FLAT REGULAR QUOTIENT RINGS

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0. **Introduction and notation.** In this paper we study the condition that the maximal right quotient (MRQ) ring Q [10, p. 106] of a right nonsingular ring R with 1 is flat as a left R-module. It is known [11, p. 134] that if Q is the classical right quotient ring of R, then Q is flat as a left R-module. This is not always the case with the MRQ ring of R: in §2 we obtain an ideal theoretic characterization (Theorem 2.1) and a module theoretic characterization (Theorem 2.2) of a right nonsingular ring R, all of whose regular right quotient rings are flat as left R-modules; we also indicate the existence of a class of commutative rings R, whose singular ideal is zero and for which the maximal quotient ring is not R-flat.

Throughout this paper R denotes an associative ring with identity. A right R-module M is denoted M_R; all R-modules are unitary.

Let N_R and M_R be modules such that $N_R \subseteq M_R$. We say that N_R is large in M_R (M_R is an essential extension of N_R) if N_R intersects nontrivially every nonzero submodule of M_R. A right ideal I or R is large in R if I_R is large in R_R.

For any module M_R, $L(M_R)$ denotes the lattice of large submodules of M_R.

Let M_R be a module. We denote by $Z(M_R)$ the singular submodule of M_R. If for any $x \in M$ we set $(0 : x) = \{r \in R \mid xr = 0\}$, then

$$Z(M_R) = \{x \in M \mid (0 : x) \in L(R_R)\}.$$

In particular $Z(R_R)$ denotes the singular (two-sided) ideal of R_R and R is right nonsingular if $Z(R_R) = (0)$.

A ring S containing R is a right quotient ring of R, if $R_R \in L(S_R)$.

Let S be a right quotient ring of R. It is easy to see that if $I_R \in L(R_R)$, then $IS \in L(S_S)$ and if $A_S \in L(S_S)$, then $(A \cap R)_R \in L(R_R)$. For a module M_R, $M \otimes_R S$ is a right R- and S-module; it follows easily from the preceding observation that $Z((M \otimes_R S)_R) = Z((M \otimes_R S)_S)$ and hereafter we write $Z(M \otimes S)$. Also for any left R-module ${}_RN$ we write $M \otimes N$ for $M \otimes_R N$ if no ambiguity arises.

A ring is regular in the sense of Von Neumann [16]. Characterizations of flatness used frequently in this paper can be found in [10, pp. 132–135]. For all homological notions the reader may consult [2].

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1. **Preliminaries.** Let R be a ring. In the following proposition we record some well-known facts about large submodules. The proof of all but part (v) can be essentially found in [7], [8], [13]. We prove part (v) below.

PROPOSITION 1.1. *Let M and N be right R-modules. The following statements are then true*:

(i) *If A_R, $B_R \in L(M_R)$, then $A \cap B$, $A+B \in L(M_R)$.*

(ii) *If $A_R \in L(M_R)$ and $B_R \subseteq M_R$ such that $A_R \subseteq B_R$, then $B_R \in L(M_R)$.*

(iii) *If $A_R \subseteq M_R$, then there exists $B_R \subseteq M_R$ such that $A \cap B = (0)$ and $A+B$ is large in M_R.*

(iv) *If $f \in \mathrm{Hom}_R(M_R, N_R)$ and $A_R \in L(N_R)$, then $f^{-1}(A_R) \in L(M_R)$.*

(v) *If $f \in \mathrm{Hom}_R(M_R, N_R)$, f an epimorphism, $Z(N_R)=(0)$ and $A_R \in L(M_R)$, then $f(A_R) \in L(N_R)$.*

(vi) *If $A_1, \ldots, A_n \in L(M_R)$ and $x_1, \ldots, x_n \in M_R$ then*

$$I = \{r \in R \mid x_i r \in A_i,\ i = 1, \ldots, n\} \in L(R_R).$$

Proof. (v). Let $0 \neq n \in N$ and pick $m \in f^{-1}(n)$. The right ideal $I_R = \{r \in R \mid mr \in A\}$ is large in R by (vi), so that $nI_R \neq (0)$ as $Z(N_R)=(0)$. There exists, hence, $r \in I_R$ such that $nr \neq 0$ and since $mr \in A$ we have $f(mr) = nr \in f(A)$. Q.E.D.

In [3, p. 459] Chase calls a right R-module M finitely related if there exists an exact sequence of right R-modules $0 \to K_R \to F_R \to M_R \to 0$ with F_R free and both F_R and K_R finitely generated. We generalize this concept below:

DEFINITION 1.2. A right R-module M is essentially finitely generated (M_R is EFG) if there exist finitely many elements $m_1, \ldots, m_n$ of M_R such that $\sum m_i R \in L(M_R)$.

DEFINITION 1.3. A right R-module M is essentially finitely related (M_R is EFR) if there exists an exact sequence $0 \to K_R \to F_R \to M_R \to 0$ with F_R finitely generated free and K_R EFG.

If M_R is EFR then any exact sequence $0 \to K_R \to F_R \to M_R \to 0$ will have the property that K_R is EFG whenever F_R is finitely generated free; this follows from a result of Schanuel's contained in [15, p. 369].

To obtain our main results (§2) we need some lemmas:

LEMMA 1.4. *Let $0 \to K_R \xrightarrow{i} M_R \xrightarrow{f} N_R \to 0$ be an exact sequence of right R-modules where $K = \ker f$ and i is the inclusion map. We have:*

(a) *If K_R, N_R are EFG and $Z(M_R)=(0)$, then M_R is EFG.*

(b) *If $Z(N_R)=(0)$ and M_R is EFG, then N_R is EFG.*

Proof. (b) follows easily from Proposition 1.1(v), so we prove (a):

Let $\{k_i : i=1, \ldots, n\} \subset K_R$, $\{n_j : j=1, \ldots, l\} \subseteq N_R$ (finite sets), such that $\sum k_i R \in L(K_R)$ and $\sum n_j R \in L(N_R)$. Choose $m_j \in f^{-1}(n_j)$ for $j=1, \ldots, l$ and set $M'_R = \sum_i k_i R + \sum_j m_j R$. We show next that $M'_R \in L(M_R)$. Let $0 \neq m \in M$. If $f(m)=0$ then $m \in K$ and there exists $r \in R$ such that $0 \neq mr \in \sum k_i R \subset M'$. Assume, next, $f(m) \neq 0$. There exists $r \in R$ such that $0 \neq f(m)r = n_1 r_1 + \cdots + n_l r_l \in \sum_j n_j R$. Set $m' = m_1 r_1 + \cdots +$

$m_l r_l$; clearly $m' \neq 0$ and $f(mr-m')=0$. If $mr-m'=0$ then $0 \neq mr = m' \in M'$, so assume $mr-m' \neq 0$. Since $mr-m' \in K$ it follows by Proposition 1.1(vi) that $I = \{x \in R \mid (mr-m')x \in \sum_i k_i R\} \in L(R_R)$ and since $Z(M_R)=(0)$, there exists $t \in I$ such that $(mr)t \neq 0$. Thus $(mr-m')t = k \in \sum_i k_i R$ and $0 \neq m(rt) = m' + k \in M'$. We have shown that for every $0 \neq m \in M$, $M' \cap mR \neq (0)$, hence $M' \in L(M_R)$. Q.E.D.

LEMMA 1.5. *Suppose R has a right quotient ring Q and K_R is an R-module such that $Z(K_R)=(0)$. If $K \otimes_R Q$ is finitely generated as a right Q-module, then K_R is EFG.*

Proof. Let $\{k_i \otimes 1 : i=1, \ldots, n, k_i \in K\}$ be a set of generators for $K \otimes Q$. Let L be the submodule of K generated by the elements $k_1, \ldots, k_n$. We show next that L_R is large in K_R. Since $Z(K_R)=(0)$, the sequence $(0) \to K \to K \otimes Q$ is exact [13, Proposition 2.2], where the mapping is the canonical one: $k \to k \otimes 1$. Now let $0 \neq k \in K$; we have $k \otimes 1 = \sum_{i=1} (k_i \otimes 1) q_i$. The right ideal $I_R = \{r \in R \mid q_i r \in R, i = 1, \ldots, n\}$ is large in R by Proposition 1.1 (vi) and since $k \otimes 1$ is a nonsingular element of $K \otimes Q$ (being the image of one under monomorphism) it follows that $(k \otimes 1) I_R \neq (0)$. Hence there is $r \in I_R$ such that $0 \neq (kr) \otimes 1 = (k \otimes 1) r = \sum_i k_i \otimes q_i r = \sum_i k_i (q_i r) \otimes 1$. It follows that $0 \neq kr = \sum k_i (q_i r) \in L$ and thus L_R is large in K_R. Q.E.D.

The following lemma due to Kaplansky [9, Lemma 4, p. 376] is stated here because of its frequent use in the sequel:

LEMMA 1.6 (KAPLANSKY). *If P is a projective right (left) module over a regular ring R, then any finitely generated submodule of P is a direct summand of P.*

We shall need the following characterization of a module M_R which is EFR.

PROPOSITION 1.7. *Let R be a ring with $Z(R_R)=(0)$ and a right quotient ring Q which is regular. A finitely generated right R-module M is EFR if and only if $M \otimes Q$ is projective as a right Q-module.*

Proof. *Only if.* Let $0 \to K_R \xrightarrow{i} F_R \xrightarrow{f} M_R \to 0$ be an exact sequence with F_R finitely generated free and K_R EFG. Let $k_1, \ldots, k_n$ be finitely many elements of K such that $K' = \sum_{i=1}^n k_i R \in L(K_R)$. It suffices to show that the exact sequence $K \otimes Q \xrightarrow{i^*} F \otimes Q \xrightarrow{f^*} M \otimes Q \to 0$ splits or equivalently $i^*(K \otimes Q)$ is a direct summand of $F \otimes Q$. Let A be the right Q-submodule of $F \otimes Q$ generated by $k_i \otimes 1$, $i=1, \ldots, n$ in $F \otimes Q$. By Lemma 1.6 A_Q is a direct summand of $F \otimes Q$. The proof of this part will be complete if we show that A_Q is large in $i^*(K \otimes Q)$. Thus let $0 \neq \sum_i x_i \otimes q_i \in i^*(K \otimes Q)$, $x_i \in K$, $q_i \in Q$. By Proposition 1.1(vi), $I = \{r \in R \mid q_i r \in R, \text{ all } i\} \in L(R_R)$ and since $Z(i^*(K \otimes Q))=(0)$ as $Z(F \otimes Q)=(0)$, there exists $t \in I$ such that $0 \neq (\sum_i x_i \otimes q_i)t = \sum_i x_i \otimes (q_i t) = \sum x_i (q_i t) \otimes 1 \in A_Q$. It follows that A_Q is large in $i^*(K \otimes Q)$.

If part. Using the notation above we see that since $M \otimes Q$ is right Q-projective, $i^*(K \otimes Q)$ is a direct summand of $F \otimes Q$ and hence it is finitely generated as a

right Q-module. Let $\{x_i \otimes 1 : x_i \in K, i=1,\ldots,n\}$ be a set of generators for $i^*(K \otimes_R Q)$. The module $K'_R = \sum_i x_i R$ is, then, large in K_R. Indeed let $0 \neq k \in K$. As $K \subseteq F$ it is clear that $k \otimes 1 \neq 0$ in $F \otimes Q$ and hence $0 \neq k \otimes 1 = \sum (x_i \otimes 1) q_i = \sum_i x_i \otimes q_i \in i^*(K \otimes Q)$. As before $I = \{r \in R \mid q_i r \in R, \text{ all } i\} \in L(R_R)$ so there exists $t \in I$ such that $0 \neq (k \otimes 1)t = \sum_i x_i(q_i t) \otimes 1$. In particular $[kt - \sum x_i(q_i t)] \otimes 1 = 0$ and hence $0 \neq kt = \sum_i x_i(q_i t) \in K'_R$. We have $K'_R \in L(K_R)$ and hence K_R is EFG or M_R is EFR. Q.E.D.

LEMMA 1.8. *Let R be a ring with $Z(R_R)=(0)$ and let Q be a right quotient ring of R. If I is any right ideal of R, then $Z(I \otimes Q) = \ker(I \otimes Q \to R \otimes Q)$.*

Proof. If A and B are right R-modules and $f \in \mathrm{Hom}_R(A_R, B_R)$, then it is clear that $f(Z(A_R)) \subseteq Z(B_R)$. Now $R \otimes Q \cong Q$ and $Z(Q_R)=(0)$ as $Z(R_R)=(0)$, so $Z(I \otimes Q) \subseteq \ker(I \otimes Q \to R \otimes Q)$.

Conversely let $u = \sum x_i \otimes q_i \in I \otimes Q$, $x_i \in I$, $q_i \in Q$ such that $\sum x_i \otimes q_i \to \sum x_i q_i = 0$. For every $t \in J = \{r \in R \mid q_i r \in R, \text{ all } i\}$ we have $(\sum x_i \otimes q_i)t = \sum x_i(q_i t) \otimes 1 = 0$. Since $J \in L(R_R)$ we have $(0 : u) \in L(R_R)$, by Proposition 1.1, and hence $u \in Z(I \otimes Q)$. Thus $\ker(I \otimes Q \to R \otimes Q) \subseteq Z(I \otimes Q)$ and this completes the proof of the lemma. Q.E.D.

We need the following characterization of flatness of a regular right quotient ring:

THEOREM 1.9. *For any ring R with $Z(R_R)=(0)$, the following statements are equivalent, for any regular right quotient ring Q of R:*

(a) *Q is flat as a left R-module.*

(b) *$Z(I \otimes Q)=(0)$ for every finitely generated right ideal I.*

(c) *$I \otimes Q$ is Q-projective for every finitely generated right ideal of R.*

Proof. (a) *implies* (b). By Lemma 1.8 and flatness of ${}_RQ$.

(b) *implies* (c). For any finitely generated right ideal I or R, the sequence $0 \to I \otimes Q \to R \otimes Q$ is exact by Lemma 1.8 and (b). Now $I \otimes Q$ is projective as a right Q-module follows from Lemma 1.6.

(c) *implies* (a). For any finitely generated right ideal I of R, $I \otimes Q$ is a submodule of a Q-free module [2, p. 6], hence $Z(I \otimes Q)=(0)$ as $Z(Q_Q)=(0)$. Now (a) follows from Lemma 1.8. Q.E.D.

The proof of the following is essentially contained in [6, p. 953]; it is given here for completeness.

LEMMA 1.10. *If M is a flat left R-module and I, J are right ideals of R, then $(I \cap J)M = IM \cap JM$ (the right-left symmetric of this also holds).*

Proof. The image of $(I \cap J) \otimes M$ in $(I+J) \otimes M$ is equal to the intersection of the images of $I \otimes M$ and $J \otimes M$ in $(I+J) \otimes M$, all images under homomorphisms induced by the obvious inclusions [1, Lemme 7, p. 32]. Since ${}_RM$ is flat, the mapping $\sum a_i \otimes m_i \to \sum a_i m_i$ from $A \otimes M$ to AM, for any right ideal A_R of R,

is an isomorphism [10, p. 132] (of abelian groups); under such an isomorphism from $(I+J)\otimes M$ to $(I+J)M$ we obtain $(I\cap J)M=IM\cap JM$. Q.E.D.

The following is a generalization of a lemma by Chase [3, Lemma 2.2, p. 462]:

LEMMA 1.11. *Let M be any right R-module, $N=\sum_{i=1}^{n} m_iR$ a finitely generated submodule of M_R and $m\in M$. Set $N^*=N+mR$ and $F=x_1R\oplus\cdots\oplus x_nR\oplus x_{n+1}R$, a free right R-module. Let $f\colon F\to N^*$ be the R-epimorphism defined $f(x_i)=m_i$, $1\leqq i\leqq n$ and $f(x_{n+1})=m$. Set $K=\ker f$, $F'=x_1R\oplus\cdots\oplus x_nR\subseteq F$ and $K'=K\cap F'$. There exists an R-epimorphism $g\colon K\to(N:m)$, where $(N:m)=\{r\in R\mid mr\in N\}$, such that* $\ker g=K'$.

Proof. If $u\in K$, then $u=x_1r_1+x_2r_2+\cdots+x_nr_n+x_{n+1}r_u$. Since $0=f(u)=m_1r_1+\cdots+m_nr_n+mr_u$, it follows that $mr_u\in N$, hence that $r_u\in(N:m)$. Define g by $g(u)=r_u$ for each $u\in K$. Q.E.D.

2. **Flat regular quotient rings.** We can now state and prove the first main result of this paper: an ideal theoretic characterization of any ring R with $Z(R_R)=(0)$ for which any regular right quotient ring is flat as a left R-module. Every ring R that satisfies $Z(R_R)=(0)$ has a regular right quotient ring, namely the MRQ ring [7].

THEOREM 2.1. *Let R be a ring with $Z(R_R)=(0)$. If Q is a regular right quotient ring of R, then the following statements are equivalent:*

(a) *Q is flat as a left R-module.*

(b) *Every finitely generated right ideal I of R is EFR.*

(c) *For any finitely generated right ideal I of R and any element a of R, $(I:a)=\{r\in R\mid ar\in I\}$ is EFG.*

(d) *For any $a\in R$, $(0:a)=\{r\in R\mid ar=0\}$ is EFG and if I_1, I_2 are finitely generated right ideals of R, then $I_1\cap I_2$ is EFG.*

Proof. (a) *implies* (d). Let $a\in R$ and $a^*\colon R\to aR$ be the R-epimorphism defined by $a^*(x)=ax$ for each $x\in R$. The sequence $0\to(0:a)\xrightarrow{i}R\xrightarrow{a^*}aR\to 0$ is then exact, where i is the inclusion map. It follows from (a) and [10, Propositon 1, p. 132] that the following sequence is also exact.

$$0\longrightarrow(0:a)Q\xrightarrow{\;i\;}Q\longrightarrow aRQ\longrightarrow 0. \tag{1}$$

By regularity of Q there exists idempotent e of Q such that $aRQ=aQ=eQ$ [16]. It follows that (1) splits and, hence, there exists idempotent f of Q such that $(0:a)Q=fQ$. Since $(0:a)Q\cong(0:a)\otimes Q$ it follows by Lemma 1.5 that $(0:a)$ is EFG.

To show the second part of (d) observe that I_jQ, $j=1,2$, are finitely generated right ideals of Q, hence $I_jQ=e_jQ$ for idempotents e_j, $j=1,2$ of Q [16]. From Lemma 1.10 and (a) we have $(I_1\cap I_2)Q=I_1Q\cap I_2Q=e_1Q\cap e_2Q$. Regularity of Q now gives $(I_1\cap I_2)Q=e_3Q$ for some idempotent e_3 of Q and Lemma 1.5 gives that $I_1\cap I_2$ is EFG. We have (d).

(d) *implies* (c). For any finitely generated right ideal I of R and element a of R the sequence $0 \to (0:a) \xrightarrow{i} (I:a) \xrightarrow{a^*} I \cap aR \to 0$ is exact ($a^*(x) = ax$ for each $x \in (I:a)$). From (d) we have $(0:a)$ and $I \cap aR$ are EFG so that $(I:a)$ is EFG follows from Lemma 1.4(a). We have (c).

(c) *implies* (b). Let $I = \sum_{i=1}^{n} a_i R$ be a finitely generated right ideal of R. We proceed by induction on n. For $n=1$, $I = a_1 R$ and I is EFR follows from the exact sequence $0 \to (0:a_1) \to R \xrightarrow{a_1^*} a_1 R \to 0$ where $(0:a_1)$ is EFG by (c). Assume now that A is EFR whenever A is a right ideal of R generated by k elements where $k < n < \infty$ and $n > 1$. Let $F = x_1 R \oplus \cdots \oplus x_n R$, a free right R-module, $F' = x_1 R \oplus \cdots \oplus x_{n-1} R \subseteq F$ and $f: F \to I$ the R-epimorphism defined by $f(x_i) = a_i$, $1 \leqq i \leqq n$. Let $K = \ker f$, $I' = a_1 R + \cdots + a_{n-1} R$ and $\bar{f}$ the restriction of f to F'. The sequences

$$0 \longrightarrow K \xrightarrow{i} F \xrightarrow{f} I \longrightarrow 0 \tag{2}$$

and

$$0 \longrightarrow K' \xrightarrow{i'} F' \xrightarrow{\bar{f}} I' \longrightarrow 0 \tag{3}$$

are exact where i and i' are inclusion maps. The following exact sequence is obtained from Lemma 1.11:

$$0 \longrightarrow K' \longrightarrow K \xrightarrow{g} (I' : a_n) \longrightarrow 0. \tag{4}$$

Now K' is EFG in (3) follows from induction assumption and the remark following Definition 1.3. In (4) $(I' : a_n)$ is EFG by (c) so K is EFG follows from Lemma 1.4(a). Now (2) gives that I is EFR. We have (b).

(b) *implies* (a). If I is any finitely generated right ideal of R (b) and Proposition 1.2 give that $I \otimes Q$ is projective as a right Q-module, hence $Z(I \otimes Q) = (0)$. Now (a) follows from Theorem 1.9.

The proof of the theorem is now complete.

REMARKS. It is easy to check that statements (c) and (d) are respectively equivalent to the following statements:

(c′) If I is any right ideal of R which is EFG and a is any element of R, then $(I : a)$ is EFG.

(d′) $(0 : a)$ is EFG for any element a of R. If I_1 and I_2 are EFG right ideals of R, then $I_1 \cap I_2$ is EFG.

Thus in statements (c) and (d) of the preceding theorem, "finitely generated" can be replaced by EFG.

The following are examples of rings belonging to the class of rings of Theorem 2.1:

(i) A right Noetherian ring R satisfies statement (d), for example, and if $Z(R_R) = (0)$ then its MRQ ring is regular, hence flat as a left R-module.

(ii) If R is right semihereditary then $Z(R_R) = (0)$ [5, p. 426] and any finitely generated right ideal of R is EFR since it is projective.

(iii) If R is finite-dimensional as a right R-module [13] and if $Z(R_R)=(0)$, then any right ideal of R is EFG [13, Theorem 1.3] so (c), for example, holds. Such rings are characterized in [13] and are those which have a semisimple (with d.c.c.) MRQ ring Q. Thus the semisimple MRQ ring Q of a ring R is flat as a left R-module [13, Theorem 2.7, p. 119].

Finally, we remark that statements (b), (c) and (d) of the theorem are internal characterizations so that if R with $Z(R_R)=(0)$ has a regular right quotient ring, which is flat as a left R-module, then every regular right quotient ring of R, in particular the MRQ ring of R, is flat as a left R-module.

We next give a module-theoretic characterization of the rings of Theorem 2.1.

THEOREM 2.2. *Let R be a ring with $Z(R_R)=(0)$. If Q is a regular right quotient ring of R, the following statements are equivalent:*

(a) *${}_RQ$ is flat.*

(b) *If M_R is a finitely generated submodule of F_R, free module of finite rank, then M_R is* EFR.

(c) *If M_R is a finitely generated submodule of F_R, free of finite rank, and $x \in F$, then $(M : x)=\{r \in R \mid xr \in M\}$ is EFG.*

(d) Let *F_R be a free module of finite rank. For any $x \in F$, $(0 : x)=\{r \in R \mid xr=0\}$ is EFG, If M_1 and M_2 are finitely generated submodules of F_R, then $M_1 \cap M_2$ is EFG.*

Proof. (a) *implies* (d). Let F_R be free of rank n and $x=(a_1, \ldots, a_n) \in F$, $a_i \in R$. Since $(0 : x) = \bigcap_{i=1}^n (0 : a_i)$, it follows from Theorem 2.1(d′) that $(0 : x)$ is EFG.

To show the second part of (d) we see that (a) implies that the sequences $0 \to M_i \otimes Q \to F \otimes Q$, $i=1, 2$, and $0 \to (M_1+M_2) \otimes Q \to F \otimes Q$ are exact. By Lemma 1.6, the right Q-modules $M_i \otimes Q$, $i=1$, 2 and $(M_1+M_2) \otimes Q$ are Q-projective. A further application of Lemma 1.6 gives that the exact sequence $0 \to M_2 \otimes Q \to (M_1+M_2) \otimes Q \to M_1+M_2/M_2 \otimes Q \to 0$ splits. It follows that $M_1/M_1 \cap M_2 \otimes Q$ is projective as a right Q-module since $M_1/M_1 \cap M_2 \otimes Q \cong M_1+M_2/M_2 \otimes Q$. Now the exact sequence $0 \to (M_1 \cap M_2) \otimes Q \to M_1 \otimes Q \to M_1/M_1 \cap M_2 \otimes Q \to 0$ splits and $(M_1 \cap M_2) \otimes Q$ is finitely generated as a right Q-module since $M_1 \otimes Q$ is. $M_1 \cap M_2$ is EFG, follows from Lemma 1.5, as $Z(M_1 \cap M_2)=(0)$. We have (d).

(d) *implies* (c). For any $x \in F$ define $x^*: (M : x) \to M \cap xR$ by $x^*(r)=xr$ for each $r \in (M : x)$, clearly an R-epimorphism. We obtain the exact sequence $0 \to (0 : x) \xrightarrow{i} (M : x) \xrightarrow{x^*} M \cap xR \to 0$ and (c) follows from Lemma 1.4(a), since $(0 : x)$ and $M \cap xR$ are both EFG by (d) and $Z((M : x))=(0)$ as $(M : x) \subseteq R_R$.

(c) *implies* (b). Let $M_R=\sum_{i=1}^n m_iR$ be a finitely generated submodule of F_R. We proceed by induction on n, the number of generators of M. For $n=1$ we have $M_R=m_1R \subseteq F$ and $(0 : m_1)$ is EFG by (c). Now M_R is EFR follows from the exact

sequence $0 \to (0 : m_1) \xrightarrow{i} R \xrightarrow{m_1^*} m_1R \to 0$ where $m_1^*(r) = m_1 r$ for each $r \in R$. Let $n > 1$ and assume that L_R is EFR whenever $L_R \subseteq F_R$ and L_R is generated by k elements, $k < n < \infty$. The argument used in showing "(c) implies (b)" for Theorem 2.1 can be applied here to show that $M_R = \sum_{i=1}^{n} m_i R$ is EFR.

(b) *implies* (a). Statement (b) of Theorem 2.1 is a special case of (b) of Theorem 2.2, so we have (a).

This completes the proof of the theorem.

We close with a word about a class of commutative nonsingular rings R for which no regular quotient ring is R-flat.

If R is a commutative ring and M a maximal ideal of R, we denote by R_M the quotient ring of R with respect to the multiplicatively closed set $R - M$ [2, p. 141] (and [4]).

Let $\{R_\alpha : \alpha \in A\}$ be a collection of valuation domains not all fields and all containing a common field K. It follows from a result of Nagata [12] that there exists a ring T with the following properties:

(a) For each $\alpha \in A$ there exists a maximal ideal M_α of T such that $T_{M_\alpha} \cong R_\alpha$.

(b) For every maximal ideal M of T, $T_M \cong K$ or $T_M \cong R_\alpha$ for some $\alpha \in A$.

(c) T is its own total quotient ring [4, p. 115].

From these we further deduce the following properties of T:

(1) T is not regular [4, Theorem 1, p. 110].

(2) T is not semihereditary [4, Theorem 2, p. 113].

(3) If GWD (T) denotes the global weak dimension of T, then GWD $(T) \leqq 1$ [4, Proposition 11, p. 116].

It follows from (3) and [5, p. 426] that $Z(T) = (0)$. Now any commutative ring T with GWD $(T) \leqq 1$ and maximal quotient ring T-flat, is semihereditary [14, Theorem 2.10]. From this and (2) it follows that the maximal quotient ring of T is not T-flat. From a remark following Theorem 2.1, no regular quotient ring of T is T-flat.

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