rho(s) A_\theta^{-1}$ for all $g \in G$. 

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COROLLARY 2. If \( \det A_\theta = 1 \), then \( \overline{\Delta}_\theta^2 = \Delta^- \) where \( \Delta^- \) is the discriminant of \( B \) restricted to \( V^- = \{ x \in V \mid A_\theta x = -x \} \). If \( \det A_\theta = -1 \), then \( \overline{\Delta}_\theta^2 = \Delta^+ \) where \( \Delta^+ \) is the discriminant of \( B \) restricted to \( V^+ = \{ x \in V \mid A_\theta x = x \} \).

2.3. Problem 1. Having the above results in hand, we are now able to give solutions to Problems 1 and 2 if \( f \) is not of inner type. As we saw in §1.4, we have reduced Problem 1 to the case where \( G_1 \) is a Steinberg group of type \( A_n, D_n, \) or \( E_6 \) and \( G \) is the corresponding Chevalley group.

Let \( G \) be a semisimple Chevalley group defined over \( K \) and let \( \Theta \) be chosen as above. Steinberg groups are just \( K \)-forms associated with continuous 1-cocycles in \( \Theta \). Indeed, let \( \{ \theta_\sigma \} \) be a continuous 1-cocycle in \( \Theta \), i.e., \( \theta_\sigma \theta_\tau = \theta_{\sigma \tau} \) for all \( \sigma, \tau \in \Gamma \) and let \( G_1 \) be the associated \( K \)-form. Let \( \Delta_1 \) be a fundamental system in \( G_1 \) corresponding to \( \Delta \). Then \( \Delta_\sigma^2 = \Delta_1 \) for all \( \sigma \in \Gamma \) and using this it can be shown that \( G_1 \) is Steinberg. Furthermore, there is a finite extension \( K_0 \) of \( K \) over which \( G_1 \) is a Chevalley group. The elements \( \sigma \in \text{Gal}(K_0/K) \) correspond to \( \theta_\sigma \in \Theta \) and if \( \sigma \neq 1 \), then \( \theta_\sigma \neq 1 \). This field \( K_0 \) is called the nuclear field of \( G_1 \) [5]. With the exception of \( D_4 \), \( K_0 \) is a quadratic extension of \( K \). As we have seen, \( \Theta = S_3 \) is \( G = D_4 \) and \( K = \mathbb{Q}_p \). Hence, in this case, \( [K_0/K] \) can be 2, 3, or 6. In stating the next theorem, we use the notation introduced in §2.1.

THEOREM II.1. Let \( G_1 \) be a Steinberg group of type \( A_n, D_n (n \neq 4), \) or \( E_6 \) defined over \( K \) (char \( K = 0 \)), let \( G \) be the corresponding Chevalley group defined over \( K \), and let \( f: G_1 \to G \) be the isomorphism between \( G_1 \) and \( G \) so that \( f \circ f^{-1} = \theta_0 \in \Theta \) for all \( \sigma \in \Gamma \). Assume that \((V, \rho, B)\) is an orthogonal representation of \( G \) defined over \( K \) such that \( \rho^*: G^* \to O(V, B) \) exists and is defined over \( K \). Then there is an orthogonal representation \((V_1, \rho_1, B_1)\) of \( G_1 \) defined over \( K \) such that \( \rho_1 \sim \rho \circ f \) and \( B_1 \) is related to \( B \) as follows:

1. \( \dim V_1 = \dim V \).
2. \( \Delta(B_1) = \alpha \Delta(B) \) if \( \det (\rho^*(\theta)) = 1 \) or \( \Delta(B_1) = \alpha \Delta(B) \) if \( \det (\rho^*(\theta)) = -1 \) where \( \alpha \in K^* \) is determined up to \((K^*)^2\) by the property that \( K_0 = K(\alpha^{1/2}) \) is the field fixed by \( \{ \sigma \mid \theta_\sigma = 1 \} \).
3. \( c(B_1) = c(B)(c_{\sigma}, z) \) where \( c_{\sigma, z} = 1 \) unless \( \theta_\sigma = \theta_z = \theta \). Then \( c_{\sigma, z} = \Delta^- \) if \( \det (\rho^*(\theta)) = 1 \) or \( \Delta^+ \) if \( \det (\rho^*(\theta)) = -1 \).

Proof. This proof is a slight generalization of that for Theorem I.1. For \( \sigma \in \Gamma \), \( (\rho \circ f)^\sigma = \rho \circ \theta_\sigma \circ f = A_\sigma (\rho \circ f) A_\sigma^{-1} \) where \( A_\sigma = \rho^*(\theta_\sigma) \). Hence, since \( \rho^* \) is defined over \( K \), \( A_\sigma^2 A_\tau = \rho^*(\theta_\sigma \theta_\tau) = \rho^*(\theta_{\sigma \tau}) = A_{\sigma \tau} \). There is an element \( X \in GL(V) \) such that \( A_\sigma = X^{-\sigma} X \) for all \( \sigma \in \Gamma \). We put \( \rho_1 = X(\rho \circ f) X^{-1} \) and \( B_1 = X^{-1} B X^{-1} \). Then it is immediate that \( \rho_1 \sim \rho \circ f \), \( \rho_1 \) is defined over \( K \), \( \rho_1 \) preserves \( B_1 \), and \( \dim V_1 = \dim V \). Since \( (\det X)^\sigma (\det X)^{-1} = \det A_\sigma = \pm 1 \) or \( -1 \), the result on \( \Delta(B_1) \) follows.

Finally, let \( h: C(B) \to M(t, \overline{K}) \) be an isomorphism of \( C(B) \) onto a total matrix algebra. (Again, if \( \dim V \equiv 1 \pmod{2} \), we should write \( C^+(B) \), but since nothing would
change in the proof below, we do not distinguish these cases.) For $\sigma \in \Gamma$, there is $Y_\sigma \in GL(t, \overline{K})$ such that $h^\sigma(x) = Y_\sigma h(x) Y_\sigma^{-1}$ for all $x \in C(B)$. The system $\{ Y_\sigma \}$ satisfies $Y_\sigma Y_\tau = b_{\sigma, \tau} Y_{\sigma \tau}$ with $b_{\sigma, \tau} \in \overline{K}^\times$ and $c(B) = (b_{\sigma, \tau})$.

Next, we use Lemma II.6. For setting $H = h \circ X^{-1}$ we have an isomorphism of $C(B_t)$ onto $M(t, \overline{K})$. For $\sigma \in \Gamma$, $H^\sigma \circ H^{-1} = I_{N_\sigma}$, where $N_\sigma = Y_\sigma h(\overline{A}_\sigma)$. Then $N_\sigma^2 N_\tau = b_{\sigma, \tau} c_{\sigma, \tau} N_{\sigma \tau}$. The elements $c_{\sigma, \tau}$ in $\overline{K}^\times$ are defined by $c_{\sigma, \tau} = c_{\tau, \sigma} = c_{\sigma, \tau}$ and (iii) follows on applying Corollary 2 of Lemma II.6. Hence, the theorem is proved.

Remark. In §2.2, we saw that if $\rho \circ \theta \sim \rho$, then $\rho^* : G^* \to O(V, B)$ exists and is defined over $K$. Furthermore, if $\rho_1$ is a representation of $G_1$ defined over $K$ and if $\rho$ is the representation of $G$ defined over $K$ such that $\rho \sim \rho_1 \circ f^{-1}$, then $\rho^*$ always exists since $\rho^* = \rho$ implies $\rho_1 \circ f^{-1} \circ \theta \sim \rho_1 \circ f^{-1}$. Therefore, Theorem II.1 is a complete reduction to the Chevalley case of the problem of finding invariant orthogonal forms on representations of Steinberg groups of type $A_n$, $D_n$ ($n \neq 4$), and $E_6$.

Groups of type $D_4$ present no new problems and we shall only outline the results.

(1) If $[K_0/K] = 2$, the situation is exactly as in Theorem II.1.

(2) If $[K_0/K] = 3$, let $\tau \in \text{Gal}(K_0/K)$ such that $\tau^3 = 1$. If $\rho^* : G^* \to O(V, B)$ exists, we have seen that $\det (A_\tau) = 1$. Furthermore, we may find $\overline{A}_\tau \in \text{Spin}(B)$ such that $\overline{A}_\tau^3 = 1$. So, $\dim V_\tau = \dim V_1, \Delta(B_\tau) = \Delta(B)$, and $c(B_\tau) = c(B)$.

(3) The case $[K_0/K] = 6$ combines the results of (1) and (2). Indeed let $\sigma, \tau \in \text{Gal}(K_0/K)$ have orders 2 and 3 respectively and let $\theta, \phi$ be the corresponding elements in $\Theta$. Then proceeding as in Theorem II.1, we get the following results: $\dim V_\tau = \dim V_1, \Delta(B_\tau) = \Delta(B)$ if $\det (\rho^*(\theta)) = 1$ and otherwise $\Delta(B_\tau) = \alpha \Delta(B)$ where $\alpha \in K^\times$ is such that $\sigma(\alpha^{1/2}) = -\alpha^{1/2}$. Finally $c(B_\tau) = c(B)$ (2-cocycle). The elements of this 2-cocycle are given in the following table:

<table>
<thead>
<tr>
<th></th>
<th>$\sigma$</th>
<th>$\tau$</th>
<th>$\sigma \tau$</th>
<th>$\sigma^2 \tau^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
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</tr>
<tr>
<td>$\sigma$</td>
<td>1</td>
<td>$\delta$</td>
<td>1</td>
<td>$\delta$</td>
</tr>
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<td>$\tau$</td>
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<td>$\sigma \tau$</td>
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</tr>
<tr>
<td>$\tau^2$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\sigma \tau^2$</td>
<td>1</td>
<td>$\delta$</td>
<td>1</td>
<td>$\delta$</td>
</tr>
<tr>
<td>$\sigma^2 \tau^2$</td>
<td>1</td>
<td>1</td>
<td>$\delta$</td>
<td>$\delta$</td>
</tr>
</tbody>
</table>

The element $\delta$ is $\Delta^+$ or $\Delta^-$ depending on whether $\det (\rho^*(\theta))$ is $-1$ or $+1$.

Remark. As in the remark above, we claim that we have reduced the case of Steinberg groups of type $D_4$ to that of Chevalley groups of type $D_4$. The verification is straightforward and we omit it.

2.4. Problem 2. Let $G_1$ be a connected group of type $A_n$, $D_n$, or $E_6$ defined over $K$ (we do not assume that $G_1$ is a Steinberg group) and let $G$ be the corresponding Chevalley group. We want to prove a theorem like Theorem I.2 under the assumption that $G$ and $G_1$ are isomorphic only (i.e. we do not require that the isomorphism be of inner type). The important fact here is that if $\rho^*$ exists, then $\rho^* : G^* \to O(V, B)$. 

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Let \( f: G_1 \to G \) be the isomorphism. Then for \( \sigma \in \Gamma \), \( f^{\sigma} \circ f^{-1} = \theta_{\sigma} \circ I_{g_{\sigma}} \) for some \( g_{\sigma} \in G \). If \((V, \rho, B)\) is an orthogonal representation of \( G \) defined over \( K \), then \((\rho \circ f)^{\sigma} = A_{\sigma}(\rho \circ f)A_{\sigma}^{-1} \) where \( A_{\sigma} = \rho(g_{\sigma})\rho^*(\theta_{\sigma}) \). Since \( A_{\sigma} \in O(V, B) \), we may prove Lemma 1.3 again. In the proof of Theorem 1.2, the only change is in \( \det(A_{\sigma}) = \det(\rho^*(\theta_{\sigma})) \) which may be \(-1\).

**Theorem II.2.** Let \( G_1 \) be a connected algebraic group of type \( A_n, D_n, \) or \( E_6 \) defined over \( K \) (char \( K = 0 \)), let \( G \) be the corresponding Chevalley group defined over \( K \), and let \( f: G_1 \to G \) be an isomorphism between \( G_1 \) and \( G \) such that \( f^{\sigma} \circ f^{-1} = \theta_{\sigma} \circ I_{g_{\sigma}} \) for all \( \sigma \in \Gamma \). Assume that \((V, \rho, B)\) is an orthogonal representation of \( G \) defined over \( K \) and assume that \( \rho^*: G^* \to O(V, B) \) exists and is defined over \( K \). Let \((V_1/K\#), \rho_1, F)\) be a skew-hermitian representation of \( G_1 \) defined over \( K \) where \( K\# = (\beta, \gamma) \) is a quaternion division algebra over \( K \). Set \( \text{Gal}(K(\beta^{1/2})/K) = \{1, \sigma\} \). Assume also that there is an absolutely irreducible representation \( \theta_1: \text{End}(V_1/K\#) \to \text{End}(V) \) defined over \( \overline{K} \) such that \( \theta_1(\rho_1(g)) = (\rho \circ f)(g) \) for all \( g \in G_1 \). Then the forms \( F \) and \( B \) are related as follows:

(i) \( \dim V_1 = 1/2 \dim V \).
(ii) \( \delta(F) = \Delta(B) \) if \( \det(\rho^*(\theta_{\sigma})) = 1 \) and \( \delta(F) = \beta \Delta(B) \) if \( \det(\rho^*(\theta_{\sigma})) = -1 \).

**References**


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