ON COMMON FIXED POINTS OF COMMUTING CONTINUOUS FUNCTIONS ON AN INTERVAL

BY

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This paper offers two methods of constructing commuting pairs of continuous functions (i.e. \( f, g \) such that \( f(g(x))=g(f(x)) \)) which map \([0, 1]\) to itself without common fixed points. Any such pair will be called "a solution to the commuting function problem".

PART I

Lemma 1. Let \((f_n \mid n \in \mathbb{N}), (g_n \mid n \in \mathbb{N})\) be two uniformly convergent sequences of continuous functions from \([0, 1]\) to itself with limits \( f, g \) respectively. If \( f_n g_{n+1} = g_n f_{n+1} \) for each \( n \in \mathbb{N} \), then \( f, g \) commute.

Lemma 2. Let \( h \) be a piecewise linear nowhere constant function (i.e. the derivative of \( h \), denoted \( Dh \), is nowhere zero) defined on \( I \); and let \( A = h(I) \) be a finite set which contains the image under \( h \) of all points in the interior of \( I \) at which \( Dh \) does not exist. If \( r, s \) are consecutive (with respect to the natural order on \( R \)) in \( h^{-1}(A) \), then \( h(r), h(s) \) are consecutive in \( A \).

Proposition 1. Let \((f_n \mid n \in \mathbb{N}), (g_n \mid n \in \mathbb{N})\) be sequences of piecewise linear functions from \([0, 1]\) to itself such that either \( |x-f_2(x)| > 1/6 \) or \( |x-g_2(x)| > 1/6 \) for each \( x \in [0, 1] \). Then the limits of the sequences \((f_n \mid n \in \mathbb{N}), (g_n \mid n \in \mathbb{N})\) form a solution to the commuting function problem provided that there exists a sequence \((A_n \mid n \in \mathbb{N})\) of finite subsets of \([0, 1]\), and for each \( n \in \mathbb{N} \), the following properties are valid:

\[ P_1(n): f_n g_{n+1} = g_n f_{n+1}; \]
\[ P_2(n): |Df_n(x)| \geq 3 \text{ and } |Dg_n(y)| \geq 3 \text{ wherever the derivatives exist}; \]
\[ P_3(n): \text{if either } Df_n(x) \text{ or } Dg_n(x) \text{ does not exist, then } x \in A_n; \]
\[ P_4(n): \text{for each pair of consecutive points } r, s \text{ of } A_{n+1} \text{ with } r < s, \]

\[ f_n([r, s]) = f_{n+1}([r, s]) \text{ and } g_n([r, s]) = g_{n+1}([r, s]); \]

\[ P_5(n): f_n^{-1}(A_n) = A_{n+1} = g_n^{-1}(A_n); \text{ and} \]
\[ P_6(n): 0, 1 \in A_n. \]

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Proof. For any $m \in \mathbb{N}^*$ and for any pair of consecutive points $r, s$ of $A_{m+1}$, there exists by $P_6(n)$ and by Lemma 2 a pair of consecutive points $r', s'$ of $A_m$ (equal to $g_m(r), g_m(s)$ respectively) such that \( \frac{1}{2} |f_{m-1}(r') - f_{m-1}(s')| \geq |f_m(r) - f_m(s)| \) since:

\[
\frac{1}{2} \cdot |f_{m-1}(r') - f_{m-1}(s')| = \frac{1}{2} \cdot |g_{m-1}(f_m(r)) - g_{m-1}(f_m(s))| \\
= \frac{1}{2} \cdot |f_m(r) - f_m(s)| \text{ [by } P_4(m-1)] \\
= \frac{1}{2} \cdot |f_m(r) - f_m(s)| \cdot \frac{|g_{m-1}(f_m(r)) - g_{m-1}(f_m(s))|}{|f_m(r) - f_m(s)|} \\
\geq \frac{1}{2} |f_m(r) - f_m(s)| \text{ [by } P_3(m), \text{ Lemma 2, } P_3(m-1), \text{ and } P_2(m-1)] \\
= |f_m(r) - f_m(s)|. 
\]

For every $n \in \mathbb{N}$ and for every $x \in [0, 1]$, there exist two consecutive points $r, s$ of $A_{n+1}$ such that $r \leq x \leq s$ (since $A_{n+1}$ is finite and contains 0, 1 by $P_6(n+1)$), and so it follows that \( |f_n(x) - f_{n+1}(x)| \leq |f_n(r) - f_n(s)| \leq (1/3)^n \) by $P_6(n)$, $P_6(n+1)$, and by iterated use of the result established in the preceding sentence. Therefore \( \|f_n - f_{n+1}\| \leq 3^{-n} \). Since \( (f_n \mid n \in \mathbb{N}), (g_n \mid n \in \mathbb{N}) \) have symmetric roles, we also have for each $n \in \mathbb{N}$ \( g_n - g_{n+1} \leq 3^{-n} \).

Now, \( (f_n \mid n \in \mathbb{N}), (g_n \mid n \in \mathbb{N}) \) are uniformly convergent sequences, since for every pair $n, m \in \mathbb{N}$ with $n \leq m$, we have:

\[
\|f_n - f_m\| \leq \sum_{i=n}^{\infty} \|f_i - f_{i+1}\| \leq \sum_{i=n}^{\infty} 3^{-i} = (2 \cdot 3^{n-1})^{-1};
\]

and similarly \( \|g_n - g_m\| \leq (2 \cdot 3^{n-1})^{-1} \). From these inequalities, the limits of the two sequences of functions have no common fixed point since \( \|f_2 - \lim_{n \to \infty} f_n\| \leq 1/6 \) and \( \|g_2 - \lim_{n \to \infty} g_n\| \leq 1/6 \). Also, the limits commute by Lemma 1.

Notation. If $f$ is a piecewise linear function defined on $I$, then let $B(f)$ denote the set of all points in the interior of $I$ at which $f$ has no derivative (i.e. at which $Df$ does not exist).

Lemma 3. Let $f$ be a piecewise linear function defined on $I$; let $g$ be a function defined on $I$; let $h$ be a linear function with range $I$; and let $l$ be a nonconstant linear function defined on $f(I)$. Then $gh(B(fh)) = g(B(f))$, $B(1f) = B(f)$, and $h^{-1}(B(f)) = B(f(h))$.

Definition. For any piecewise linear function $f$, if there exists an $s \in \mathbb{R}^+$ such that $|Df(x)| = s$ for each $x$ for which $Df(x)$ exists, then $f$ has a derivative of constant absolute value which we will denote by $f$ has DCAV; also, we will denote $s$ by $\text{slope } f$.

Notation. If $I$ is an interval with endpoints $r, s$, then let $T_r: I \to I$ denote the linear function defined for each $x \in I$ by $T_r(x) = r + s - x$.

Proposition 2. Let $f: [a, a'] \to [b, b']$ and $g: [c, c'] \to [b, b']$ be piecewise linear functions. If $\{f(a), f(a')\} = \{b, b\} = \{g(c), g(c')\}$ and if $f, g$ have DCAV, then there
exist piecewise linear functions \( j: [d, d'] \to [a, a'] \) and \( k: [d, d'] \to [c, c'] \) satisfying the properties: \( j(d) = a, j(d') = a' \); \( k(d), k(d') = \{c, c'\}; j, k \) have DCAV; \( j(B(k)) \in B(f), k(B(j)) \in B(g) \); and \( fj = gk \). The following diagram commutes:

\[
\begin{array}{ccc}
[d, d'] & \xrightarrow{j} & [a, a'] \\
\downarrow{k} & & \downarrow{f} \\
[c, c'] & \xrightarrow{g} & [b, b']
\end{array}
\]

**Proof.** First Proposition 2 will be proved under the additional assumption that \( f(a) = b = g(c) \) (and hence \( f(a') = b' = g(c') \)). We proceed by induction on the number of distinct points in \( B(f) \cup B(g) \). Pick an interval \([d, d']\).

If \( B(f) \cup B(g) \) is empty, then \( f \) and \( g \) are linear, so define \( j, k \) to be the unique linear maps on \([d, d']\) such that \( j(d) = a, j(d') = a', k(d) = c, \) and \( k(d') = c' \). Observe that \( fj \) and \( gk \) are linear functions on \([d, d']\) such that \( fj(d) = b = gk(d) \) and \( fj(d') = gk(d') \). Therefore \( fj = gk \); also \( B(j) \) and \( B(k) \) are empty. So \( j, k \) satisfy Proposition 2.

Now let \( n \in N \) and assume the induction hypothesis: Proposition 2 is valid whenever \( f(a) = b = g(c) \) and \( B(f) \cup B(g) \) has no more than \( n \) elements. Suppose we are given \( f, g \) satisfying the hypothesis of Proposition 2 such that \( f(a) = b = g(c) \) and \( B(f) \cup B(g) \) has \( n + 1 \) elements. Since \( B(f) \cup B(g) \) is finite and nonempty, let \( b'' \) denote the smallest element of \( f(B(f)) \cup g(B(g)) \). Since \( f \) and \( g \) play symmetric roles, assume without loss of generality that \( b'' \in f(B(f)) \). Let \( a'' \) denote any point in \( B(f) \) such that \( f(a'') = b'' \). \( B(f) \) equals the set of points which are local minimum or local maximum points for \( f \) except for \( a, a' \), because \( f \) has DVAC; also, \( a'' \) is a local minimum for \( f \) since \( f \) has a local maximum at the endpoint \( a' \) of \([a, a']\). Now \( f \) has local minima at \( a \) and \( a'' \), so \( f \) must have a local maximum between \( a \) and \( a'' \), i.e. \( B(f) \mid_{[a, a'']} \) is nonempty. Let \( b'' \) denote the maximum element of \( f(B(f) \mid_{[a, a'']} ) \), and let \( a'' \) be any element of \( B(f) \mid_{[a, a'']} \) such that \( f(a'') = b'' \). Let \( c'' \) be the minimum element of \([c, c']\) such that \( g(c'') = b'' \), and let \( c'' \) be the maximum element of \([c, c'']\) such that \( g(c'') = b'' \). Let \( d'' = (2d + d')/3 \) and let \( d'' = (d + 2d')/3 \). Observe that the induction hypothesis is applicable to each of the three pairs of functions: (i) \( f \mid_{[a, a'']} \), \( g \mid_{[c, c'']} \); (ii) \( f \mid_{[a, a'']} \), \( g \mid_{[c, c'']} \); and (iii) \( f \mid_{[a, a'']} \), \( g \mid_{[c, c'']} \). Therefore, for (i) there exists \( j', k' \) defined on \([d, d'']\), for (ii) there exists \( j'', k'' \) defined on \([d', d'']\), and for (iii) there exists \( j'', k'' \) defined on \([d'', d'']\) satisfying Proposition 2 (as stated for \( f, g, j, k \) respectively). See Figure 1.

Now define the two points \( d^*, d^{**} \):

\[
d^* = d + (d' - d) \cdot \frac{\text{slope } j'}{\text{slope } j' + \text{slope } j''}.
\]

\[
d^{**} = d + (d' - d) \cdot \frac{\text{slope } j'' + \text{slope } j'''}{\text{slope } j' + \text{slope } j'' + \text{slope } j'''}.
\]
Let $S'$ be the linear function defined from $[d, d^*]$ onto $[d, d']$ such that $S'(d) = d$, let $S''$ be the linear function defined from $[d^*, d^{**}]$ onto $[d'', d']$ such that $S''(d^*) = d''$, and let $S'$ be the linear function defined from $[d^{**}, d']$ onto $[d'', d']$ such that $S''(d') = d'$. Let $j = j'S' \cup T_{a' \rightarrow a} S'' \cup j'''S''$ (considering a function to be a set of ordered pairs), and let $k = k'S' \cup k''S'' \cup k'''S''$. By checking the points $d^*, d^{**}$, it can be seen that $j, k$ are well defined functions (i.e., $j(d^*) = a^*, j(d^{**}) = a^*$, $k(d^*) = c^*$, and $k(d^{**}) = c^*$). It is clear that $j, k$ are piecewise linear functions. Also, $j, k$ have DCAV with

slope $j = \frac{1}{2}$(slope $j' + \text{slope } j'' + \text{slope } j''')$ and slope $k = \frac{1}{2}$(slope $k' + \text{slope } k'' + \text{slope } k'''$).

Observe that $j(d) = a$, $j(d') = a'$, $k(d) = c$, and $k(d') = c'$. We also have:

$$j(B(k)) \subseteq \{ j(d^*) \cup j(B(k'S'))) \cup j(B(k''S'')) \cup j(B(k''S''')) \}
= \{ a^*, a^* \} \cup \{ j(B(k)) \cup T_{a' \rightarrow a} S''(B(k''S'')) \cup j'''S''(B(k''S'')) \}
= \{ a^*, a^* \} \cup \{ j(B(k)) \cup T_{a' \rightarrow a} S''(B(k''S'')) \}
[\text{by Lemma 3}]
\subseteq \{ a^*, a^* \} \cup B(f|_{a^* \rightarrow a^*}) \cup B(fT_{a^* \rightarrow a^*} - T_{a^* \rightarrow a^* - 1}) \cup B(f|_{a^* \rightarrow a^*})
= B(f).$$

Therefore, $j(B(k)) \subseteq B(f)$. Now, since $j$ has DCAV, $x \in B(j)$ only if $x$ is either a
local maximum or a local minimum of \( j \). Neither \( d^* \) nor \( d^{**} \) is a local maximum or a local minimum of \( j \), so neither is in \( B(j) \). So:

\[
k(B(j)) = k(B(j'S')) \cup k(B(T_a\alpha \cdot j''S'')) \cup k(B(j'S'''))
\]

\[
= k'(S(B(j'S'))) \cup k''S''(B(T_a\alpha \cdot j''S')) \cup k'''S''(B(j'S'''))
\]

\[
= k'(B(j')) \cup k''(B(j'')) \cup k'''(B(j''')) \quad \text{[by Lemma 3]}
\]

\[
\subseteq B(g|x_{c,-c_{-1}}|) \cup B(g|x_{c,-c_{-1}}|) \cup B(g|x_{c,-c_{-1}}|)
\]

Therefore, \( k(B(j)) \subseteq B(g) \). It is clear that \( f_j = gk \). Hence, \( j, k \) satisfy Proposition 2. By induction, Proposition 2 has been proved provided \( f(a) = b = g(c) \).

Now assume that \( f, g \) satisfy the hypothesis of Proposition 2.

Case 1. If \( f(a) = b = g(c) \), then \( j, k \) exist satisfying Proposition 2 as defined above.

Case 2. If \( f(a) = b = g(c) \), then apply Case 1 to \( f, g \)-functions to get two functions \( j, k \); now \( j, T_{cc}k \) satisfy Proposition 2.

Case 3. If \( f(a) = b = g(c') \), then apply Case 1 to \( f, g \)-functions to get two functions \( j, k \); now \( T_{aa}jT_{dd}, kT_{dd} \) satisfy Proposition 2.

Case 4. If \( f(a) = b = g(c) \), then apply Case 1 to \( f, g \)-functions to get two functions \( j, k \); now \( T_{aa}jT_{dd}, T_{cc}kT_{dd} \) satisfy Proposition 2.

This completes the proof of Proposition 2.

**Lemma 4.** Let the two functions \( h, k \) map \([a, a']\) onto \([b, b']\) and let \( h \) be linear and \( k \) be piecewise linear. If \( k \) has DCAV, then slope \( h \leq \) slope \( k \).

**The Construction.** We are now ready to construct (using Proposition 2) sequences \((f_n \mid n \in \mathbb{N})\), \((g_n \mid n \in \mathbb{N})\), \((A_n \mid n \in \mathbb{N})\) which satisfy the conditions of Proposition 1. Specifically we desire two sequences \((f_n \mid n \in \mathbb{N})\), \((g_n \mid n \in \mathbb{N})\) of piecewise linear functions mapping \([0, 1]\) into itself such that for every \( x \in [0, 1] \) either \( |x - f_2(x)| > \frac{1}{4} \) or \( |x - g_2(x)| > \frac{1}{4} \), and a sequence \((A_n \mid n \in \mathbb{N})\) of finite subsets of \([0, 1]\) such that \( 0, 1 \in A_0 \), and such that the following properties are satisfied for every \( n \in \mathbb{N} \):

- **P₁(n):** for \( i = 1, 2, 3, 4, 5 \) as in Proposition 1;
- **P₂(n):** \( f_n |\lambda_{n+1} = f_{n+1} |\lambda_{n+1} \) and \( g_n |\lambda_{n+1} = g_{n+1} |\lambda_{n+1} \);
- **P₃(n):** \( f_n(B(g_n)) \cup g_n(B(f_n)) \subseteq A_n \);
- **P₄(n):** \( A_n \subseteq A_{n+1} \) and;
- **P₁₀(n):** for all \( r', s' \) consecutive in \( A_n \) such that \( r' < s' \),

and \( f_n |[r', s'] \) and \( g_n |[r', s'] \) each have DCAV.

Now define \( A_0 = \{0, 1\} \), \( A_1 = \{0, 1/3, 2/3, 1\} \), and

\[
A_2 = \{0, 1/9, 2/9, 1/3, 6/15, 7/15, 8/15, 9/15, 2/3, 7/9, 8/9, 9/9, 1\}.
\]

Also, define \( f_0, g_0, f_1, g_1, f_2, \) and \( g_2 \) at each \( x \in [0, 1] \) as follows:

if \( 0 \leq x < 1/3 \), let \( f_0(x) = 3x \), and let \( g_0(x) = 1 - 3x \);
if \( 1/3 \leq x < 2/3 \), let \( f_0(x) = 2 - 3x \), and let \( g_0(x) = 3x - 1 \);
if \(2/3 \leq x \leq 1\), let \(f_0(x) = 3x - 2\), and let \(g_0(x) = 3 - 3x\); if \(0 \leq x < 1/3\) or \(2/3 \leq x \leq 1\), let \(f_1(x) = f_0(x)\), and let \(g_1(x) = g_0(x)\); if \(1/3 \leq x \leq 6/15\), let \(f_1(x) = (8/3) - 5x\); if \(6/15 \leq x < 7/15\), let \(f_2(x) = 5x - (4/3)\); if \(7/15 \leq x < 2/3\), let \(f_2(x) = (10/3) - 5x\); if \(1/3 \leq x \leq 8/15\), let \(g_1(x) = 5x - (5/3)\); if \(8/15 \leq x < 9/15\), let \(g_1(x) = (11/3) - 5x\); if \(9/15 \leq x \leq 2/3\), let \(g_1(x) = 5x - (7/3)\); if \(0 \leq x < 6/15\) or \(7/15 \leq x \leq 1\), let \(f_2(x) = f_1(x)\); if \(6/15 \leq x < 31/75\), let \(f_2(x) = (25/3)x - (8/3)\); if \(31/75 \leq x \leq 32/75\), let \(f_2(x) = (38/9) - (25/3)x\); if \(32/75 \leq x < 7/15\), let \(f_2(x) = (25/3)x - (26/9)\); if \(0 \leq x < 8/15\) or \(9/15 \leq x \leq 1\), let \(g_2(x) = g_1(x)\); if \(8/15 \leq x < 41/75\), let \(g_2(x) = (49/9) - (25/3)x\); if \(41/75 \leq x \leq 42/75\), let \(g_2(x) = (25/3)x - (11/3)\); if \(42/75 \leq x < 9/15\), let \(g_2(x) = (17/3) - (25/3)x\). Observe that \(f_i, g_i, A_i\) for \(i = 0, 1, 2\) have been defined satisfying the desired properties. By Figure 2 for each \(x \in [0, 1]\), \(|x - f_2(x)| > 1/6\) or \(|x - g_2(x)| > 1/6\). Also, the properties \(P_1(i-1), P_2(i), P_3(i-1), P_4(i-1), P_5(i-1), P_6(i), P_7(i-1), P_8(i-1), P_9(i-1), P_{10}(x)\) are satisfied for \(i=1, 2\). Proceed to define \((f_n \mid n \in N), (g_n \mid n \in N), \) and \((A_n \mid n \in N)\) by induction; let \(n \in N, n \geq 2\), and assume that \(f_n, g_n, A_i\) (for \(i = 0, 1, 2, \ldots, n\)) have been defined satisfying the desired properties, especially: \(P_1(i-1), P_2(i), P_3(i-1), P_4(i-1), P_5(i-1), P_6(i), P_7(i-1), P_8(i-1), P_9(i-1), P_{10}(i)\) for \(i = 1, 2, \ldots, n\). Define \(A_{n+1} = f_n^{-1}(A_n)\). Notice that

\[
A_{n+1} = f_n^{-1}(A_n) = f_n^{-1}g_n^{-1}(A_n-1) = g_n^{-1}f_n^{-1}(A_n-1) = g_n^{-1}(A_n),
\]

so \(P_0(n)\) is satisfied. For each \(x \in [0, 1]\) define \(f_{n+1}(x)\) and \(g_{n+1}(x)\) in the following manner. There exist two consecutive points \(d, d'\) of \(A_{n+1}\) such that \(d \leq x \leq d'\) (since \(A_{n+1}\) is finite and contains 0, 1). We can set \(\{a, a'\} = \{g_n(d), g_n(d')\}, \{b, b'\} = \{f_n^{-1}g_n(d), f_n^{-1}g_n(d')\}, \) and \(\{c, c'\} = \{f_n(d), f_n(d')\}\) such that \(a < a', b < b', \) and \(c < c'\) by Lemma 2. Now observe that \(f_{n\mid[a, a']}, g_{n\mid[c, c']}\) satisfy the hypothesis of Proposition 2 for \(f, g\) respectively; so let \(j, k\) be defined by Proposition 2, \(j\) mapping \([d, d']\) onto \([a, a']\) and \(k\) mapping \([d, d']\) onto \([c, c']\). If \(g_n(d) = a,\) then define \(f_{n+1}(x) = k(x),\) and \(g_{n+1}(x) = j(x);\) if \(g_n(d) = a',\) then define \(f_{n+1}(x) = kT_{dd}(x),\) and \(g_{n+1}(x) = jT_{dd}(x).\) In the above procedure, for a fixed \(x \in [0, 1]\), the points \(d, d'\) (defined to be consecutive points in \(A_{n+1}\) such that \(d \leq x \leq d'\)) are not necessarily unique. However, \(f_{n+1}\) and \(g_{n+1}\) are well defined functions since \(g_{n+1}|_{A_{n+1}} = g_n|_{A_{n+1}}\) by definition, and hence \(f_{n+1}|_{A_{n+1}} = f_n|_{A_{n+1}}.\) Therefore \(P_1(n)\) is satisfied. \(P_1(n)\) is satisfied since for each \(x \in [0, 1]\) there exist \(d, d'\) consecutive in \(A_{n+1}\) such that \(d \leq x \leq d'\) and either \(f_n g_{n+1}(x) = f_j T_{dd}(x) = g_n kT_{dd}(x) = g_n f_{n+1}(x),\) or \(f_n g_{n+1}(x) = f_k j(x) = g_n f_{n+1}(x)\) where \(j, k\) are defined by Proposition 2 as above; hence \(f_n g_{n+1}\).
Figure 2. The graphs of $f_n$, $g_n$ and the diagonal; $f_n$ or $g_n$ lies between the dotted lines wherever $|x-f_n(x)| \leq \frac{1}{k}$ or $|x-g_n(x)| \leq \frac{1}{k}$ respectively.

Comment: Simultaneous to and independent of the author’s preceding work(1), W. M. Boyce [1], [2] constructed essentially the same solution to the commuting function problem.

Remark. Simultaneous to and independent of the author’s preceding work(1), W. M. Boyce [1], [2] constructed essentially the same solution to the commuting function problem.

(1) Compare [1] and [3].
function problem defined above. These functions are nowhere Lipschitzian; smoother solutions are described in Part II below.

**Part II**

**Notation.** For any real valued mapping \( h \) defined on a subset of the reals, let \( h^\ast \) denote the map: \( h^\ast(x) = 1 - h(1 - x) \) for each \( x \) for which \( h(1 - x) \) is defined; also, \( h \) will be called \( s \)-Lipschitzian provided \( s \) is a real number and for each \( x, y \) in the domain of \( h \), \( |h(x) - h(y)| \leq s \cdot |x - y| \). Now pick any \( b \) in \([0, \frac{1}{2}]\), and define

\[
 s = \frac{3 - 2b + (6 - 4b)^{1/2}}{1 - 2b}.
\]

Define the three linear functions:

- \( h_1: [b, (1 - b + sb)/s] \to [b, 1] \) by \( h_1(x) = sx - sb + b \);
- \( h_2: [(1 - b + sb)/s, (2 - b + sb)/s] \to [0, 1] \) by \( h_2(x) = 2 - sx + sb - b \);
- \( h_3: [(2 - b + sb)/s, (3 - 2b + sb)/s] \to [0, 1 - b] \) by \( h_3(x) = -2 + sx - sb + b \).

And define the piecewise linear function \( h: [b, (3 - 2b + sb)/s] \to [0, 1] \) by \( h = h_1 \cup h_2 \cup h_3 \). Let \( C_b \) denote all continuous functions from \([0, b]\) to \([0, b]\) which have \( b \) as a fixed point.

**Definition.** For each \( g \) in \( C_b \) let \( \bar{g} \) denote the unique extension of \( g \) defined by:

1. \( \bar{g}(x) = g(x) \) whenever \( 0 \leq x \leq b \);
2. \( \bar{g}(x) = h(x) \) whenever \( b \leq x \leq h_3^{-1}(1 - b) \);
3. \( \bar{g}(x) = h_1^{-1}\bar{g}(h^\ast(x)) \) whenever \( h_3^{-1}(1 - b) \leq x \leq h_3^{-1}(h_3^{-1}(0)) \);
4. \( \bar{g}(x) = h_2^{-1}\bar{g}(h^\ast(x)) \) whenever \( h_2^{-1}(h_2^{-1}(0)) \leq x \leq 1 - b \); and
5. \( \bar{g}(x) = \) the fixed point of \( h^\ast \) whenever \( 1 - b \leq x \leq 1 \).

See Figure 3.

**Remark.** That the preceding definition is consistent can be checked by direct mechanical methods, or (as suggested by Felix Albrecht) by a Zorn’s Lemma argument. The following sketch of a proof that \( \bar{g} \) is uniquely defined above for any \( s \)-Lipschitzian \( g \) in \( C_b \) was suggested by David Boyd.

**Proof.** Let \( g \) be an \( s \)-Lipschitzian function in \( C_b \), and let \( L \) denote the set of \( s \)-Lipschitzian functions from \([0, 1]\) to itself which satisfy properties 1, 2, and 5 for \( g \) in the Definition. Now define the mapping \( T \) from \( L \) to \( L \): for any \( f \) in \( L \), let

\[
 T(f)(x) = f(x) \text{ whenever } 0 \leq x \leq h_3^{-1}(1 - b) \text{ or whenever } 1 - b \leq x \leq 1; \\
 T(f)(x) = h_1^{-1}f(h^\ast(x)) \text{ whenever } h_3^{-1}(1 - b) \leq x \leq h_3^{-1}(h_3^{-1}(0)); \\
 T(f)(x) = h_2^{-1}f(h^\ast(x)) \text{ whenever } h_2^{-1}(h_2^{-1}(0)) \leq x \leq 1 - b.
\]

To see that \( T(f) \) is in \( L \), observe that \( T(f) \) is \( s \)-Lipschitzian by a system of inequalities using the facts that \( h_1^{-1} \) and \( h_2^{-1} \) are linear and \( 1/s \)-Lipschitzian, \( h_3^{-1}(0) = h_3^{-1}(0), h^\ast \) is \( s \)-Lipschitzian, and the fixed points of \( h^\ast \) are \( h_3^{-1}(1 - b), (s - 1)(1 - b)/(s + 1) \), and \( 1 - b \).

\( L \) is clearly a complete metric space with respect to the supremum norm metric, and \( T \) is a contraction of this metric space with constant \( 1/s \) (since \( h_1^{-1} \) and \( h_2^{-1} \) are linear and \( 1/s \)-Lipschitzian).
are 1/s-Lipschitzian). Hence there is a unique function $g$ in $L$ such that $T(g) = g$, and this function satisfies the definition of $g$.

**Lemma 5.** For any $f, g$ in $C_0$ and any $x$ in $[\frac{1}{2}, 1]$,

1. $f^*(x) = x$ implies $g(x) \neq x$, and
2. $\overline{f}^*(g(x)) = g(f^*(x))$.

**Proof of (1).** As a comment to clarify notation, $\overline{f}^*(x) = (\overline{f})^*(x)$. Observe that the domain of definition of $h^*$ is $[h_{3}^{-1}(b), 1 - b]$, and

$$h_{3}^{-1}(b) = \frac{-3 + 2b + s - sb}{s} = 1 - b + \frac{(1 - 2b)(2b - 3)}{3 - 2b + 6 - 4b} < \frac{1}{2}$$

since $0 \leq b < \frac{1}{2}$. So on $[\frac{1}{2}, 1]$, the fixed points of $\overline{f}^*$ are either in $[1 - b, 1]$ or are the fixed points of $h^*$. By definition of $\overline{g}$ on $[1 - b, 1]$, $\overline{g}(x)$ equals the fixed point of $h_{3}^*$ which equals $(1 - b)(s - 1)/(1 + s) < 1 - b \leq x$ for each $x \in [1 - b, 1]$. Therefore $\overline{g}$
and \( f^* \) have no common fixed point in \([1 - b, 1]\). The only fixed points of \( h^* \) are the fixed points of \( h^*_i \) for \( i = 1, 2, 3 \). The fixed point of \( h^*_i \) is \( 1 - b \), which (as has just been seen) is not a fixed point of \( g \). Denote the fixed point of \( h^*_i \) by \( x_2 \);
\[
x_2 = \frac{(1 - b)(s - 1)}{s + 1} < \frac{b - 2 + s + s^2 - s^2 b - 2s}{s^2} = h^*_2 - 1(h^*_2 - 1(0)),
\]
and so
\[
\bar{g}(x_2) \in h^*_1 - 1([0, 1]) = [(s - 1 + b - sb)/s, (s + b - sb)/s].
\]
However,
\[
x_2 = \frac{(1 - b)(s - 1)}{s + 1} < \frac{s - 1 + b - sb}{s} \leq \bar{g}(x_2),
\]
so \( x_2 \), the fixed point of \( h^*_2 \), is not a fixed point for \( g \). Let \( x_3 \) denote the fixed point of \( h^*_3 \);
\[
x_3 = \frac{3 - s + sb - b}{1 - s} = \frac{3 - 2b + sb}{s} = h^*_3 - 1(1 - b).
\]
Therefore \( \bar{g}(x_3) = g(h^*_3 - 1(1 - b)) = 1 - b > x_3 \). Therefore \( f^* \) and \( g \) have no common fixed point in \([1, 1]\).

**Proof of (2).** For each \( x \in [1 - b, 1] \), \( f^*(g(x)) = f^*(x_2) = h^*_2(x_2) = x_2 \), and \( g(f^*(x)) = g(1 - f(1 - x)) = g(1 - f(1 - x)) = x_2 \) since \( f(1 - x) \in [0, b] \). Therefore \( f^*(g(x)) = \bar{g}(f^*(x)) \) for each \( x \in [1 - b, 1] \). For each \( x \in [h^*_2 - 1(h^*_2 - 1(0)), 1 - b] \),
\[
f^*(\bar{g}(x)) = f^*(h^*_2 - 1\bar{g}(h^*(x))) = h^*_2 h^*_2 - 1\bar{g}(h^*(x)) = \bar{g}(f^*(x));
\]
hence, \( f^*(\bar{g}(x)) = \bar{g}(f^*(x)) \). For each \( x \in [h^*_3 - 1(1 - b), h^*_3 - 1(h^*_3 - 1(0))] \),
\[
f^*(\bar{g}(x)) = f^*(h^*_3 - 1\bar{g}(h^*(x))) = h^*_3 h^*_3 - 1\bar{g}(h^*(x)) = \bar{g}(f^*(x));
\]
hence \( f^*(\bar{g}(x)) = \bar{g}(f^*(x)) \). Now \( h^*_3 - 1(1 - b) \) is the fixed point of \( h^*_3 \) (as has been seen above), so
\[
h^*_3([h^*_3 - 1(b), h^*_3 - 1(1 - b)]) = [b, h^*_3 - 1(1 - b)]
\]
which equals the domain of definition of \( h \); also \( h^*_3 - 1(1 - b) \) is the fixed point of \( h_3 \), so
\[
h_3([h^*_3 - 1(b), h^*_3 - 1(1 - b)]) = [h^*_3 - 1(b), 1 - b]
\]
which equals the domain of definition of \( h^* \). Observe that the two piecewise linear functions
\[
h^*(h|_{[h^*_3 - 1(b), h^*_3 - 1(1 - b)]}), \quad (h^*(h|_{[h^*_3 - 1(0), h^*_3 - 1(1 - b)]}))^*
\]
are each the union of three linear functions and map the points \( h^*_3 - 1(b) \), \( h^*_3 - 1(1) \), \( h^*_3 - 1(0) \), \( h^*_3 - 1(1 - b) \) to \( b, 1, 0, 1 - b \) respectively; hence the two functions coincide. Therefore for each \( x \in [\frac{1}{2}, h^*_3 - 1(1 - b)] \), \( x \in [h^*_3 - 1(b), h^*_3 - 1(1 - b)] \),
\[
f^*(\bar{g}(x)) = f^*(h_3(x)) = h^*h(x) = (h^*h)^*(x) = 1 - h^*(h(1 - x))
\]
\[
h(1 - h(1 - x)) = h(h^*(x)) = g(h^*_3(x)) = \bar{g}(f^*(x)).
\]
Therefore \( f^* \) and \( g \) commute on \([\frac{1}{2}, 1]\) and have no common fixed point in \([\frac{1}{2}, 1]\).
Proposition 3. For any \( f, g \) in \( C_b \), \( f^* \) and \( g^* \) form a solution to the commuting function problem.

Proof. Let \( f, g \) be in \( C_b \). Then by Lemma 5, \( f^* \) and \( g^* \) commute without common fixed point on \([\frac{1}{2}, 1]\); also \( f^* \) and \( g \) commute without common fixed point on \([\frac{1}{2}, 1]\). Therefore \( f^{**} \) and \( g^* \) commute without common fixed point on \([0, \frac{1}{2}]\). But \( f^{**} = f \), so \( f \) and \( g^* \) form a solution to the commuting function problem.

Corollary. If \( f, g \) are in \( C_b \), then:
1. \( f, g^* \) form a solution to the commuting function problem;
2. \( f, g^* \) are s-Lipschitzian if and only if \( f, g \) are s-Lipschitzian;
3. \( f, g^* \) are linear on each component of a dense open subset of \([0, 1]\) if and only if \( f, g \) are linear on each component of a dense open subset of \([0, 1]\); and
4. \( f, g^* \) are differentiable almost everywhere if and only if \( f, g \) are differentiable almost everywhere.

References

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